Acta Mathematica et Informatica Universitatis Ostraviensis

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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 10 (2002), No. 1, 35--41

Persistent URL: http://dml.cz/dmlcz/120583

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A notě on a páper of Oppenheim and Salát concerning series of Cantor type

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Abstract. The main resuíts of this páper are two criteria for irrational series which consist of rational numbers where the denominators are speciál integers and numerators are too large. Several applications for second-order linear recurrences are included.

1. Introduction

There exist many papers concerning the irrationality of infinite series. Erdós and Straus [3] proved that if A is a positive integer greater than oné and *{c_n}'^L_l* is a sequence of integers such that *J2^=i* knl^ -2" < oo then the number

*=£A2« .

is irrational. A similar criterion can be found in [5]. Duverney [2] proved another type of criterion for irrationality of infinite series.

If the series consists of rational numbers in the reduced form and the denominator of the previous term divides the denominator of the next term then we call the series of Cantor type. Oppenheim in [6] proved the following theorem.

Theorem 1.1. *Let* {a_n}^°=1 *and* {b_n}'^L_1 *be two sequences of integers such that* $a_n > 1$ *and* $|b_n| < a_n$ *hold for every sufficiently large n. Suppose that*

nnfM±i = 0.

Then the number

:Y\ t'i n?=i ªJ

Received: March 11, 2002.
1991 *Mathematics Subject Classification:* 11J72.
Key words and phrases: Irrationality, infinite series.
Supported by the grant no. 201/01/0471 of the Czech Grant Agency.

is rational iff $b_n = 0$ *for every sufficiently large positive integer* n.

This theorem and its proof can also be found in a paper of Šalát [7]. As a consequence of a more general theorem, Bundschuh and Pethö [1] proved the following result.

Theorem 1.2. *Let* $\{R_n\}_{n=1}^{\infty}$ *be a second-order linear recurrence with characteristic polynomial* $x^2 - A_1 x - 1$ *where* A_1 *is a positive integer and* $R_0 = 0$, $R_1 = 1$. *Suppose that* $\{b_n\}_{n=1}^{\infty}$ *is a sequence of integers such that* $|b_n|$ *is not a constant for every large positive integer* n *and there is a positive real number* ϵ *such that for every sufficiently large* n

$$|b_n| \le R_{2^n-1}^{1-\epsilon}.$$

Then the number

$$\sum_{n=1}^{\infty} \frac{b_n}{R_{2^n}}$$

is transcendental.

Hančl and Kiss [4] proved the irrationality of this series under weaker conditions.

2. Main results

The main results of this paper are Theorems 2.1 and 2.2. These theorems deal with criteria for the irrationality of infinite convergent series of Cantor type. The terms of these series consist of special rational numbers which do not depend on arithmetical properties such as divisibility.

Theorem 2.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers greater than one. Suppose that* $\{b_n\}_{n=1}^{\infty}$ *is a sequence of integers such that*

$$\liminf_{n\to\infty} \frac{|b_n|+1}{a_n} = 0 \tag{18}$$

and for every sufficiently large positive integer n

$$|b_{n+1}| \le \frac{1}{2}\max(|b_n|, 1)a_{n+1}. \tag{19}$$

Then the number

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{\prod_{j=1}^{n} a_j}$$

is rational iff $b_n = 0$ *for every sufficiently large positive integer* n.

Theorem 2.2. *Let* K *be a real number with* $0 < K < 1$. *Suppose that* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of positive integers greater than one and* $\{b_n\}_{n=1}^{\infty}$ *is a sequence of nonnegative integers such that*

$$\liminf_{n\to\infty} \frac{b_n+1}{a_n} = 0, \tag{20}$$

and for every sufficiently large positive integer n

$$b_{n+1} \le K\max(b_n, 1)a_{n+1}. \tag{21}$$

Then the number

$$\beta = \sum_{n=1}^{\infty} \frac{b_n}{\prod_{j=1}^{n} a_j}$$

is rational iff $b_n = 0$ *for every sufficiently large positive integer* n.

Corollary 2.1. *Let* t *be a positive integer. Assume that* $\{R_n\}_{n=1}^{\infty}$ *is the second-order linear recurrence with characteristic polynomial* $Q(x) = x^2 - A_1 x - A_2$ *where* A_1 *and* A_2 *are integers and* $R_0 = 0$, $R_1 = 1$. *Suppose that the polynomial* $Q(x)$ *has zeros* α_1 *and* α_2 *with* $|\alpha_1| > |\alpha_2|$. *Assume that* $\{b_n\}_{n=1}^{\infty}$ *is a sequence of integers such that*

$$\liminf_{n\to\infty} \frac{b_n}{\alpha_1^{t2^{n-1}}} = 0 \tag{22}$$

and for every sufficiently large positive integer n

$$|b_{n+1}| \leq \frac{1}{2} \max(|b_n|, 1)|\alpha_1^{t2^n} + \alpha_2^{t2^n}|. \tag{23}$$

Then the number

$$\sum_{n=1}^{\infty} \frac{b_n}{R_{t2^n}}$$

is rational iff $b_n = 0$ *for every sufficienty large positive integer* n.

Example 2.1. *Let* $\pi(n)$ *be the number of primes less than or equal* n *and let* $T(n)$ *be the greatest prime cube less than or equal* n. *As an immediate consequence of Theorem 2.1 we obtain that the number*

$$\sum_{n=1}^{\infty} (-1)^{\pi(n)} \frac{\frac{n!}{(T(n))!} + \pi(n)}{3^n n!}$$

is irrational.

Example 2.2. *Let the numbers* t, A_1, A_2, α_1, α_2 *and the sequence* $\{R_n\}_{n=1}^{\infty}$ *satisfy all conditions stated in Corollary 2.1. Assume that* s *is a positive integer with* $|\alpha_1|^s > 2$. *Let* $P(n)$ *denote the greatest prime less than* n. *As an immediate consequence of Corollary 2.1 we obtain that the number*

$$\sum_{n=1}^{\infty} \frac{R_{t(2^n - 2^{P(n)}) - sn}}{R_{t2^n}}$$

is irrational.

Example 2.3. *Let the numbers* t, A_1, A_2, α_1, α_2 *and the sequence* $\{R_n\}_{n=1}^{\infty}$ *satisfy all conditions stated in Corollary 2.1. Assume that* $A_1^2 - 4A_2 > 0$. *Let* $Q(n)$ *be the greatest prime square less than* n. *As an immediate consequence of Theorem 2.2 we obtain that the number*

$$\sum_{n=1}^{\infty} \frac{R_{t(2^n - 2^{Q(n)}) - 2n}}{R_{t2^n}}$$

is irrational.

Example 2.4. *Let t be a positive integer. Assume that $\{F_n\}_{n=1}^{\infty}$ is the Fibonacci sequence. Let $S(n)$ be the greatest Fibonacci number less than n. As an immediate consequence of Theorem 2.2 we obtain that the number*

$$\sum_{n=1}^{\infty} \frac{F_{t(2^n-2^{S(n)})-n}}{F_{t2^n}}$$

is irrational.

Open problem 2.1. *Let $\{F_n\}_{n1}^{\infty}$ be the Fibonacci sequence. Is the number*

$$\sum_{n=1}^{\infty} \frac{F_{2^n-n}}{F_{2^n}+1}$$

irrational?

3. Proofs of theorems and of corollary

Proof. (of Theorem 2.1) Assume that the number α is rational and there exist infinitely many n such that $b_n \neq 0$. Then there are integers p and q with $q > 0$ such that $\alpha = \frac{p}{q}$. ¿From this we obtain that for every positive integer N

$$\alpha = \frac{p}{q} = \sum_{n=1}^{\infty} \frac{b_n}{\prod_{j=1}^{n} a_j} = \sum_{n=1}^{N} \frac{b_n}{\prod_{j=1}^{n} a_j} + \sum_{n=N+1}^{\infty} \frac{b_n}{\prod_{j=1}^{n} a_j}.$$

Thus for every positive integer N the number

$$I_N = q(\prod_{j=1}^{N} a_j)(\frac{p}{q} - \sum_{n=1}^{N} \frac{b_n}{\prod_{j=1}^{n} a_j}) = q \sum_{n=N+1}^{\infty} \frac{b_n}{\prod_{j=N+1}^{n} a_j} \tag{24}$$

is an integer. We also have

$$|b_{n+k}| \leq \frac{1}{2^k} \max(|b_n|, 1) \prod_{j=n+1}^{n+k} a_j \tag{25}$$

for every positive integer k and for every sufficiently large positive integer n which can be proved by mathematical induction using inequality (19). From (18) we obtain that there exist infinitely many positive integers M such that

$$\frac{|b_M|+1}{a_M} \leq \frac{1}{3q}. \tag{26}$$

Without loss of generality assume that the number M is sufficiently large.

Now we prove that $|I_{M-1}| < 1$. Equation (24) and inequalities (25) and (26) imply that

$$|I_{M-1}| = q|\sum_{n=M}^{\infty} \frac{b_n}{\prod_{j=M}^{n} a_j}| \leq q \sum_{n=M}^{\infty} \frac{|b_n|}{\prod_{j=M}^{n} a_j} \leq$$

$$q\frac{|b_M|}{a_M} + q \sum_{n=M+1}^{\infty} \frac{\frac{1}{2^{n-M}} \max(|b_M|, 1) \prod_{j=M+1}^{n} a_j}{\prod_{j=M}^{n} a_j} =$$

$$q\frac{|b_M|}{a_M}+\frac{q(\max(|b_M|,1))}{a_M}\sum_{j=1}^{\infty}\frac{1}{2^j}=q\frac{|b_M|+\max(|b_M|,1)}{a_M}\leq$$

$$2q\frac{|b_M|+1}{a_M}\leq\frac{2}{3}<1. \tag{27}$$

Now we prove that $|I_{M-1}|>0$. Let P be the least positive integer greater or equal to M such that $b_P\neq 0$. Hence

$$1\leq |b_P|. \tag{28}$$

¿From (24), (25) and (28) we obtain that

$$|I_{M-1}|=q|\sum_{n=M}^{\infty}\frac{b_n}{\prod_{j=M}^{n}a_j}|=q|\sum_{n=P}^{\infty}\frac{b_n}{\prod_{j=M}^{n}a_j}|\geq$$

$$q(\frac{|b_P|}{\prod_{j=M}^{P}a_j}-\sum_{n=P+1}^{\infty}\frac{|b_n|}{\prod_{j=M}^{n}a_j})\geq$$

$$q(\frac{|b_P|}{\prod_{j=M}^{P}a_j}-\sum_{n=P+1}^{\infty}\frac{\frac{1}{2^{n-P}}\max(|b_P|,1)\prod_{j=P+1}^{n}a_j}{\prod_{j=M}^{n}a_j})=$$

$$q(\frac{|b_P|}{\prod_{j=M}^{P}a_j}-\frac{(\max(|b_P|,1)}{\prod_{j=M}^{P}a_j}\sum_{j=1}^{\infty}\frac{1}{2^j})=q(\frac{|b_P|-\max(|b_P|,1)}{\prod_{j=M}^{P}a_j})=0. \tag{29}$$

Inequality (29) implies that if $I_{M-1}=0$ then for every $N\geq P+1$

$$b_N=\frac{1}{2^{N-P}}\max(|b_P|,1)\prod_{j=P+1}^{N}a_j=\frac{1}{2^{N-P}}|b_P|\prod_{j=P+1}^{N}a_j.$$

Thus

$$\liminf_{n\to\infty}\frac{1+|b_n|}{a_n}\geq\liminf_{n\to\infty}\frac{|b_n|}{a_n}=$$

$$\liminf_{n\to\infty}\frac{\frac{1}{2^{n-P}}|b_P|\prod_{j=P+1}^{n}a_j}{a_n}=\liminf_{n\to\infty}\frac{1}{2^{n-P}}|b_P|\prod_{j=P+1}^{n-1}a_j\geq$$

$$\liminf_{n\to\infty}\frac{1}{2^{n-P}}|b_P|2^{n-P-2}=\frac{1}{4}|b_P|$$

which contradicts (18). Hence $|I_{M-1}|>0$. ¿From this and (27) we obtain $0<I_{M-1}<1$. This contradicts the fact that the number I_{M-1} is an integer and the proof of Theorem 2.1 is complete. □

Proof. (of Theorem 2.2) Assume that the number β is rational and there exist infinitely many positive integers n such that $b_n\neq 0$. Then there are positive integers p and q such that

$$\beta=\frac{p}{q}=\sum_{n=1}^{\infty}\frac{b_n}{\prod_{j=1}^{n}a_j}.$$

This implies that for every positive integer N the number

$$I_N = q(\prod_{j=1}^{N} a_j)(\frac{p}{q} - \sum_{n=1}^{N} \frac{b_n}{\prod_{j=1}^{n} a_j}) = q \sum_{n=N+1}^{\infty} \frac{b_n}{\prod_{j=N+1}^{n} a_j} \tag{30}$$

is a positive integer. We also have

$$|b_{n+k}| \leq K^k \max(|b_n|, 1) \prod_{j=n+1}^{n+k} a_j \tag{31}$$

for every positive integer k and for every sufficiently large positive integer n which can be proved by mathematical induction using inequality (21). From (20) we obtain that there exists sufficiently large positive integer M such that

$$\frac{b_M + 1}{a_M} \leq \frac{1}{3q(1-K)}. \tag{32}$$

Equation (30) and inequalities (31) and (32) imply that

$$I_{M-1} = q \sum_{n=M}^{\infty} \frac{b_n}{\prod_{j=M}^{n} a_j} \leq q \sum_{n=M}^{\infty} \frac{K^{n-M} \max(b_M, 1) \prod_{j=M+1}^{n} a_j}{\prod_{j=M}^{n} a_j} \leq$$

$$q \frac{\max(b_M, 1)}{a_M} \sum_{j=0}^{\infty} K^j = q \frac{\max(b_M, 1)}{a_M} \frac{1}{1-K} < q \frac{b_M + 1}{a_M} \frac{1}{1-K} < \frac{1}{3}.$$

This contradicts the fact that the number I_{M-1} is a positive integer and the proof of Theorem 2.2 is complete. □

Proof. (of Corollary 2.1) We can write the terms of the sequence $\{R_n\}_{n=1}^{\infty}$ in the form

$$R_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

This implies that

$$R_{t2^n} = \frac{\alpha_1^{2t} - \alpha_2^{2t}}{\alpha_1 - \alpha_2} \prod_{j=1}^{n-1} (\alpha_1^{t2^j} + \alpha_2^{t2^j}) =$$

$$R_{2t} \prod_{j=1}^{n-1} (\alpha_1^{t2^j} + \alpha_2^{t2^j}) \tag{33}$$

where for every $j = 1, 2, \ldots, n-1$ the number

$$\alpha_1^{t2^j} + \alpha_2^{t2^j}$$

is an integer. From (22), (23) and (33) we obtain (18) and (19). Now we apply Theorem 2.1 and the proof of Corollary 2.1 is complete. □

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