Gérard Boutteaux; Stéphane Louboutin

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The class number one problem for the non-normal sextic CM-fields. Part 2.

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Abstract. We delineate the determination of all the non-normal sextic CM-fields with class number one and whose maximal totally real subfields are non-normal cubic field. There are 367 non-isomorphic such fields. Since we had already proved elsewhere that there are 19 non-isomorphic non-normal sextic CM-fields with class number one and whose maximal totally real subfields are cyclic cubic fields, we are now in a position to conclude that there are 386 non-isomorphic non-normal sextic CM-fields which have class number one.

1. Introduction

Lately, great progress have been made towards the determination of all the normal CM-fields with class number one. Due to the work of S.-H. Kwon, Y. Lefeuvre, F. Lemmermeyer, S. Louboutin, R. Okazaki, Y.-H. Park and Y.-S. Yang, all the normal CM-fields of degrees less than or equal to 48 with class number one are known (with only partial solutions in the special cases of normal fields of degree 32 and 48). In contrast, up to now the determination of all the non-normal CM-fields with class number one and of a given degree has only been solved for quartic fields (see [LO]). The present piece of work is an abridged version of half the work to be completed in [Bou] (the PhD thesis of the first author under the supervision of the second auhtor): the determination of all the non-normal sextic CM-fields with class number one, regardless whether their maximal totally real subfield is a non-normal totally real cubic field (the situation dealt with in the present paper) or a real cyclic cubic field (the situation dealt with in [BL]).

Let K be a CM-field, i.e. K is a totally imaginary number field, hence of even degree $2n \geq 2$, and K is a quadratic extension of its maximal totally real subfield k, hence k is of degree n. We let h_K^-, $Q_K \in \{1,2\}$ and w_K denote its relative class number, its Hasse unit index and the number of complex roots of unity contained

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in K, respectively. Note that $w_K = w_A$ where A is the maximal abelian subfield of K. We have (see [Wa, Chapter 4]):

$$h_K^- = \frac{Q_K w_K}{(2\pi)^n}\sqrt{\frac{d_K}{d_k}}\frac{\mathrm{Res}_{s=1}(\zeta_K)}{\mathrm{Res}_{s=1}(\zeta_k)}, \tag{1}$$

where d_K and d_k denote the absolute values of the discriminants of K and k, respectively.

2. On CM-fields of odd class numbers

Proposition 1. *Let K be a CM-field of degree $2n$, $n > 1$ odd, and let $\mathcal{F}_{K/k}$ denote the finite part of the conductor of the quadratic extension K/k.*

(1) *At least one prime ideal of k is ramified in the quadratic extension K/k. Therefore, $d_K \geq 3d_k^2$ and the narrow class number h_k^+ of k divides the class number h_K of K. Consequently, if h_K is odd then h_k^+ is odd, $h_k^+ = h_k$ and every totally positive unit of k is the square of some unit of k.*

(2) *Assume that h_K^- is odd. Then, exactly one prime ideal $\mathcal{Q}$ of k is ramified in the quadratic extension K/k and $Q_K = 1$. Finally, if $\mathcal{Q}$ is above an odd rational prime q, then $\mathcal{F}_{K/k} = \mathcal{Q}$, $q \equiv 3 \pmod 4$ and the inertia degree f of $\mathcal{Q}$ is odd.*

Proof.

(1) Let $\mathcal{F}_{K/k}$ denote the finite part of the conductor of the quadratic extension K/k and let χ be the quadratic character associated with this quadratic extension K/k. According to class field theory, there exists some primitive quadratic character χ_0 on the multiplicative group $(A_k/\mathcal{F}_{K/k})^*$ (where A_k denotes the ring of algebraic integers of k) such that for any $\alpha \in A_k$ we have $\chi((\alpha)) = \nu(\alpha)\chi_0(\alpha)$ where $\nu(\alpha) \in \{\pm 1\}$ denotes the sign of the norm $N_{K/\mathbf{Q}}(\alpha)$ of α. In particular, if K/k is unramified at all the finite places of k then for any algebraic unit ϵ of k we must have $\nu(\epsilon) = +1$. Taking $\epsilon = -1$ for which $\nu(\epsilon) = (-1)^n$, we obtain that n must be even.

(2) Let t denote the number of primes ideals of k which are ramified in the quadratic extension K/k. According to the first point of this Proposition, we have $t \geq 1$. Assume that h_K^- is odd. Since 2^{t-1} divides h_K^- (see [Lou3, Proposition 6]), we have $t = 1$. Moreover, since 2^t divides h_K^- if $Q_K = 2$ (see [Lou3, Proposition 6]), we have $Q_K = 1$. Now, if q is odd, then K/k is tamely ramified, hence $\mathcal{F}_{K/k} = \mathcal{Q}$. Since the multiplicative group $(A_k/\mathcal{Q})^*$ is cyclic of order $q^f - 1$ and since $\nu(-1) = (-1)^n = -1$ (by the first point), we have $1 = \chi((-1)) = \nu(-1)\chi_0(-1) = -\chi_0(-1) = -(-1)^{(q^f-1)/2}$, which yields $q^f \equiv 3 \pmod 4$. Hence, $q \equiv 3 \pmod 4$ and f odd. •

Lemma 2. *Let $K/k = k(\sqrt{\alpha})/k$ be a quadratic extension of number fields, where α is an algebraic integer of k. There exists some integral ideal $\mathcal{I}$ of k such that $(4\alpha) = \mathcal{I}^2\mathcal{F}_{K/k}$.*

Proof. Let A_k denote the ring of algebraic integers of k and $\mathcal{D}_{K/k}$ denote the different of the quadratic extension K/k. Since $\mathcal{D}_{K/k} = \gcd\{(2\sqrt{\alpha});\ \alpha \in A_k$ and $K =$

$k(\sqrt{\alpha})\}$ (see [Lan, Chapter III, Prop. 8]), for any $\alpha \in A_k$ such that $K = k(\sqrt{\alpha})$ there exists an integral ideal $\mathcal{I}_K$ of K such that $(2\sqrt{\alpha}) = \mathcal{I}_K^2 \mathcal{D}_{K/k}$. Taking relative norms, we do get $(4\alpha) = \mathcal{I}^2 N_{K/k}(\mathcal{D}_{K/k}) = \mathcal{I}^2 \mathcal{F}_{K/k}$ where $\mathcal{I} = N_{K/k}(\mathcal{I}_K)$. •

Proposition 3. *Let K be a CM-field of degree $2n$, $n > 1$ odd, let $\mathcal{F}_{K/k}$ denote the finite part of the conductor of the quadratic extension K/k, assume that the class number h_K of K is odd, let $\mathcal{Q}$ denote the unique prime ideal $\mathcal{Q}$ of k which is ramified in the quadratic extension K/k (see* Proposition 1*) and let $q \geq 2$ denote the rational prime below $\mathcal{Q}$.*

(1) *If $q > 2$ then $\mathcal{F}_{K/k} = \mathcal{Q}$.*
(2) *If $q = 2$ and $\sqrt{-1} \notin K$, then there exists $r \geq 1$ odd such that $\mathcal{F}_{K/k} = \mathcal{Q}^r$.*
(3) *If [$q = 2$, $\sqrt{-1} \notin K$ and $\sqrt{-q} \notin K$] or if [$q > 2$ and $\sqrt{-q} \notin K$], then q is neither totally ramified in $k/\mathbf{Q}$ nor inert in $k/\mathbf{Q}$.*

Proof. Let α be any totally positive algebraic element of k such that $K = k(\sqrt{-\alpha})$. There exists some integral ideal $\mathcal{I}$ of k such that $(4\alpha) = \mathcal{I}^2 \mathcal{F}_{K/k}$, by Lemma 2. Set $h := h_k^+ = h_k$, which is odd by Proposition 1.

(1) K/k is tamely ramified.
(2) If $\mathcal{F}_{K/k} = \mathcal{Q}^r$ with r even, then $(4\alpha) = \mathcal{J}^2$ with $\mathcal{J} := \mathcal{I}\mathcal{Q}^{r/2}$. Since h is odd and since both $\mathcal{J}^2$ and $\mathcal{J}^h$ are principal in the narrow sense, $\mathcal{J}$ is principal in the narrow sense and there exists some totally positive unit ϵ of k such that $4\alpha = \epsilon\beta^2$. Since ϵ is the square of some unit of k (see Point 1 of Proposition 1), we obtain $4\alpha = \gamma^2$ for some $\gamma \in k$ and $K = k(\sqrt{-\alpha}) = k(\sqrt{-1})$, a contradiction.
(3) Assume that $(q) = \mathcal{Q}^n$ is totally ramified in $k/\mathbf{Q}$ or that $(q) = \mathcal{Q}$ is inert in $k/\mathbf{Q}$. Since $\mathcal{F}_{K/k} = \mathcal{Q}^r$ for some odd r (by the previous two points) and since n is odd, we obtain $(4q\alpha) = \mathcal{J}^2$ for some integral ideal $\mathcal{J}$ of k. As in the previous point, we obtain $4q\alpha = \gamma^2$ for some $\gamma \in k$ and $K = k(\sqrt{-\alpha}) = k(\sqrt{-q})$, a contradiction. •

3. Bounds on residues

Lemma 4. *Let K be a totally imaginary number field of degree $2n \geq 2$. Set $\lambda_n = n(\gamma + \log(2\pi)) - 1$, where $\gamma = 0.577\cdots$ denotes Euler's constant. Then, $\frac{1}{2} \leq \beta < 1$ and $\zeta_K(\beta) \leq 0$ imply*

$$\mathrm{Res}_{s=1}(\zeta_K) \geq (1-\beta) d_K^{(\beta-1)/2} (1 + \lambda_n(1-\beta))(1 - \frac{2\pi n d_K^{(1-\beta)/2n}}{d_K^{1/2n}}).$$

Proof. Let K be a totally imaginary number field of degree $2n \geq 2$. Assume that $\zeta_K(\beta) \leq 0$ for some β satisfying $\frac{1}{2} \leq \beta < 1$. According to the proof of [Lou2, Prop A], Hecke's integral representations of Dedekind zeta functions yield

$$\mathrm{Res}_{s=1}(\zeta_K) \geq (1-\beta) d_K^{(\beta-1)/2} f_n(\beta)(1 - 2\pi n d_K^{-\beta/2n}).$$

where $f_n(\beta) = \beta(\Gamma(\beta)/(2\pi)^{\beta-1})^n$ is positive and log-convex in the range $\beta > 0$, hence convex in the same range. Since $f_n'(1) = (f_n'/f_n)(1) = -\lambda_n$, we obtain $f_n(\beta) \geq f_n(1) + (\beta - 1)f_n'(1) = 1 + \lambda_n(1-\beta)$ in the range $0 < \beta < 1$. •

To get the term $(1-\beta)d_K^{(\beta-1)/2}$ as large as possible, we would like to be allowed to choose $\beta = 1-2/\log d_K$, and $\zeta_K(1-(2/\log d_K)) \leq 0$ would imply $\mathrm{Res}_{s=1}(\zeta_K) \geq (2+o(1))/\log d_K$ where $o(1)$ is an explicit error term which approaches to zero as d_K goes to infinity. Discarding this error term, we would obtain (see the proofs of Theorems 8 and 12): Let K be a sextic CM-field and assume that $\zeta_K(1-(2/\log d_K)) \leq 0$. Then,

$$h_K^- \geq \frac{4\sqrt{d_K/d_F}}{e\pi^3(\log d_F)^2\log d_K}, \tag{2}$$

and $h_K^- = 1$ would imply $d_K \leq 2\cdot 10^{23}$, $\rho_K := d_K^{1/6} \leq 7500$ and $d_F \leq 2\cdot 10^{11}$. Since we do not know how to prove that $\zeta_K(1-(2/\log d_K)) < 0$ (which would hold true if ζ_K had no real zero in the range $1-(2/\log d_K) \leq s < 1$), we will have to be a little more clever and we will obtain worse bounds (see Theorems 8 and 12 below).

Proposition 5.

(1) *Let $a > 0$ be given and let K range over CM-field sextic fields.*
   (a) *There exists d_a effective such that $d_K \geq d_a$ and $\zeta_K(1-(1/a\log d_K)) \leq 0$ imply*

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{1}{ae^{1/2a}\log d_K}. \tag{3}$$

   (b) *For any $c > 1$, there exists $d_{a,c}$ effective such that $d_K \geq d_{a,c}$, $1-(1/a\log d_K) \leq \beta < 1$ and $\zeta_K(\beta) \leq 0$ imply*

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{1-\beta}{ce^{1/2a}}. \tag{4}$$

(2) *(See* [Lou7]*). If F is totally real cubic number field, then*

$$\mathrm{Res}_{s=1}(\zeta_F) \leq \frac{1}{8}\log^2 d_F, \tag{5}$$

and $\frac{1}{2} \leq \beta < 1$ and $\zeta_F(\beta) = 0$ imply

$$\mathrm{Res}_{s=1}(\zeta_F) \leq \frac{1-\beta}{48}\log^3 d_F. \tag{6}$$

Proof. According to Lemma 4, $\zeta_K(1-(1/a\log d_K)) \leq 0$ implies

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{F_a(d_K)}{ae^{1/2a}\log d_K}$$

where $F_a(x) = (1+(\lambda_3/a\log x))(1-6\pi e^{1/6a}/x^{1/6})$ is clearly ≥ 1 for $x \geq d_a$ large enough (notice that $\lambda_3 = 6.245278\cdots$). In the same way, $1-(1/a\log d_K) \leq \beta < 1$ and $\zeta_K(\beta) \leq 0$ imply

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{1-\beta}{e^{1/2a}}G_a(d_K)$$

where $G_a(x) = 1-6\pi e^{1/6a}/x^{1/6}$ is $\geq 1/c$ for $x \geq d_{a,c} = (\frac{6\pi c}{c-1})^6 e^{1/a}$ large enough. •

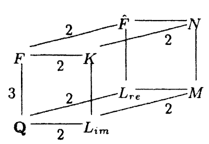
Lemma 6. *(See* [LLO, Lemma 15]*). Set* $c_1 = (3+2\sqrt{2})/2 < 3$ *and* $c_2 = (2+\sqrt{3})/4 < 1$. *The Dedekind zeta function of a number field* M *has at most one real zero in the range* $1 - (1/c_1 \log d_M) \leq s < 1$ *and at most two real zeros in the range* $1 - (1/c_2 \log d_M) \leq s < 1$.

4. Non-normal sextic CM-fields with non-normal maximal totally real real cubic subfields

From now on, we let K denote a non-normal sextic CM-field whose maximal totally real subfield $F = k$ is a non-normal cubic field. We write $K = F(\sqrt{-\delta})$ where δ is a totally positive algebraic element of F. We let $\hat{F}$ denote the normal closure of F. Hence, $\hat{F}$ is a real non-abelian normal sextic field with Galois group the dihedral group of order six, and we let L_{re} denote the only quadratic subfield of $\hat{F}$. Hence, $L_{re} = \mathbf{Q}(\sqrt{d_F})$ and $\hat{F} = FL_{re} = F(\sqrt{d_F})$. Recall that there exists some integer $f \geq 1$ such that $d_F = d_{L_{re}} f^2$ and $d_{\hat{F}} = d_{L_{re}}^3 f^4 = d_{L_{re}} d_F^2$ (see [Mar]). In particular, $d_{L_{re}}^2 \mid d_F^2 \mid d_K$ and $d_{\hat{F}} \mid d_F^3$.

4.1. First case

We assume that K contains an imaginary quadratic subfield $L_{im} = \mathbf{Q}(\sqrt{-d_{L_{im}}})$ of discriminant $-d_{L_{im}} < 0$. Then $w_K = w_{L_{im}}$, $Q_K = Q_{L_{im}} = 1$ and $K = FL_{im} = F(\sqrt{-d_{L_{im}}})$, i.e. we can choose $\delta \in \mathbf{Q}$. We set $M = L_{re}L_{im}$ and $N = FM$. Hence, M is an imaginary biquadratic bicyclic number field, N is a dihedral CM-field of degree 12 and we have the following (incomplete) lattice of subfields:



Lemma 7. *Recall that we have set* $c_1 = (3 + 2\sqrt{2})/2$.

(1) *Since* $\zeta_N/\zeta_M = (\zeta_K/\zeta_{L_{im}})^2$, *any real zero of the entire function* $\zeta_K/\zeta_{L_{im}}$ *is at least a double zero of* ζ_N, *hence is less than* $1 - (1/c_1 \log d_N)$. *Moreover,* d_N *divides* d_K^3. *Hence,* $(\zeta_K/\zeta_{L_{im}})(s) \geq 0$ *for* $1 - (1/3c_1 \log d_K) \leq s < 1$.

(2) $Q_K = 1$, $w_K = w_{L_{im}}$ *and* $h_{L_{im}}$ *divides* h_K^-. *In particular,* $h_K^- = 1$ *implies* $h_{L_{im}} = 1$.

(3) *Assume that* $h_K^- = 1$. *Then, any real zero of* ζ_K *is at least a double zero of* ζ_N. *Hence,* $\zeta_K(s) \leq 0$ *in the range* $1 - (1/3c_1 \log d_K) \leq s < 1$, *by Lemma 6.*

Proof. Since $d_N/d_M = (d_K/d_{L_{im}})^2$ and since d_M divides $(d_{L_{re}} d_{L_{im}})^2$, (use the conductor-discriminant formula), we deduce that $d_N \mid d_K^2 d_{L_{re}}^2 \mid d_K^3$. Since K/L_{im} is a cubic extension, then $Q_K = Q_{L_{im}} = 1$ and $h_{L_{im}} = h_{L_{im}}^-$ divides h_K^- (see [LOO, Theorem 5]). Therefore, if $h_K^- = 1$ then L_{im} must be one of the nine imaginary quadratic fields of class number one (i.e. $d_{L_{im}} \in \{4, 8, 3, 7, 11, 19, 43, 67, 163\}$), which implies $\zeta_{L_{im}}(s) < 0$ for $0 < s < 1$, and using $\zeta_N = (\zeta_K/\zeta_{L_{im}})^2\zeta_M$, we obtain the last assertion. •

Theorem 8. *(Compare with* Theorem 12 *below). Let $K = FL_{im}$ be a non-normal sextic CM-field which is a compositum of a non-normal totally real cubic field F and of an imaginary quadratic field L_{im}. Assume that $\zeta_{L_{im}}(s) \le 0$ in the range $0 < s < 1$. Then,*

$$h_K^- \ge \frac{\sqrt{d_K/d_F}}{144(\log d_F)^2 \log d_K} \quad \text{for } d_K \ge 10^{19}. \tag{7}$$

Moreover, $h_K^- = 1$ implies $d_K \le 2 \cdot 10^{27}$, $\rho_K := d_K^{1/6} \le 34500$ and $d_F \le 3 \cdot 10^{13}$.

Proof. Set $c_3 = 3\pi^3 c_1 e^{1/6c_1} = 287.031\cdots$ (recall that $c_1 = (3 + 2\sqrt{2})/2$). We first use (1) and the second point of Lemma 7 to obtain

$$h_K^- \ge \frac{w_K}{8\pi^3}\sqrt{\frac{d_K}{d_F}}\frac{\text{Res}_{s=1}(\zeta_K)}{\text{Res}_{s=1}(\zeta_F)}.$$

We then use (3) with $a = 3c_1$ for which $d_a = 10^{19}$ (see the third point of Lemma 7) and we finally use (5). We obtain

$$h_K^- \ge \frac{w_K\sqrt{d_K/d_F}}{c_3(\log d_F)^2 \log d_K} \quad \text{for } d_K \ge 10^{19}$$

As for the last assertion, we notice that since $N_{F/\mathbf{Q}}(\mathcal{F}_{K/F}) \ge 3$ (use Proposition 1), we have $d_K \ge 3d_F^2$ which implies

$$h_K^- \ge \frac{2\sqrt{3d_F}}{c_3(\log d_F)^2 \log(3d_F^2)} > 1$$

for $d_F \ge 3 \cdot 10^{13}$, and $d_F \le \sqrt{d_K/3}$ which implies

$$h_K^- \ge \frac{8(3d_K)^{1/4}}{c_3(\log(d_K/3))^2 \log d_K} > 1$$

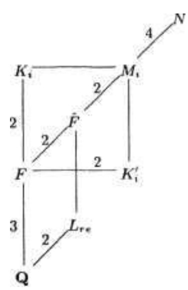
for $d_K \ge 2 \cdot 10^{27}$. •

By using Theorem 8, necessary conditions for class numbers of CM-fields to be equal to one (see [BL], [Oka] and [Bou]) and the technique developed in [LOO, Sections 4.4 and 4.7] for computing class numbers of such non-normal sextic CM-fields, we obtain:

Corollary 9. *There are* 134 *non-normal CM sextic fields $K = FL_{im}$ with class number one which are composita of a non-normal totally real cubic field F and of an imaginary quadratic field L_{im}, the ones given in* TABLES 1, 2 *and* 3 *below.*

4.2. Second čase

We assume that *K = F(yf^ó)* contains no imaginary quadratic field (where *5* is a totally positive algebraic integer of F), i.e. we assume that we cannot choose *6 £ Z.* We let *ó\ — ó, 6*$_2$ and *63* denote the three conjugates of <5 and set *d* = M2Ó3 = $N_{F/Q}$*(6).* We then set K_t = F(</=^), *K[= F{yf^d'*$_F$*T*$_i$*) = F{y/-d*$_L$*Ji),* *Mi* = Fíx/[1]^"). Hence, *M/F* is bicyclic biquadratic and *Ki/F, F/F* and ATJ/F are the three quadratic subextensions (with base field *F)* of the extension *Mi/F.* We also set *N* = F (\/-<$i, >/-<$2, V~<5$_3$) and M_0 = *F{sf^d)* (recall that F denotes the normál closure of *F).* Hence, *N* is a normál CM-field with maximal totally reál subfield N^+ *— F (y/Si&2, y/^i&3)* > *N* is the normál closure of *K* and the Mj's are conjugate subfields of *N.* We háve the following incomplete lattice of subfields:



We will set *M = Mi* and *K — K\.*

Lemma 10. *For* 1 < *i* < 3, č/ie *Mi are pairwise distinct but isomorphic. Hence, [N* : F] = 8 and *Gfd(NfF) = C*$_2$ *x C*$_2$ *x C*$_2$*.*

Proof. Assume that two of them, say *M* and M_2 are equal. Then for some OJ e F we háve Ó2 = a[2]^. Hence, d = <5$_3$a^2<5^2 and M_3 = *F{\J—<S*$_3$*)* = F(%/—ď) is normál. Since the Mj's are conjugate subfields of TV, we obtain *M ~ M\ — M*$_2$ — M_3 = *F{y/~^d) = F(y/dp, \f^-d).* Since iť/F is one of the three quadratic subextensions of *M/F* and since if / *F{\/dp) — F* (for /í is totally imaginary), we obtain that *K — F{y/~^d')* oř *K = F(y/—ddp),* contrary to the hypothesis that *K* contains no imaginary quadratic subfield. •

Lemma 11. *Recall that we háve set c*$_2$ (2 + \/3)/4. *We háve the following factorization of Dedekind zeta functions:*

(8) CN/CA ***Í[(CMJC*$_p$*)***

(9) = (CMO/CF)(CWC/0^3

and

(10) $$\zeta_M/\zeta_{\hat{F}} = (\zeta_K/\zeta_F)(\zeta_{K'}/\zeta_F).$$

Therefore, any complex zero of the entire function ζ_K/ζ_F is at least a triple zero of the Dedekind zeta function ζ_N, hence is less than $1-(1/c_2\log d_N)$, by Lemma 6. Moreover, d_N divides d_K^{24}. Hence, $(\zeta_K/\zeta_F)(s)\geq 0$ for $1-(1/24c_2\log d_K)\leq s<1$.

Proof. Since $\mathrm{Gal}(N/\hat{F})$ is the elementary 2-group $\mathcal{C}_2\times\mathcal{C}_2\times\mathcal{C}_2$, using abelian L-functions we easily obtain (8). To deduce (9), we notice that $\zeta_{M_3}=\zeta_{M_2}=\zeta_{M_1}$, for the M_i are pairwise isomorphic for $1\leq i\leq 3$. In the same way, since M/F is bicyclic biquadratic with quadratic subextensions K/F, K'/F and $\hat{F}/F$, we obtain (10). Let us finally prove that d_N divides d_K^{24}. To begin with, let f denote the norm of the conductor of any one the quadratic extensions $M_i/\hat{F}$ (since the M_i are pairwise isomorphic for $1\leq i\leq 3$, we have $f_1=f_2=f_3$). Using the conductor-discriminant formula, we obtain that d_N divides $d_{\hat{F}}^8(f_1f_2f_3)^4=d_{\hat{F}}^8f^{12}$. Since $d_M=d_{\hat{F}}^2f$, we obtain that d_N divides $d_M^{12}/d_{\hat{F}}^{16}$. Now, according to [Sta, Lemma 6], the different $\mathcal{D}_{M/\hat{F}}$ divides the different $\mathcal{D}_{K/F}$. Hence, $d_M=d_{\hat{F}}^2N_{M/\mathbf{Q}}(\mathcal{D}_{M/\hat{F}})$ divides $d_{\hat{F}}^2N_{M/\mathbf{Q}}(\mathcal{D}_{K/F})=d_{\hat{F}}^2(N_{K/\mathbf{Q}}(\mathcal{D}_{K/F}))^2=d_{\hat{F}}^2d_K^2/d_F^4$. Hence, d_N divides $(d_{\hat{F}}^2d_K^2/d_F^4)^{12}/d_{\hat{F}}^{16}=d_{\hat{F}}^8d_K^{24}/d_F^{48}$ and noticing that $d_{\hat{F}}$ divides d_F^3, we finally obtain that d_N divides $(d_K/d_F)^{24}$, hence divides d_K^{24}. •

Theorem 12. *(Compare with* Theorem 8 *above). Let K be a non-normal sextic CM-field with maximal totally real subfield a non-normal cubic field F and assume that K contains no imaginary quadratic subfield. We have*

(11) $$h_K^-\geq\frac{\sqrt{d_K/d_F}}{355(\log d_F)^2\log d_K}\quad \textit{for } d_K\geq 10^{22}.$$

Moreover, $h_K^-=1$ implies $d_K\leq 2\cdot10^{29}$, $\rho_K:=d_K^{1/6}\leq 76500$ and $d_F\leq 3\cdot10^{14}$.

Proof. Set $c_4=12\pi^3c_2e^{1/48c_2}=354.989\cdots$ (recall that $c_2=(2+\sqrt{3})/4$). Now, there are two cases to consider.
First, assume that ζ_F has a real zero β in $[1-(1/24c_2\log d_K),1[$. Then, $\zeta_K(\beta)=0\leq 0$. Therefore (use (4) with $a=24c_2$ and $c=48$ for which $d_{a,c}=6\cdot10^7$),

(12) $$\mathrm{Res}_{s=1}(\zeta_K)\geq\frac{1-\beta}{48e^{1/48c_2}}\quad\text{for } d_K\geq 6\cdot10^7.$$

Using (1), (6), (12) and $Q_Kw_K\geq 2$, we get

(13) $$h_K^-\geq\frac{\sqrt{d_K/d_F}}{4\pi^3e^{1/48c_2}\log^3d_F}\geq\frac{\sqrt{d_K/d_F}}{2\pi^3e^{1/48c_2}(\log d_F)^2(\log d_K)}\quad\text{for } d_K\geq 6\cdot10^7$$

(for $d_K\geq d_F^2$).
Second, assume that ζ_F has no real zero in $[1-(1/24c_2\log d_K),1[$. Then $\zeta_F(1-(1/24c_2\log d_K))<0$ and according to Lemma 11, we conclude that

$$\zeta_K(1-(1/24c_2\log d_K))\leq 0.$$

Therefore (use (3) with $a = 24c_2$ for which $d_a = 10^{22}$),

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{1}{24c_2 e^{1/48c_2}\log d_K} \quad \text{for } d_K \geq 10^{22}. \tag{14}$$

Using (1), (5), (14) and $Q_K w_K \geq 2$, we get

$$h_K^- \geq \frac{\sqrt{d_K/d_F}}{12\pi^3 c_2 e^{1/48c_2}(\log d_F)^2 \log d_K} \quad \text{for } d_K \geq 10^{22}. \tag{15}$$

Since the right hand side of (13) is always greater than or equal to the right hand side of (15) (for $c_2 > 1/2$), the lower bound (15) always holds.

Finally, the proof of the last assertion of this Theorem is similar to the proof of the last assertion of Theorem 8. •

Lemma 13. *Assume that h_K is odd, let $\mathcal{Q}$ be the only prime ideal of F which is ramified in K/F (see* Proposition 1*) and let $q \geq 2$ be the rational prime in $\mathcal{Q}$. Assume that $(q) = \mathcal{Q}_1\mathcal{Q}_2^2$ is partially ramified in $F/\mathbf{Q}$. Then, $\mathcal{F}_{K/F} = \mathcal{Q}_2^r$ for some odd $r \geq 1$.*

Proof. Let α be any totally positive algebraic integer of F such that $K = F(\sqrt{-\alpha})$. There exists some integral ideal $\mathcal{I}$ of F such that $(4\alpha) = \mathcal{I}^2\mathcal{F}_{K/k}$ (see Lemma 2) and $\mathcal{F}_{K/F} = \mathcal{Q}_1^r$ or $\mathcal{Q}_2^r$ for some odd $r \geq 1$ (see Proposition 3). Now, $\mathcal{F}_{K/F} = \mathcal{Q}_1^r$ would imply $(4q^r\alpha) = \mathcal{J}^2$ with $\mathcal{J} = \mathcal{I}\mathcal{Q}_1^r\mathcal{Q}_2^r$. Since the narrow class number of F is odd (by Point 1 of Proposition 1), as in the proof of Proposition 3 we would obtain $4q^r\alpha = \gamma^2$ for some $\gamma \in F$. Hence, $K = F(\sqrt{-\alpha}) = F(\sqrt{-q^r}) = F(\sqrt{-q})$ would contain an imaginary quadratic subfield, a contradiction. •

Lemma 14. *Let k be a number field of degree $n \geq 1$, let A_k denote its ring of algebraic integers and let $\mathcal{Q}$ be a prime ideal of k above the rational prime $q = 2$. Let $e \geq 1$ denote the ramification index of $\mathcal{Q}$. If there exists a primitive quadratic character χ_0 on the multiplicative group $(A_k/\mathcal{Q}^r)^*$, then $2 \leq r \leq 2e+1$, which implies $2 \leq r \leq 2n+1$.*

Proof. If $m = \sum_{i\geq 0} a_i 2^i$, $a_i \in \{0,1\}$ is the binary expansion of a non-negative integer $m \geq 0$, then it is well know that the 2-adic valuation $\nu_2(m!)$ of $m!$ is equal to $m - S(m)$ where $S(m) = \sum_{i\geq 0} a_i$. Hence,

$$\nu_2\left(\frac{(2m)!}{4^m(m!)^2}\right) = S(m) - 2m$$

(notice that $S(2m) = S(m)$). Let $k_{\mathcal{Q}}$ be the $\mathcal{Q}$-adic completion of k and let $\nu_{\mathcal{Q}}$ denote the associated valuation. The Taylor series expansion

$$\sqrt{1+x} = 1 + \sum_{m\geq 1}(-1)^{m-1}\frac{(2m)!}{4^m(m!)^2}x^m$$

is convergent if and only if

$$\lim_{m\longrightarrow+\infty}\nu_{\mathcal{Q}}\left(\frac{(2m)!}{4^m(m!)^2}x^m\right) = \lim_{k\longrightarrow+\infty} e(S(m) - 2m) + m\nu_{\mathcal{Q}}(x) = +\infty,$$

hence if and only if $\nu_{\mathcal{Q}}(x) > 2e$. Therefore, if $\alpha \in A_k$ satisfies $\alpha \equiv 1 \pmod{\mathcal{Q}^{2e+1}}$, there exists $\beta \in A_k$ such that $\alpha \equiv \beta^2 \pmod{\mathcal{Q}^{2e+1}}$, which implies $\chi_0(\alpha) = +1$. Since there must exist some $\alpha \in A_k$ satisfying $\alpha \equiv 1 \pmod{\mathcal{Q}^{r-1}}$ and $\chi_0(\alpha) \neq 1$ is χ_0 is primitive, we do obtain that r must satisfy $r - 1 < 2e + 1$. Finally, since the order of the group $(A_k/\mathcal{Q}^r)^*$ must be even, we have $r \geq 2$. •

Lemma 15. *If* $(2) = \mathcal{Q}_1\mathcal{Q}_2^2$ *is partially ramified in* $F/\mathbf{Q}$ *and if there exists a primitive quadratic character* χ_0 *on the multiplicative group* $(A_k/\mathcal{Q}_2^r)^*$, *then* $r = 5$.

Proof. Since $2 \leq r \leq 5$ and r is odd, it suffices to prove that a quadratic character on the multiplicative group $(A_k/\mathcal{Q}_2^3)^*$ is not primitive. Since $\ker(A_k/\mathcal{Q}_2^3)^* \longrightarrow (A_k/\mathcal{Q}_2^2)^* = \{\pm 1\}$, it suffices to prove that -1 is a square in $(A_k/\mathcal{Q}_2^3)^*$. This follows from the fact that if $\pi \in \mathcal{Q}_2 \setminus \mathcal{Q}_2^2$, then $\pi \in \mathcal{Q}_2^2 \setminus \mathcal{Q}_2^3$, hence $\pi^2 \equiv 2 \pmod{\mathcal{Q}_2^3}$ and $(1+\pi)^2 \equiv -1 \pmod{\mathcal{Q}_2^3}$. •

Theorem 16. *Let* K *be a non normal sextic CM-field. Let* F *denote its totally real cubic subfield. Assume that* F *is non-normal, that only one prime ideal* $\mathcal{Q}$ *of* F *is ramified in the quadratic extension* K/F *and let* $q \geq 2$ *be the rational prime in* $\mathcal{Q}$.

(1) *Assume that* $q > 2$. *Then, either*
  - (a) $(q) = \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3$ *splits completely in* F *and* $\mathcal{F}_{K/k} = \mathcal{Q}_1$, $\mathcal{Q}_2$ *or* $\mathcal{Q}_3$ *(and the three corresponding sextic fields may be non-isomorphic (e.eg. cases* 11 *and* 12 *in* Table 4 *below)).*
  - (b) $(q) = \mathcal{Q}_1\mathcal{Q}_2^2$ *is partially ramified in* F *and* $\mathcal{F}_{K/k} = \mathcal{Q}_2$.
  - (c) $(q) = \mathcal{Q}_1\mathcal{Q}_2$ *in* F *with* $f_{\mathcal{Q}_1} = 1$ *and* $f_{\mathcal{Q}_2} = 2$, *then* $\mathcal{F}_{K/k} = \mathcal{Q}_1$.

(2) *Assume that* $q = 2$. *Then, either*
  - (a) $(2) = \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3$ *splits completely in* F *and* $\mathcal{F}_{K/k} = \mathcal{Q}_1^3$, $\mathcal{Q}_2^3$ *or* $\mathcal{Q}_3^3$ *(and the three corresponding sextic fields may be non-isomorphic).*
  - (b) $(2) = \mathcal{Q}_1\mathcal{Q}_2^2$ *is partially ramified in* F *and* $\mathcal{F}_{K/k} = \mathcal{Q}_2^5$.
  - (c) $(2) = \mathcal{Q}_1\mathcal{Q}_2$ *in* F *and* $\mathcal{F}_{K/k} = \mathcal{Q}_1^3$ *or* $\mathcal{Q}_2^3$ *(and the two corresponding sextic fields* K *are not isomorphic).*

Proof. Use Proposition 3 and Lemmas 13 and 15. •

By using Theorem 12, necessary conditions for class numbers of CM-fields to be equal to one (see [BL], [Oka] and [Bou]), by using Theorem 16 for constructing the quadratic characters associated with the quadratic extensions K/F and by adapting the technique developed in [LOO, Sections 4.4 and 4.7] for computing class numbers of non-normal sextic CM-fields containing imaginary quadratic subfields, we obtain:

Corollary 17. *There are* 233 *non-normal CM sextic fields* K *with class number one which contain no imaginary quadratic subfield and whose maximal totally real cubic subfields* F *are non-normal, the ones given in* TABLES 4, 5 *and* 6 *below.*

5. Tables

TABLE 1. $K = FL_{im}$ and $(q) = \mathcal{Q}$ is inert in F

case	$d_{L_{im}}$	d_F	d_L	f	$P_F(x)$	d_K	ρ_K
1	3	148	37	2	x^3+x^2-3x-1	$2^4\cdot 3^3\cdot 37^2$	$9.16\cdots$
2	3	316	316	1	x^3+x^2-4x-2	$2^4\cdot 3^3\cdot 79^2$	$11.79\cdots$
3	3	568	568	1	x^3-x^2-6x-2	$2^6\cdot 3^3\cdot 71^2$	$14.34\cdots$
4	3	940	940	1	x^3-7x-4	$2^4\cdot 3^3\cdot 5^2\cdot 47^2$	$16.96\cdots$
5	4	321	321	1	x^3+x^2-4x-1	$2^6\cdot 3^2\cdot 107^2$	$13.69\cdots$

TABLE 2. $K = FL_{im}$ and $(q) = \mathcal{Q}^3$ is totally ramified in F

case	$d_{L_{im}}$	d_F	d_L	f	$P_F(x)$	d_K	ρ_K
1	3	621	69	3	x^3-6x-3	$3^7\cdot 23^2$	$10.24\cdots$
2	3	756	21	6	x^3-6x-2	$2^4\cdot 3^7\cdot 7^2$	$10.94\cdots$
3	3	837	93	3	x^3-6x-1	$3^7\cdot 31^2$	$11.31\cdots$
4	3	1620	5	18	$x^3-12x-14$	$2^4\cdot 3^9\cdot 5^2$	$14.10\cdots$
5	3	1944	24	9	x^3-9x-6	$2^6\cdot 3^{11}$	$14.98\cdots$
6	3	2241	249	3	x^3-9x-5	$3^7\cdot 83^2$	$15.71\cdots$
7	3	2700	12	15	$x^3-15x-20$	$2^4\cdot 3^7\cdot 5^4$	$16.72\cdots$
8	3	2808	312	3	x^3-9x-2	$2^6\cdot 3^7\cdot 13^2$	$16.94\cdots$
9	3	3132	348	3	$x^3-18x-20$	$2^4\cdot 3^7\cdot 29^2$	$17.57\cdots$
10	3	4104	456	3	$x^3-18x-16$	$2^6\cdot 3^7\cdot 19^2$	$19.22\cdots$
11	3	4860	60	9	$x^3-18x-12$	$2^4\cdot 3^{11}\cdot 5^2$	$20.34\cdots$
12	3	5940	165	6	$x^3-12x-6$	$2^4\cdot 3^7\cdot 5^2\cdot 11^2$	$21.74\cdots$
13	3	6588	732	3	$x^3-15x-16$	$2^4\cdot 3^7\cdot 61^2$	$22.51\cdots$
14	3	8289	921	3	$x^3-21x-12$	$3^7\cdot 307^2$	$24.30\cdots$
15	3	9153	113	9	$x^3-21x-4$	$3^9\cdot 113^2$	$25.12\cdots$
16	3	11880	1320	3	$x^3-27x-34$	$2^6\cdot 3^7\cdot 5^2\cdot 11^2$	$27.40\cdots$
17	3	12744	1416	3	$x^3-21x-30$	$2^6\cdot 3^7\cdot 59^2$	$28.05\cdots$
18	3	20385	2265	3	$x^3-33x-48$	$3^7\cdot 5^2\cdot 151^2$	$32.80\cdots$
19	4	148	37	2	x^3+x^2-3x-1	$2^8\cdot 37^2$	$8.39\cdots$
20	4	404	101	2	x^3-x^2-5x-1	$2^8\cdot 101^2$	$11.73\cdots$
21	4	564	141	2	x^3+x^2-5x-3	$2^8\cdot 3^2\cdot 47^2$	$13.11\cdots$
22	4	756	21	6	x^3-6x-2	$2^8\cdot 3^6\cdot 7^2$	$14.46\cdots$
23	4	1524	381	2	x^3+x^2-7x-1	$2^8\cdot 3^2\cdot 127^2$	$18.26\cdots$
24	4	3124	781	2	$x^3-16x-12$	$2^8\cdot 11^2\cdot 71^2$	$23.20\cdots$
25	8	148	37	2	x^3+x^2-3x-1	$2^{11}\cdot 37^2$	$11.87\cdots$
26	11	1573	143	11	x^3+x^2-7x-2	$11^5\cdot 13^2$	$17.34\cdots$

TABLE 3. $K = FL_{im}$ and $(q) = \mathcal{Q}_1\mathcal{Q}_2^2$ is partially ramified in F

case	$d_{L_{im}}$	d_F	d_L	f	$P_F(x)$	d_K	ρ_K
1	3	321	321	1	x^3+x^2-4x-1	$3^3\cdot 107^2$	$8.22\cdots$
2	3	564	141	2	x^3+x^2-5x-3	$2^4\cdot 3^3\cdot 47^2$	$9.92\cdots$
3	3	993	993	1	x^3+x^2-6x-3	$3^3\cdot 331^2$	$11.98\cdots$
4	3	1101	1101	1	$x^3+x^2-9x-12$	$3^3\cdot 367^2$	$12.40\cdots$
5	3	1425	57	5	x^3-x^2-8x-3	$3^3\cdot 5^4\cdot 19^2$	$13.51\cdots$
6	3	1524	381	2	x^3+x^2-7x-1	$2^4\cdot 3^3\cdot 127^2$	$13.82\cdots$
7	3	2505	2505	1	$x^3-x^2-10x-5$	$3^3\cdot 5^2\cdot 167^2$	$16.31\cdots$
8	3	3144	3144	1	$x^3-x^2-16x-8$	$2^6\cdot 3^3\cdot 131^2$	$17.59\cdots$
9	3	3252	813	2	x^3+x^2-9x-3	$2^4\cdot 3^3\cdot 271^2$	$17.79\cdots$
10	3	3540	885	2	$x^3-x^2-15x-15$	$2^4\cdot 3^3\cdot 5^2\cdot 59^2$	$18.30\cdots$
11	3	3576	3576	1	$x^3+x^2-15x-3$	$2^6\cdot 3^3\cdot 149^2$	$18.36\cdots$
12	3	3873	3873	1	$x^3-x^2-16x-17$	$3^3\cdot 1291^2$	$18.85\cdots$
13	3	4692	1173	2	$x^3-x^2-17x-3$	$2^4\cdot 3^3\cdot 17^2\cdot 23^2$	$20.10\cdots$
14	3	4764	4764	1	$x^3-x^2-12x-6$	$2^4\cdot 3^3\cdot 397^2$	$20.20\cdots$
15	3	6420	1605	2	$x^3+x^2-25x-55$	$2^4\cdot 3^3\cdot 5^2\cdot 107^2$	$22.32\cdots$
16	3	7032	7032	1	$x^3+x^2-14x-18$	$2^6\cdot 3^3\cdot 293^2$	$23.00\cdots$
17	3	7464	7464	1	$x^3-x^2-24x-24$	$2^6\cdot 3^3\cdot 311^2$	$23.46\cdots$
18	3	8220	8220	1	$x^3+x^2-20x-12$	$2^4\cdot 3^3\cdot 5^2\cdot 137^2$	$24.23\cdots$
19	3	8472	8472	1	$x^3-x^2-27x-33$	$2^6\cdot 3^3\cdot 353^2$	$24.48\cdots$
20	3	9192	9192	1	$x^3+x^2-18x-30$	$2^6\cdot 3^3\cdot 383^2$	$25.15\cdots$
21	3	10200	408	5	$x^3+x^2-23x-27$	$2^6\cdot 3^3\cdot 5^4\cdot 17^2$	$26.04\cdots$
22	3	10641	10641	1	$x^3+x^2-22x-16$	$3^3\cdot 3547^2$	$26.41\cdots$
23	3	10812	10812	1	$x^3-x^2-23x-9$	$2^4\cdot 3^3\cdot 17^2\cdot 53^2$	$26.55\cdots$
24	3	12216	12216	1	$x^3-x^2-35x-57$	$2^6\cdot 3^3\cdot 509^2$	$27.65\cdots$
25	3	14520	120	11	$x^3-33x-22$	$2^6\cdot 3^3\cdot 5^2\cdot 11^4$	$29.29\cdots$
26	3	16689	16689	1	$x^3+x^2-26x-24$	$3^3\cdot 5563^2$	$30.69\cdots$
27	3	16860	16860	1	$x^3+x^2-16x-10$	$2^4\cdot 3^3\cdot 5^2\cdot 281^2$	$30.79\cdots$
28	3	18201	18201	1	$x^3+x^2-30x-48$	$3^3\cdot 6067^2$	$31.59\cdots$
29	3	19020	19020	1	$x^3+x^2-20x-30$	$2^4\cdot 3^3\cdot 5^2\cdot 317^2$	$32.05\cdots$
30	3	20073	20073	1	$x^3-x^2-30x-24$	$3^3\cdot 6691^2$	$32.63\cdots$
31	4	316	316	1	x^3+x^2-4x-2	$2^6\cdot 79^2$	$8.58\cdots$
32	4	940	940	1	x^3-7x-4	$2^6\cdot 5^2\cdot 47^2$	$12.34\cdots$
33	4	1708	1708	1	x^3-x^2-8x-2	$2^6\cdot 7^2\cdot 61^2$	$15.06\cdots$
34	4	1772	1772	1	$x^3-14x-12$	$2^6\cdot 443^2$	$15.24\cdots$
35	4	2636	2636	1	$x^3-x^2-16x-18$	$2^6\cdot 659^2$	$17.40\cdots$
36	4	2700	12	15	$x^3-15x-20$	$2^6\cdot 3^6\cdot 5^4$	$17.54\cdots$
37	4	3596	3596	1	$x^3-11x-8$	$2^6\cdot 29^2\cdot 31^2$	$19.30\cdots$
38	4	4364	4364	1	$x^3+x^2-19x-27$	$2^6\cdot 1091^2$	$20.58\cdots$
39	4	4844	4844	1	$x^3+x^2-12x-14$	$2^6\cdot 7^2\cdot 173^2$	$21.31\cdots$
40	4	5356	5356	1	$x^3-22x-28$	$2^6\cdot 13^2\cdot 103^2$	$22.04\cdots$
41	4	7084	7084	1	$x^3+x^2-19x-11$	$2^6\cdot 7^2\cdot 11^2\cdot 23^2$	$24.19\cdots$
42	4	7244	7244	1	$x^3+x^2-20x-38$	$2^6\cdot 1811^2$	$24.37\cdots$
43	4	7404	7404	1	$x^3+x^2-12x-6$	$2^6\cdot 3^2\cdot 617^2$	$24.55\cdots$

TABLE 3 (continued).

case	$d_{L_{im}}$	d_F	d_L	f	$P_F(x)$	d_K	ρ_K
44	4	8556	8556	1	$x^3+x^2-27x-51$	$2^6\cdot 3^2\cdot 23^2\cdot 31^2$	$25.76\cdots$
45	4	9676	9676	1	$x^3-22x-12$	$2^6\cdot 41^2\cdot 59^2$	$26.84\cdots$
46	4	11884	11884	1	$x^3-19x-24$	$2^6\cdot 2971^2$	$28.75\cdots$
47	4	14316	14316	1	$x^3-x^2-27x-21$	$2^6\cdot 3^2\cdot 1193^2$	$30.59\cdots$
48	4	14956	14956	1	$x^3+x^2-32x-6$	$2^6\cdot 3739^2$	$31.04\cdots$
49	4	16044	16044	1	$x^3+x^2-36x-54$	$2^6\cdot 3^2\cdot 7^2\cdot 191^2$	$31.77\cdots$
50	4	18604	18604	1	$x^3-x^2-36x-18$	$2^6\cdot 4651^2$	$33.38\cdots$
51	4	21324	21324	1	$x^3-x^2-44x-84$	$2^6\cdot 3^2\cdot 1777^2$	$34.93\cdots$
52	7	469	469	1	x^3+x^2-5x-4	$7^3\cdot 67^2$	$10.74\cdots$
53	7	756	21	6	x^3-6x-2	$2^4\cdot 3^6\cdot 7^3$	$12.59\cdots$
54	7	2177	2177	1	x^3+x^2-8x-5	$7^3\cdot 311^2$	$17.92\cdots$
55	7	2233	2333	1	x^3+x^2-8x-1	$7^3\cdot 11^2\cdot 29^2$	$18.07\cdots$
56	7	4193	4193	1	$x^3-x^2-12x-7$	$7^3\cdot 599^2$	$22.30\cdots$
57	7	4641	4641	1	$x^3+x^2-14x-21$	$3^2\cdot 7^3\cdot 13^2\cdot 17^2$	$23.07\cdots$
58	7	5089	5089	1	$x^3-x^2-14x-11$	$7^3\cdot 727^2$	$23.78\cdots$
59	7	5369	5369	1	$x^3-17x-23$	$7^3\cdot 13^2\cdot 59^2$	$24.21\cdots$
60	7	6153	6153	1	$x^3-x^2-12x-3$	$3^2\cdot 7^3\cdot 293^2$	$25.34\cdots$
61	7	6601	6601	1	$x^3-13x-9$	$7^3\cdot 23^2\cdot 41^2$	$25.94\cdots$
62	7	7665	7665	1	$x^3-27x-19$	$3^2\cdot 5^2\cdot 7^3\cdot 73^2$	$27.27\cdots$
63	7	8113	8113	1	$x^3-13x-5$	$7^3\cdot 19^2\cdot 61^2$	$27.79\cdots$
64	7	8505	105	9	$x^3-27x-51$	$3^{10}\cdot 5^2\cdot 7^3$	$28.23\cdots$
65	7	9905	9905	1	$x^3-17x-19$	$5^2\cdot 7^3\cdot 283^2$	$29.70\cdots$
66	7	10353	10353	1	$x^3+x^2-22x-43$	$3^2\cdot 7^3\cdot 17^2\cdot 29^2$	$30.14\cdots$
67	7	14385	14385	1	$x^3-x^2-36x-69$	$3^2\cdot 5^2\cdot 7^3\cdot 137^2$	$33.63\cdots$
68	7	16737	16737	1	$x^3-x^2-36x-27$	$3^2\cdot 7^3\cdot 797^2$	$35.37\cdots$
69	7	20545	20545	1	$x^3+x^2-40x-67$	$5^2\cdot 7^3\cdot 587^2$	$37.88\cdots$
70	7	25249	25249	1	$x^3-19x-9$	$7^3\cdot 3607^2$	$40.57\cdots$
71	7	30849	30849	1	$x^3+x^2-42x-45$	$3^2\cdot 7^3\cdot 13^2\cdot 113^2$	$43.37\cdots$
72	8	568	568	1	x^3-x^2-6x-2	$2^9\cdot 71^2$	$11.71\cdots$
73	8	1304	1304	1	$x^3-x^2-11x-1$	$2^9\cdot 163^2$	$15.45\cdots$
74	8	1944	24	9	x^3-9x-6	$2^9\cdot 3^{10}$	$17.65\cdots$
75	8	3736	3736	1	$x^3+x^2-14x-22$	$2^9\cdot 467^2$	$21.94\cdots$
76	8	8920	8920	1	$x^3-22x-16$	$2^9\cdot 5^2\cdot 223^2$	$29.32\cdots$
77	8	11032	11032	1	$x^3+x^2-14x-10$	$2^9\cdot 7^2\cdot 197^2$	$31.48\cdots$
78	8	11608	11608	1	$x^3-x^2-23x-5$	$2^9\cdot 1451^2$	$32.02\cdots$
79	8	14296	14296	1	$x^3+x^2-24x-4$	$2^9\cdot 1787^2$	$34.32\cdots$
80	8	14680	14680	1	$x^3+x^2-30x-70$	$2^9\cdot 5^2\cdot 367^2$	$34.62\cdots$
81	8	28504	28504	1	$x^3-34x-40$	$2^9\cdot 7^2\cdot 509^2$	$43.19\cdots$
82	11	473	473	1	x^3-5x-1	$11^3\cdot 43^2$	$11.61\cdots$
83	11	2024	2024	1	$x^3-x^2-10x-6$	$2^6\cdot 11^3\cdot 23^2$	$18.86\cdots$
84	11	2101	2101	1	$x^3-x^2-11x-8$	$11^3\cdot 191^2$	$19.10\cdots$
85	11	2233	2233	1	x^3+x^2-8x-1	$7^2\cdot 11^3\cdot 29^2$	$19.49\cdots$
86	11	5368	5368	1	$x^3+x^2-19x-23$	$2^6\cdot 11^3\cdot 61^2$	$26.11\cdots$

TABLE 3 (continued).

case	$d_{L_{im}}$	d_F	d_L	f	$P_F(x)$	d_K	ρ_K
87	11	7084	7084	1	$x^3+x^2-19x-11$	$2^4\cdot 7^2\cdot 11^3\cdot 23^2$	$28.64\cdots$
88	11	15961	15961	1	$x^3-x^2-30x-32$	$11^3\cdot 1451^2$	$37.54\cdots$
89	11	17116	17116	1	$x^3+x^2-16x-2$	$2^4\cdot 11^3\cdot 389^2$	$38.43\cdots$
90	11	27049	27049	1	$x^3+x^2-34x-56$	$11^3\cdot 2459^2$	$44.76\cdots$
91	11	37609	37609	1	$x^3+x^2-34x-32$	$11^3\cdot 13^2\cdot 263^2$	$49.96\cdots$
92	19	1425	57	5	x^3-x^2-8x-3	$3^2\cdot 5^4\cdot 19^3$	$18.38\cdots$
93	19	3021	3021	1	x^3+x^2-9x-6	$3^2\cdot 19^3\cdot 53^2$	$23.61\cdots$
94	19	3496	3496	1	$x^3-13x-14$	$2^6\cdot 19^3\cdot 23^2$	$24.79\cdots$
95	19	4104	456	3	$x^3-18x-16$	$2^6\cdot 3^6\cdot 19^3$	$26.15\cdots$
96	19	5529	5529	1	$x^3+x^2-20x-39$	$3^2\cdot 19^3\cdot 97^2$	$28.88\cdots$
97	19	5624	5624	1	$x^3-x^2-24x-28$	$2^6\cdot 19^3\cdot 37^2$	$29.04\cdots$
98	19	39064	39064	1	$x^3-x^2-46x-26$	$2^6\cdot 19^3\cdot 257^2$	$55.42\cdots$
99	43	473	473	1	x^3-5x-1	$11^2\cdot 43^3$	$14.58\cdots$
100	43	7224	7224	1	$x^3+x^2-19x-7$	$2^6\cdot 3^2\cdot 7^2\cdot 43^3$	$36.18\cdots$
101	43	11137	11137	1	$x^3+x^2-22x-8$	$7^2\cdot 37^2\cdot 43^3$	$41.79\cdots$
102	67	469	469	1	x^3+x^2-5x-4	$7^2\cdot 67^3$	$15.65\cdots$
103	163	1304	1304	1	$x^3-x^2-11x-1$	$2^6\cdot 163^3$	$25.53\cdots$

TABLE 4. K does not contain any imaginary quadratic field and $(q) = \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3$ splits in F

case	d_F	q	$P_K(X)$	d_K	ρ_K
1	148	107	$x^6+49x^4+223x^2+107$	$2^4\cdot 37^2\cdot 107$	$11.52\cdots$
2	148	139	$x^6+77x^4+235x^2+139$	$2^4\cdot 37^2\cdot 139$	$12.03\cdots$
3	148	491	$x^6+89x^4+1787x^2+491$	$2^4\cdot 37^2\cdot 491$	$14.85\cdots$
4	148	691	$x^6+65x^4+459x^2+691$	$2^4\cdot 37^2\cdot 691$	$15.72\cdots$
5	316	211	$x^6+65x^4+427x^2+211$	$2^4\cdot 79^2\cdot 211$	$16.61\cdots$
6	316	307	$x^6+53x^4+711x^2+307$	$2^4\cdot 79^2\cdot 307$	$17.69\cdots$
7	321	59	$x^6+31x^4+86x^2+59$	$3^2\cdot 59\cdot 107^2$	$13.50\cdots$
8	321	79	$x^6+141x^4+291x^2+79$	$3^2\cdot 79\cdot 107^2$	$14.18\cdots$
9	321	163	$x^6+27x^4+126x^2+163$	$3^2\cdot 107^2\cdot 163$	$16.00\cdots$
10	404	43	$x^6+57x^4+247x^2+43$	$2^4\cdot 43\cdot 101^2$	$13.83\cdots$
11	404	179	$x^6+33x^4+235x^2+179$	$2^4\cdot 101^2\cdot 179$	$17.54\cdots$
12	404	179	$x^6+1073x^4+1335x^2+179$	$2^4\cdot 101^2\cdot 179$	$17.54\cdots$
13	404	283	$x^6+41x^4+251x^2+283$	$2^4\cdot 101^2\cdot 283$	$18.94\cdots$
14	469	263	$x^6+20x^4+128x^2+263$	$7^2\cdot 67^2\cdot 263$	$19.66\cdots$
15	564	331	$x^6+33x^4+291x^2+331$	$2^4\cdot 3^2\cdot 47^2\cdot 331$	$21.73\cdots$
16	568	83	$x^6+33x^4+203x^2+83$	$2^6\cdot 71^2\cdot 83$	$17.29\cdots$
17	621	151	$x^6+36x^4+144x^2+151$	$3^6\cdot 23^2\cdot 151$	$19.68\cdots$
18	733	127	$x^6+29x^4+163x^2+127$	$127\cdot 733^2$	$20.21\cdots$
19	733	127	$x^6+80x^4+892x^2+127$	$127\cdot 733^2$	$20.21\cdots$
20	756	211	$x^6+33x^4+279x^2+211$	$2^4\cdot 3^6\cdot 7^2\cdot 211$	$22.22\cdots$
21	785	23	$x^6+30x^4+97x^2+23$	$5^2\cdot 23\cdot 157^2$	$15.55\cdots$
22	993	139	$x^6+186x^4+8505x^2+139$	$3^2\cdot 139\cdot 331^2$	$22.70\cdots$
23	1101	31	$x^6+68x^4+776x^2+31$	$3^2\cdot 31\cdot 367^2$	$18.30\cdots$
24	1345	7	$x^6+14x^4+45x^2+7$	$5^2\cdot 7\cdot 269^2$	$15.26\cdots$
25	1373	71	$x^6+32x^4+252x^2+71$	$71\cdot 1373^2$	$22.61\cdots$
26	1373	71	$x^6+32x^4+108x^2+71$	$71\cdot 1373^2$	$22.61\cdots$
27	1425	43	$x^6+551x^4+422x^2+43$	$3^2\cdot 5^4\cdot 19^2\cdot 43$	$21.06\cdots$
28	1524	19	$x^6+629x^4+4763x^2+19$	$2^4\cdot 3^2\cdot 19\cdot 127^2$	$18.79\cdots$
29	1901	31	$x^6+24x^4+108x^2+31$	$31\cdot 1901^2$	$21.95\cdots$
30	1901	31	$x^6+93x^4+451x^2+31$	$31\cdot 1901^2$	$21.95\cdots$

TABLE 4 (continued).

case	d_F	q	$P_K(X)$	d_K	ρ_K
31	1944	139	$x^6+21x^4+111x^2+139$	$2^6\cdot 3^{10}\cdot 139$	$28.40\cdots$
32	2101	23	$x^6+60x^4+128x^2+23$	$11^2\cdot 23\cdot 191^2$	$21.59\cdots$
33	2228	19	$x^6+81x^4+727x^2+19$	$2^4\cdot 19\cdot 557^2$	$21.33\cdots$
34	2300	107	$x^6+749x^4+567x^2+107$	$2^4\cdot 5^4\cdot 23^2\cdot 107$	$28.76\cdots$
35	2557	23	$x^6+16x^4+60x^2+23$	$23\cdot 2557^2$	$23.06\cdots$
36	2713	3	$x^6+33x^4+155x^2+3$	$3\cdot 2713^2$	$16.74\cdots$
37	2857	19	$x^6+222x^4+137x^2+19$	$19\cdot 2857^2$	$23.17\cdots$
38	3325	23	$x^6+116x^4+2672x^2+23$	$5^4\cdot 7^2\cdot 19^2\cdot 23$	$25.17\cdots$
39	3356	107	$x^6+89x^4+1659x^2+107$	$2^4\cdot 107\cdot 839^2$	$32.61\cdots$
40	3941	71	$x^6+20x^4+96x^2+71$	$7^2\cdot 71\cdot 563^2$	$32.14\cdots$
41	4409	11	$x^6+467x^4+690x^2+11$	$11\cdot 4409^2$	$24.45\cdots$
42	6133	7	$x^6+68x^4+328x^2+7$	$7\cdot 6133^2$	$25.31\cdots$
43	6133	7	$x^6+229x^4+83x^2+7$	$7\cdot 6133^2$	$25.31\cdots$
44	7668	19	$x^6+1473x^4+363x^2+19$	$2^4\cdot 3^6\cdot 19\cdot 71^2$	$32.31\cdots$
45	7753	3	$x^6+129x^4+59x^2+3$	$3\cdot 7753^2$	$23.76\cdots$
46	7753	19	$x^6+30x^4+129x^2+19$	$19\cdot 7753^2$	$32.33\cdots$
47	9076	3	$x^6+1017x^4+143411x^2+3$	$2^4\cdot 3\cdot 2269^2$	$25.05\cdots$
48	10333	7	$x^6+36x^4+96x^2+7$	$7\cdot 10333^2$	$30.12\cdots$
49	10077	31	$x^6+36189732x^4$ $+28035372408x^2+31$	$3^2\cdot 31\cdot 3359^2$	$38.28\cdots$
50	11853	7	$x^6+3141x^4+627x^2+7$	$3^6\cdot 7\cdot 439^2$	$31.53\cdots$
51	15061	7	$x^6+12x^4+32x^2+7$	$7\cdot 15061^2$	$34.15\cdots$
52	15641	2	$x^6+72x^4+37x^2+2$	$2^3\cdot 15641^2$	$35.36\cdots$
53	15700	3	$x^6+81x^4+107x^2+3$	$2^4\cdot 3\cdot 5^4\cdot 157^2$	$30.07\cdots$
54	16649	2	$x^6+2316144x^4+404773789x^2+2$	$2^3\cdot 16649^2$	$36.11\cdots$
55	17929	3	$x^6+2090x^4+7069x^2+3$	$3\cdot 17929^2$	$31.43\cdots$
56	20252	11	$x^6+377x^4+35259x^2+11$	$2^4\cdot 11\cdot 61^2\cdot 83^2$	$40.64\cdots$
57	21281	2	$x^6+33528x^4+565x^2+2$	$2^3\cdot 13^2\cdot 1637^2$	$39.19\cdots$
58	21913	3	$x^6+230x^4+11941x^2+3$	$3\cdot 17^2\cdot 1289^2$	$33.60\cdots$
59	22229	7	$x^6+340x^4+18784x^2+7$	$7\cdot 22229^2$	$38.88\cdots$
60	23417	2	$x^6+32x^4+45x^2+2$	$2^3\cdot 23417^2$	$40.45\cdots$
61	30169	3	$x^6+19955x^4+598x^2+3$	$3\cdot 30169^2$	$37.38\cdots$
62	30868	3	$x^6+569x^4+1327x^2+3$	$2^4\cdot 3\cdot 7717^2$	$37.67\cdots$
63	31165	7	$x^6+2277x^4+76371x^2+7$	$5^2\cdot 7\cdot 23^2\cdot 271^2$	$43.52\cdots$
64	32421	7	$x^6+48x^4+180x^2+7$	$3^2\cdot 7\cdot 101^2\cdot 107^2$	$44.10\cdots$
65	37437	7	$x^6+556749204x^4+136272x^2+7$	$3^2\cdot 7\cdot 12479^2$	$46.26\cdots$
66	46996	3	$x^6+3467981x^4+3660259x^2+3$	$2^4\cdot 3\cdot 31^2\cdot 379^2$	$43.33\cdots$
67	51316	3	$x^6+13361x^4+907x^2+3$	$2^4\cdot 3\cdot 12829^2$	$44.62\cdots$
68	85300	3	$x^6+310760391641x^4$ $+33331797907x^2+3$	$2^4\cdot 3\cdot 5^4\cdot 853^2$	$52.86\cdots$
69	126664	3	$x^6+82841249297885x^4$ $+30290200633872943x^2+3$	$2^6\cdot 3\cdot 71^2\cdot 223^2$	$60.31\cdots$
70	142876	3	$x^6+9749333x^4+106807x^2+3$	$2^4\cdot 3\cdot 23^2\cdot 1553^2$	$62.78\cdots$

TABLE 5. K does not contain any imaginary quadratic field and $(q) = \mathcal{Q}_1\mathcal{Q}_2$ in F

case	d_F	q	$P_K(X)$	d_K	ρ_K
1	148	23	$x^6+25x^4+51x^2+23$	$2^4\cdot 23\cdot 37^2$	$8.92\cdots$
2	148	31	$x^6+29x^4+67x^2+31$	$2^4\cdot 31\cdot 37^2$	$9.37\cdots$
3	148	227	$x^6+25x^4+175x^2+227$	$2^4\cdot 37^2\cdot 227$	$13.06\cdots$
4	148	331	$x^6+33x^4+227x^2+331$	$2^4\cdot 37^2\cdot 331$	$13.91\cdots$
5	148	467	$x^6+49x^4+615x^2+467$	$2^4\cdot 37^2\cdot 467$	$14.73\cdots$
6	148	499	$x^6+33x^4+299x^2+499$	$2^4\cdot 37^2\cdot 499$	$14.89\cdots$
7	148	547	$x^6+41x^4+319x^2+547$	$2^4\cdot 37^2\cdot 547$	$15.12\cdots$
8	316	23	$x^6+25x^4+79x^2+23$	$2^4\cdot 23\cdot 79^2$	$11.48\cdots$
9	316	131	$x^6+21x^4+119x^2+131$	$2^4\cdot 79^2\cdot 131$	$15.34\cdots$
10	316	547	$x^6+113x^4+1195x^2+547$	$2^4\cdot 79^2\cdot 547$	$19.47\cdots$
11	321	23	$x^6+37x^4+131x^2+23$	$3^2\cdot 23\cdot 107^2$	$11.54\cdots$
12	321	31	$x^6+38x^4+89x^2+31$	$3^2\cdot 31\cdot 107^2$	$12.13\cdots$
13	321	307	$x^6+81x^4+315x^2+307$	$3^2\cdot 107^2\cdot 307$	$17.78\cdots$
14	404	7	$x^6+9x^4+19x^2+7$	$2^4\cdot 7\cdot 101^2$	$10.22\cdots$
15	404	67	$x^6+25x^4+79x^2+67$	$2^4\cdot 67\cdot 101^2$	$14.89\cdots$
16	404	347	$x^6+33x^4+211x^2+347$	$2^4\cdot 101^2\cdot 347$	$19.59\cdots$
17	469	2	$x^6+9x^4+14x^2+4$	$2^6\cdot 7^2\cdot 67^2$	$15.53\cdots$
18	469	11	$x^6+36x^4+168x^2+11$	$7^2\cdot 11\cdot 67^2$	$11.58\cdots$
19	469	47	$x^6+20x^4+104x^2+47$	$7^2\cdot 47\cdot 67^2$	$14.75\cdots$
20	469	223	$x^6+29x^4+195x^2+223$	$7^2\cdot 67^2\cdot 223$	$19.13\cdots$
21	469	239	$x^6+24x^4+164x^2+239$	$7^2\cdot 67^2\cdot 239$	$19.35\cdots$
22	473	71	$x^6+30x^4+229x^2+71$	$11^2\cdot 43^2\cdot 71$	$15.85\cdots$
23	473	139	$x^6+31x^4+202x^2+139$	$11^2\cdot 43^2\cdot 139$	$17.73\cdots$
24	564	163	$x^6+57x^4+435x^2+163$	$2^4\cdot 3^2\cdot 47^2\cdot 163$	$19.31\cdots$
25	564	211	$x^6+21x^4+123x^2+211$	$2^4\cdot 3^2\cdot 47^2\cdot 211$	$20.15\cdots$
26	568	139	$x^6+53x^4+463x^2+139$	$2^6\cdot 71^2\cdot 139$	$18.84\cdots$
27	621	19	$x^6+33x^4+75x^2+19$	$3^6\cdot 19\cdot 23^2$	$13.93\cdots$
28	621	47	$x^6+45x^4+483x^2+47$	$3^6\cdot 23^2\cdot 47$	$16.20\cdots$
29	621	103	$x^6+60x^4+504x^2+103$	$3^6\cdot 23^2\cdot 103$	$18.47\cdots$
30	733	2	$x^6+15x^4+11x^2+2$	$2^3\cdot 733^2$	$12.75\cdots$
31	733	11	$x^6+32x^4+188x^2+11$	$11\cdot 733^2$	$13.44\cdots$
32	733	71	$x^6+37x^4+339x^2+71$	$71\cdot 733^2$	$18.34\cdots$
33	837	2	$x^6+27x^4+15x^2+2$	$2^3\cdot 3^6\cdot 31^2$	$13.32\cdots$
34	837	79	$x^6+45x^4+195x^2+79$	$3^6\cdot 31^2\cdot 79$	$19.52\cdots$
35	993	7	$x^6+11x^4+26x^2+7$	$3^2\cdot 7\cdot 331^2$	$13.79\cdots$
36	1101	223	$x^6+288x^4+564x^2+223$	$3^2\cdot 223\cdot 367^2$	$25.42\cdots$
37	1300	11	$x^6+13x^4+43x^2+11$	$2^4\cdot 5^4\cdot 11\cdot 13^2$	$16.27\cdots$
38	1300	67	$x^6+49x^4+587x^2+67$	$2^4\cdot 5^4\cdot 13^2\cdot 67$	$21.99\cdots$
39	1304	11	$x^6+41x^4+75x^2+11$	$2^6\cdot 11\cdot 163^2$	$16.29\cdots$
40	1345	19	$x^6+42x^4+65x^2+19$	$5^2\cdot 19\cdot 269^2$	$18.03\cdots$
41	1373	3	$x^6+60x^4+136x^2+3$	$3\cdot 1373^2$	$13.34\cdots$
42	1373	151	$x^6+92x^4+1664x^2+151$	$151\cdot 1373^2$	$25.64\cdots$

TABLE 5 (continued).

case	d_F	q	$P_K(X)$	d_K	ρ_K
43	1573	71	$x^6 + 20x^4 + 104x^2 + 71$	$11^4 \cdot 13^2 \cdot 71$	$23.66\cdots$
44	2101	31	$x^6 + 40x^4 + 356x^2 + 31$	$11^2 \cdot 31 \cdot 191^2$	$22.70\cdots$
45	2589	79	$x^6 + 24x^4 + 84x^2 + 79$	$3^2 \cdot 79 \cdot 863^2$	$28.44\cdots$
46	2636	3	$x^6 + 193x^4 + 59x^2 + 3$	$2^4 \cdot 3 \cdot 659^2$	$16.58\cdots$
47	2708	11	$x^6 + 17x^4 + 51x^2 + 11$	$2^4 \cdot 11 \cdot 677^2$	$20.78\cdots$
48	2808	67	$x^6 + 81x^4 + 171x^2 + 67$	$2^6 \cdot 3^6 \cdot 13^2 \cdot 67$	$28.43\cdots$
49	3021	2	$x^6 + 28x^4 + 17x^2 + 2$	$2^3 \cdot 3^2 \cdot 19^2 \cdot 53^2$	$20.44\cdots$
50	3356	59	$x^6 + 29x^4 + 183x^2 + 59$	$2^4 \cdot 59 \cdot 839^2$	$29.54\cdots$
51	3508	11	$x^6 + 13x^4 + 27x^2 + 11$	$2^4 \cdot 11 \cdot 877^2$	$22.65\cdots$
52	3604	19	$x^6 + 13x^4 + 35x^2 + 19$	$2^4 \cdot 17^2 \cdot 19 \cdot 53^2$	$25.04\cdots$
53	3736	19	$x^6 + 97x^4 + 1739x^2 + 19$	$2^6 \cdot 19 \cdot 467^2$	$25.34\cdots$
54	3957	2	$x^6 + 93x^4 + 258x^2 + 4$	$2^6 \cdot 3^2 \cdot 1319^2$	$31.63\cdots$
55	4765	2	$x^6 + 79x^4 + 1087x^2 + 2$	$2^3 \cdot 5^2 \cdot 953^2$	$23.79\cdots$
56	4841	3	$x^6 + 46x^4 + 29x^2 + 3$	$3 \cdot 47^2 \cdot 103^2$	$20.31\cdots$
57	4853	2	$x^6 + 191x^4 + 207x^2 + 2$	$2^3 \cdot 23^2 \cdot 211^2$	$23.94\cdots$
58	4933	7	$x^6 + 37x^4 + 275x^2 + 7$	$7 \cdot 4933^2$	$23.54\cdots$
59	4933	19	$x^6 + 113x^4 + 123x^2 + 19$	$19 \cdot 4933^2$	$27.80\cdots$
60	5081	3	$x^6 + 14x^4 + 25x^2 + 3$	$3 \cdot 5081^2$	$20.64\cdots$
61	5685	31	$x^6 + 93x^4 + 195x^2 + 31$	$3^2 \cdot 5^2 \cdot 31 \cdot 379^2$	$31.63\cdots$
62	6185	3	$x^6 + 10x^4 + 17x^2 + 3$	$3 \cdot 5^2 \cdot 1237^2$	$22.04\cdots$
63	6401	3	$x^6 + 257x^4 + 59x^2 + 3$	$3 \cdot 37^2 \cdot 173^2$	$22.29\cdots$
64	6557	7	$x^6 + 36x^4 + 312x^2 + 7$	$7 \cdot 79^2 \cdot 83^2$	$25.88\cdots$
65	7028	11	$x^6 + 65x^4 + 399x^2 + 11$	$2^4 \cdot 7^2 \cdot 11 \cdot 251^2$	$28.56\cdots$
66	7796	3	$x^6 + 49x^4 + 395x^2 + 3$	$2^4 \cdot 3 \cdot 1949^2$	$23.81\cdots$
67	8372	3	$x^6 + 13x^4 + 35x^2 + 3$	$2^4 \cdot 3 \cdot 7^2 \cdot 13^2 \cdot 23^2$	$24.38\cdots$
68	8837	23	$x^6 + 336x^4 + 7540x^2 + 23$	$23 \cdot 8837^2$	$34.86\cdots$
69	8909	2	$x^6 + 11x^4 + 15x^2 + 2$	$2^3 \cdot 59^2 \cdot 151^2$	$29.31\cdots$
70	8909	7	$x^6 + 1792x^4 + 1444x^2 + 7$	$7 \cdot 59^2 \cdot 151^2$	$28.67\cdots$
71	9293	2	$x^6 + 300x^4 + 97x^2 + 2$	$2^3 \cdot 9293^2$	$29.73\cdots$
72	9413	11	$x^6 + 105x^4 + 139x^2 + 11$	$11 \cdot 9413^2$	$31.48\cdots$
73	9749	2	$x^6 + 99x^4 + 31x^2 + 2$	$2^3 \cdot 9749^2$	$30.21\cdots$
74	9805	7	$x^6 + 21317x^4 + 464371x^2 + 7$	$5^2 \cdot 7 \cdot 37^2 \cdot 53^2$	$29.60\cdots$
75	9812	3	$x^6 + 407569x^4 + 50756231x^2 + 3$	$2^4 \cdot 3 \cdot 11^2 \cdot 223^2$	$25.71\cdots$
76	9813	7	$x^6 + 101x^4 + 83x^2 + 7$	$3^2 \cdot 7 \cdot 3271^2$	$29.61\cdots$
77	9869	7	$x^6 + 1044x^4 + 5424x^2 + 7$	$7 \cdot 71^2 \cdot 139^2$	$29.66\cdots$
78	10261	7	$x^6 + 20x^4 + 24x^2 + 7$	$7 \cdot 31^2 \cdot 331^2$	$30.05\cdots$
79	10292	3	$x^6 + 473x^4 + 143x^2 + 3$	$2^4 \cdot 3 \cdot 31^2 \cdot 83^2$	$26.12\cdots$
80	10721	3	$x^6 + 11x^4 + 26x^2 + 3$	$3 \cdot 71^2 \cdot 151^2$	$26.48\cdots$
81	12092	19	$x^6 + 52261x^4 + 280087x^2 + 19$	$2^4 \cdot 19 \cdot 3023^2$	$37.49\cdots$
82	12269	3	$x^6 + 97x^4 + 155x^2 + 3$	$3 \cdot 12269^2$	$27.69\cdots$
83	12309	7	$x^6 + 60x^4 + 768x^2 + 7$	$3^2 \cdot 7 \cdot 11^2 \cdot 373^2$	$31.93\cdots$
84	12333	31	$x^6 + 62820x^4 + 69024x^2 + 31$	$3^2 \cdot 31 \cdot 4111^2$	$40.94\cdots$

TABLE 5 (continued).

case	d_F	q	$P_K(X)$	d_K	ρ_K
85	12401	3	$x^6+194x^4+245x^2+3$	$3\cdot 12401^2$	$27.79\cdots$
86	12788	11	$x^6+2997x^4+6019x^2+11$	$2^4\cdot 11\cdot 23^2\cdot 139^2$	$34.87\cdots$
87	14165	23	$x^6+147536x^4+1835892x^2+23$	$5^2\cdot 23\cdot 2833^2$	$40.80\cdots$
88	14229	7	$x^6+12x^4+24x^2+7$	$3^6\cdot 7\cdot 17^2\cdot 31^2$	$33.51\cdots$
89	14420	11	$x^6+185x^4+123x^2+11$	$2^4\cdot 5^2\cdot 7^2\cdot 11\cdot 103^2$	$36.29\cdots$
90	15212	3	$x^6+3313x^4+203x^2+3$	$2^4\cdot 3\cdot 3803^2$	$29.75\cdots$
91	15252	19	$x^6+208953x^4+11619x^2+19$	$2^4\cdot 3^2\cdot 19\cdot 31^2\cdot 41^2$	$40.51\cdots$
92	15284	3	$x^6+41x^4+419x^2+3$	$2^4\cdot 3\cdot 3821^2$	$29.80\cdots$
93	16532	3	$x^6+17x^4+23x^2+3$	$2^4\cdot 3\cdot 4133^2$	$30.59\cdots$
94	16721	3	$x^6+83x^4+1322x^2+3$	$3\cdot 23^2\cdot 727^2$	$30.70\cdots$
95	17165	23	$x^6+2336x^4+176292x^2+23$	$5^2\cdot 23\cdot 3433^2$	$43.50\cdots$
96	17273	3	$x^6+94x^4+53x^2+3$	$3\cdot 23^2\cdot 751^2$	$31.04\cdots$
97	17684	11	$x^6+641x^4+291x^2+11$	$2^4\cdot 11\cdot 4421^2$	$38.85\cdots$
98	17717	2	$x^6+15x^4+55x^2+2$	$2^3\cdot 7^2\cdot 2531^2$	$36.86\cdots$
99	17780	3	$x^6+61x^4+35x^2+3$	$2^4\cdot 3\cdot 5^2\cdot 7^2\cdot 127^2$	$31.34\cdots$
100	19112	3	$x^6+29x^4+47x^2+3$	$2^6\cdot 3\cdot 2389^2$	$32.10\cdots$
101	19544	11	$x^6+149x^4+5103x^2+11$	$2^6\cdot 7^2\cdot 11\cdot 349^2$	$40.17\cdots$
102	19869	7	$x^6+69x^4+1011x^2+7$	$3^2\cdot 7\cdot 37^2\cdot 179^2$	$37.46\cdots$
103	21853	7	$x^6+236x^4+8944x^2+7$	$7\cdot 13^2\cdot 41^4$	$38.66\cdots$
104	23612	3	$x^6+209x^4+107x^2+3$	$2^4\cdot 3\cdot 5903^2$	$34.45\cdots$
105	24884	3	$x^6+113x^4+2507x^2+3$	$2^4\cdot 3\cdot 6221^2$	$35.06\cdots$
106	26612	3	$x^6+565x^4+3035x^2+3$	$2^4\cdot 3\cdot 6653^2$	$35.85\cdots$
107	30356	3	$x^6+3025x^4+56423x^2+3$	$2^4\cdot 3\cdot 7589^2$	$37.46\cdots$
108	33885	7	$x^6+24x^4+132x^2+7$	$3^6\cdot 5^2\cdot 7\cdot 251^2$	$44.75\cdots$
109	34172	11	$x^6+1145x^4+699x^2+11$	$2^4\cdot 11\cdot 8543^2$	$48.39\cdots$
110	38204	3	$x^6+186193x^4+2027x^2+3$	$2^4\cdot 3\cdot 9551^2$	$40.44\cdots$
111	38420	3	$x^6+113x^4+2891x^2+3$	$2^4\cdot 3\cdot 5^2\cdot 17^2\cdot 113^2$	$40.52\cdots$
112	39704	3	$x^6+34513x^4+11627x^2+3$	$2^6\cdot 3\cdot 7^2\cdot 709^2$	$40.96\cdots$
113	41240	3	$x^6+591661x^4+19373647295x^2+3$	$2^6\cdot 3\cdot 5^2\cdot 1031^2$	$41.49\cdots$
114	43368	19	$x^6+39750177x^4+618651147x^2+19$	$2^6\cdot 3^2\cdot 13^2\cdot 19\cdot 139^2$	$57.39\cdots$
115	44072	11	$x^6+7957605x^4+2047039x^2+11$	$2^6\cdot 7^2\cdot 11\cdot 787^2$	$52.67\cdots$
116	45884	3	$x^6+14833x^4+765899x^2+3$	$2^4\cdot 3\cdot 11471^2$	$42.99\cdots$
117	51404	3	$x^6+659338673x^4+49539876107x^2+3$	$2^4\cdot 3\cdot 71^2\cdot 181^2$	$44.65\cdots$
118	56264	3	$x^6+788594737x^4+20558987x^2+3$	$2^6\cdot 3\cdot 13^2\cdot 541^2$	$46.01\cdots$
119	58904	3	$x^6+2225x^4+342539x^2+3$	$2^6\cdot 3\cdot 37^2\cdot 199^2$	$46.72\cdots$
120	88700	3	$x^6+127921x^4+43787x^2+3$	$2^4\cdot 3\cdot 5^4\cdot 887^2$	$53.55\cdots$
121	93260	3	$x^6+121861x^4+2454407x^2+3$	$2^4\cdot 3\cdot 5^2\cdot 4663^2$	$54.46\cdots$

TABLE 6. K does not contain any imaginary quadratic field and $(q) = \mathcal{Q}_1\mathcal{Q}_2^2$ in F

case	d_F	q	$P_K(X)$	d_K	ρ_K
1	568	2	$x^6 + 7x^4 + 10x^2 + 2$	$2^{11} \cdot 71^2$	$14.75\cdots$
2	1101	3	$x^6 + 8x^4 + 12x^2 + 3$	$3^3 \cdot 367^2$	$12.40\cdots$
3	2024	2	$x^6 + 11x^4 + 30x^2 + 2$	$2^{11} \cdot 11^2 \cdot 23^2$	$22.53\cdots$
4	2233	11	$x^6 + 74x^4 + 77x^2 + 11$	$7^2 \cdot 11^3 \cdot 29^2$	$19.49\cdots$
5	2505	3	$x^6 + 11x^4 + 30x^2 + 3$	$3^3 \cdot 5^2 \cdot 167^2$	$16.31\cdots$
6	2589	3	$x^6 + 16x^4 + 36x^2 + 3$	$3^3 \cdot 863^2$	$16.49\cdots$
7	3124	11	$x^6 + 193x^4 + 1859x^2 + 11$	$2^4 \cdot 11^3 \cdot 71^2$	$21.80\cdots$
8	3252	3	$x^6 + 281x^4 + 3183x^2 + 3$	$2^4 \cdot 3^3 \cdot 271^2$	$17.79\cdots$
9	3596	2	$x^6 + 9x^4 + 16x^2 + 2$	$2^9 \cdot 29^2 \cdot 31^2$	$27.29\cdots$
10	3941	7	$x^6 + 37x^4 + 147x^2 + 7$	$7^3 \cdot 563^2$	$21.84\cdots$
11	4844	2	$x^6 + 65x^4 + 24x^2 + 2$	$2^9 \cdot 7^2 \cdot 173^2$	$30.14\cdots$
12	5901	7	$x^6 + 452x^4 + 2744x^2 + 7$	$3^2 \cdot 7^3 \cdot 281^2$	$24.99\cdots$
13	6153	3	$x^6 + 10x^4 + 21x^2 + 3$	$3^3 \cdot 7^2 \cdot 293^2$	$22.00\cdots$
14	6237	7	$x^6 + 96x^4 + 252x^2 + 7$	$3^8 \cdot 7^3 \cdot 11^2$	$25.45\cdots$
15	8637	3	$x^6 + 1804x^4 + 612x^2 + 3$	$3^3 \cdot 2879^2$	$24.64\cdots$
16	8745	3	$x^6 + 130x^4 + 117x^2 + 3$	$3^3 \cdot 5^2 \cdot 11^2 \cdot 53^2$	$24.74\cdots$
17	10868	11	$x^6 + 117x^4 + 979x^2 + 11$	$2^4 \cdot 11^3 \cdot 13^2 \cdot 19^2$	$33.03\cdots$
18	12660	3	$x^6 + 25x^4 + 147x^2 + 3$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 211^2$	$27.98\cdots$
19	14189	7	$x^6 + 229x^4 + 11347x^2 + 7$	$7^3 \cdot 2027^2$	$33.48\cdots$
20	15252	3	$x^6 + 25x^4 + 51x^2 + 3$	$2^4 \cdot 3^3 \cdot 31^2 \cdot 41^2$	$29.78\cdots$
21	16116	3	$x^6 + 25297x^4 + 47595x^2 + 3$	$2^4 \cdot 3^3 \cdot 17^2 \cdot 79^2$	$30.33\cdots$
22	17889	3	$x^6 + 19x^4 + 90x^2 + 3$	$3^3 \cdot 67^2 \cdot 89^2$	$31.40\cdots$
23	19113	3	$x^6 + 175x^4 + 4026x^2 + 3$	$3^3 \cdot 23^2 \cdot 277^2$	$32.10\cdots$
24	22425	3	$x^6 + 47671x^4 + 822x^2 + 3$	$3^3 \cdot 5^4 \cdot 13^2 \cdot 23^2$	$33.86\cdots$
25	23028	3	$x^6 + 49x^4 + 267x^2 + 3$	$2^4 \cdot 3^3 \cdot 19^2 \cdot 101^2$	$34.16\cdots$
26	29553	3	$x^6 + 19x^4 + 18x^2 + 3$	$3^3 \cdot 9851^2$	$37.12\cdots$
27	31101	7	$x^6 + 1857804x^4 + 471072x^2 + 7$	$3^2 \cdot 7^3 \cdot 1481^2$	$43.49\cdots$
28	32073	3	$x^6 + 322x^4 + 23073x^2 + 3$	$3^3 \cdot 10691^2$	$38.15\cdots$
29	34152	3	$x^6 + 145x^4 + 171x^2 + 3$	$2^6 \cdot 3^3 \cdot 1423^2$	$38.96\cdots$
30	36872	11	$x^6 + 229059609x^4 + 855080347x^2 + 11$	$2^6 \cdot 11^3 \cdot 419^2$	$49.63\cdots$
31	37176	3	$x^6 + 375539437x^4 + 32882926911x^2 + 3$	$2^6 \cdot 3^3 \cdot 1549^2$	$40.08\cdots$
32	38580	3	$x^6 + 241x^4 + 75x^2 + 3$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 643^2$	$40.57\cdots$
33	41172	3	$x^6 + 29329x^4 + 1047339x^2 + 3$	$2^4 \cdot 3^3 \cdot 47^2 \cdot 73^2$	$41.46\cdots$
34	41865	3	$x^6 + 102958x^4 + 30321x^2 + 3$	$3^3 \cdot 5^2 \cdot 2791^2$	$41.70\cdots$
35	47004	3	$x^6 + 433x^4 + 267x^2 + 3$	$2^4 \cdot 3^3 \cdot 3917^2$	$43.34\cdots$
36	52764	3	$x^6 + 28549x^4 + 3128295x^2 + 3$	$2^4 \cdot 3^3 \cdot 4397^2$	$45.04\cdots$
37	53736	3	$x^6 + 56317x^4 + 86415x^2 + 3$	$2^6 \cdot 3^3 \cdot 2239^2$	$45.31\cdots$
38	73176	3	$x^6 + 23867281x^4 + 2424100779x^2 + 3$	$2^6 \cdot 3^3 \cdot 3049^2$	$50.23\cdots$
39	76200	3	$x^6 + 61x^4 + 207x^2 + 3$	$2^6 \cdot 3^3 \cdot 5^4 \cdot 127^2$	$50.91\cdots$
40	98844	3	$x^6 + 2293x^4 + 6999x^2 + 3$	$2^4 \cdot 3^3 \cdot 8237^2$	$55.52\cdots$
41	108325	7	$x^6 + 12962664x^4 + 309216012x^2 + 7$	$5^4 \cdot 7^3 \cdot 619^2$	$65.93\cdots$
42	201129	3	$x^6 + 137137x^4 + 7947x^2 + 3$	$3^3 \cdot 67043^2$	$70.36\cdots$

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Institut de Mathématiques de Luminy, UPR 9016, 163, avenue de Luminy, Case 907, 13288 Marseille Cedex 9, FRANCE

E-mail address: `louboutiQiml.univ-mrs.fr`

E-mail address: `gboutteauxQnet-up.com`