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Abstract. There is in the literature a lot of deteminant formulae involving the relative class number of an imaginary abelian number field. Most of these formulae can be obtained in a unique way by means of the Stickelberger ideal, as shown in [K]. Some papers giving the relative class number formula for intermediate fields of the cyclotomic $\mathbb{Z}_p$-extension of an imaginary abelian field in the form of a product of determinants have appeared recently (see [H], [T]). The aim of this note is to show that it is not essential to assume that we deal with a layer in the cyclotomic $\mathbb{Z}_p$-extension, the similar construction can be done for any extension of abelian fields.

1. Group determinants

Let G be a finite abelian group with a fixed element $\tau \in G$ of order 2 (i.e. $\tau \neq 1$ and $\tau^2 = 1$). Let $\widehat{G}$ be the group of characters of G (i.e. the group of all homomorphisms from G to the multiplicative group of nonzero complex numbers). We shall denote

$$\widehat{G}_- = \{\chi \in \widehat{G};\ \chi(\tau) = -1\}$$

the set of odd characters on G and

$$\widehat{G}_+ = \{\chi \in \widehat{G};\ \chi(\tau) = 1\}$$

the set of even characters on G. Let $f\colon G \to \mathbb{C}$ be any complex valued mapping which is odd, i.e. f satisfies $f(\tau\sigma) = -f(\sigma)$ for each $\sigma \in G$. Then we have the following well-known determinant formula:

Lemma 1. *For any system C of representatives of $G/\{1,\tau\}$ we have*

$$\det(f(\sigma\rho^{-1}))_{\sigma,\rho\in C} = \prod_{\chi\in\widehat{G}_-} \frac{1}{2}\sum_{\sigma\in G} \chi(\sigma) f(\sigma).$$

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Proof. Let us define the following element of the complex group ring

$$\theta = \frac{1}{2}\sum_{\sigma\in G} f(\sigma)\sigma^{-1} \in \mathbb{C}[G].$$

Since f is odd, we have $\theta \in (1-\tau)\mathbb{C}[G]$. We shall consider the linear transfomation of $(1-\tau)\mathbb{C}[G]$ given by the multiplication by θ. The matrix of this transformation with respect to the basis $\{(1-\tau)\sigma;\ \sigma \in C\}$ is $f(\sigma\rho^{-1}))_{\sigma,\rho\in C}$ because

$$\begin{aligned}(1-\tau)\sigma\theta &= \frac{1-\tau}{2}\,\sigma\sum_{\rho\in G} f(\sigma\rho^{-1})(\sigma\rho^{-1})^{-1} \\ &= \frac{1-\tau}{2}\sum_{\rho\in C}(f(\sigma\rho^{-1})\rho + f(\sigma\rho^{-1}\tau)\rho\tau) \\ &= \sum_{\rho\in C} f(\sigma\rho^{-1})(1-\tau)\rho.\end{aligned}$$

The matrix of this transformation with respect to the basis of orthogonal idempotents $\{e_\chi;\ \chi \in \widehat{G}_-\}$, where $e_\chi = \frac{1}{|G|}\sum_{\rho\in G}\chi(\rho)\rho^{-1}$, is diagonal with entries $\frac{1}{2}\sum_{\sigma\in G}\chi^{-1}(\sigma)f(\sigma)$ since

$$\begin{aligned}e_\chi\theta &= \frac{1}{2|G|}\sum_{\sigma\in G} f(\sigma)\sigma^{-1}\sum_{\rho\in G}\chi(\rho\sigma^{-1})(\rho\sigma^{-1})^{-1} \\ &= \frac{1}{2|G|}\sum_{\sigma\in G} f(\sigma)\chi(\sigma^{-1})\sum_{\rho\in G}\chi(\rho)\rho^{-1} \\ &= \frac{1}{2}\sum_{\sigma\in G}\chi^{-1}(\sigma)f(\sigma)e_\chi.\end{aligned}$$

The lemma follows due to the fact that $\chi \mapsto \chi^{-1}$ defines a bijection on $\widehat{G}_-$.
□

Let H be a subgroup of G and let

$$H^\perp = \{\chi \in \widehat{G};\ \forall h \in H\colon \chi(h) = 1\}$$

denote the subgroup of all characters of G which are trivial on H. Let $\{\psi_1, \dots, \psi_k\}$ be a fixed system of representatives of $\widehat{G}/H^\perp$. If $\tau \in H$ then $H^\perp \subseteq \widehat{G}_+$ and $\{\psi_1, \dots, \psi_k\}$ consists of the same number of odd and even characters, so we shall assume that $\psi_1, \dots, \psi_{k/2}$ are odd and $\psi_{(k/2)+1}, \dots, \psi_k$ are even. If $\tau \notin H$ then $H^\perp$ contains both odd and even characters and we shall assume that $\psi_1, \dots, \psi_k$ are all even.

For any $\chi \in \widehat{G}$ and any coset $T \in G/H$ we define

$$s_\chi(T) = \sum_{\sigma\in T}\chi(\sigma)f(\sigma).$$

Lemma 2. *If $\tau \in H$ then*

$$\det(f(\sigma\rho^{-1}))_{\sigma,\rho\in C} = \prod_{i=1}^{k/2} \det\left(\tfrac{1}{2}s_{\psi_i}(TS^{-1})\right)_{T,S\in G/H}.$$

If $\tau \notin H$ then

$$\det(f(\sigma\rho^{-1}))_{\sigma,\rho\in C} = \prod_{i=1}^{k} \det\left(s_{\psi_i}(TS^{-1})\right)_{T,S\in U},$$

where U is a system of representatives of $(G/H)/\{H,\tau H\}$.

Proof. At first, let us assume $\tau \in H$. For any $i = 1,\dots,k/2$, using [W, Lemma 5.26], we have

$$\det\left(\tfrac{1}{2}s_{\psi_i}(TS^{-1})\right)_{T,S\in G/H} = \prod_{\chi\in\widehat{G/H}} \frac{1}{2}\sum_{T\in G/H}\chi(T)s_{\psi_i}(T).$$

But there is a natural isomorphism $\widehat{G/H} \simeq H^{\perp}$ (e.g., see [W, page 23]). Hence

$$\begin{aligned}\det\left(\tfrac{1}{2}s_{\psi_i}(TS^{-1})\right)_{T,S\in G/H} &= \prod_{\chi\in H^{\perp}} \frac{1}{2}\sum_{T\in G/H}\sum_{\sigma\in T}\chi(\sigma)\psi_i(\sigma)f(\sigma)\\ &= \prod_{\chi\in\psi_i\cdot H^{\perp}} \frac{1}{2}\sum_{T\in G/H}\sum_{\sigma\in T}\chi(\sigma)f(\sigma).\end{aligned}$$

But $\widehat{G}_-$ equals to the disjoint union of cosets $\psi_i\cdot H^{\perp}$ for $i = 1,\dots,k/2$ and Lemma 1 implies the first equality.

Now, let us assume $\tau \notin H$. Lemma 1 gives for any $i = 1,\dots,k$,

$$\begin{aligned}\det\left(s_{\psi_i}(TS^{-1})\right)_{T,S\in U} &= \prod_{\chi\in\widehat{G/H}_-} \frac{1}{2}\sum_{T\in G/H}\chi(T)s_{\psi_i}(T)\\ &= \prod_{\chi\in\widehat{G/H}_-} \frac{1}{2}\sum_{T\in G/H}\chi(T)\sum_{\sigma\in T}\psi_i(\sigma)f(\sigma)\\ &= \prod_{\chi\in(H^{\perp})_-} \frac{1}{2}\sum_{T\in G/H}\sum_{\sigma\in T}\chi(\sigma)\psi_i(\sigma)f(\sigma)\\ &= \prod_{\chi\in\psi_i\cdot(H^{\perp})_-} \frac{1}{2}\sum_{T\in G/H}\sum_{\sigma\in T}\chi(\sigma)f(\sigma).\end{aligned}$$

The lemma follows using the fact that $\widehat{G}_-$ equals to the disjoint union of $\psi_i\cdot(H^{\perp})_-$ for $i = 1,\dots,k$. □

2. Relative class number formulae

For any abelian field L, let $G_L = \mathrm{Gal}(L/\mathbb{Q})$ be its Galois group, X_L the group of primitive Dirichlet characters corresponding to L, and $X_L^+ = \{\chi \in X_L;\, \chi(-1) = 1\}$ the subgroup of even characters and $X_L^- = \{\chi \in X_L;\, \chi(-1) = -1\}$ the subset odd characters. For any integer t relatively prime to the conductor m of L, let $(t, L) \in G_L$ be the automorphism given by the Artin map, i.e., (t, L) is the restriction to L of the automorphism of the m-th cyclotomic field which sends each root of unity to its t-th power.

Let us fix an imaginary abelian field K of conductor m. For any divisor $n > 1$ of m and any $\sigma \in G_K$, let us choose and fix a rational number $a_{n,\sigma}$. We put

$$t_\sigma = \sum_{1<n|m} \sum_{t \bmod^\times m} \left(\tfrac{1}{2} - \left\langle \tfrac{t}{n} \right\rangle\right) a_{n,(t,K)\sigma^{-1}}$$

for any $\sigma \in G_K$, where the second sum is taken over a reduced residue system modulo m. Let

$$b_{n,\chi} = \frac{\phi(m)}{\phi(n)} \sum_{\sigma \in G_K} a_{n,\sigma}\chi(\sigma),$$

for any $1 < n \mid m$ and any $\chi \in X_K$, where ϕ means the Euler ϕ-function. Let $\tau \in G_K$ be the complex conjugation. Then for any system C of representatives of $G_K/\{1, \tau\}$, Theorem 1 of [K] gives the following formula

$$\det(t_{\sigma\rho^{-1}})_{\sigma,\rho \in C} = \frac{h_K^-}{Q_K w_K} \prod_{\chi \in X_K^-} \sum_{n|m,\, f_\chi|n} b_{n,\chi} \prod_{p|n} (1 - \chi(p)), \tag{2.1}$$

where h_K^-, Q_K, and w_K are the relative class number of K, the Hasse unit index of K, and the number of roots of unity in K, respectively, the product is taken over all prime divisors of n, and f_χ is the conductor of χ.

We have shown in [K] that by a suitable choice of parameters $a_{n,\sigma}$ one can obtain many of the known relative class number formulae. Recently, for the cyclotomic $\mathbb{Z}_p$-extension of an imaginary abelian field, Tsumura and Hirabayashi (see [T] and [H]) has constructed formulae, where the relative class number of an intermediate field of the $\mathbb{Z}_p$-extension is given in the form of a product of determinants. Now, we shall show that the same job can be done in a simpler setting: for any extension of abelian fields.

We shall keep all notation of this section and assume that L is a subfield of K. For any $\sigma \in G_L$ and any $\chi \in X_K$ let

$$s_\chi(\sigma) = \sum_{\rho \in G_K,\, \rho|_L = \sigma} t(\rho)\chi(\rho),$$

where the sum is taken over all automorphisms ρ of K whose restriction to L is σ. Let $\{\psi_1, \ldots, \psi_k\}$ be a system of representatives of X_K/X_L. If L is real then this system consists of the same number of odd and even characters, so we shall assume that $\psi_1, \ldots, \psi_{k/2}$ are odd and $\psi_{(k/2)+1}, \ldots, \psi_k$ are even. If L is imaginary then we shall assume that $\psi_1, \ldots, \psi_k$ are all even.

Theorem. *If L is real then*

$$\det(t(\sigma\rho^{-1}))_{\sigma,\rho\in C} = \prod_{i=1}^{k/2} \det\left(\tfrac{1}{2}s_{\psi_i}(\sigma\rho^{-1})\right)_{\sigma,\rho\in G_L}. \tag{2.2}$$

If L is imaginary then

$$\det(t(\sigma\rho^{-1}))_{\sigma,\rho\in C} = \prod_{i=1}^{k} \det\left(s_{\psi_i}(\sigma\rho^{-1})\right)_{\sigma\rho\in U}, \tag{2.3}$$

where U is a system of representatives of $G_L/\{1,\tau|_L\}$.

Proof. This follows from Lemma 2. □

Now, putting (2) or (3) together with (1), we arrive at the relative class number formula in the form of the product of determinants. For example, if L is an imaginary abelian field and K an intermediate field in its cyclotomic $\mathbb{Z}_p$-extension, then (1) and (3) for $a_{n,\sigma}$ as in [K, Example 1] we obtain the main result of [H].

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