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On divisibility of one special type of numbers

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Abstract: We will deal with numbers given by the relation ${}^k\mathfrak{J}_n = \sum_{i=0}^{n-2} \binom{n}{i} k^{n-2-i}$, where k is any nonnegative integer and n is any positive integer greater than 1, with ${}^k\mathfrak{J}_0 = {}^k\mathfrak{J}_1 = 0$. The special type of these numbers for $k = 1$ was investigated before in [1]. In this paper some results about divisibility of numbers ${}^k\mathfrak{J}_n$ are found. In addition certain properties of their divisibility are used for finding primes of the type ${}^k\mathfrak{J}_n$ for $k \leqq 13$ and $n \leqq 4500$.

Key Words: Special type of numbers, divisibility, primality

Mathematics Subject Classification: 11A51, 11A07, 11Y11

1. Introduction

In [1] the numbers $\mathfrak{J}_n$ were studied. They are created from the polynomials $\mathcal{J}_n(x)$ in the following way:

$$\mathfrak{J}_n = \mathcal{J}_n(1) = 2^n - n - 1 ,$$

where n is any positive integer.

We can recall that these polynomials are defined by the relation

$$\mathcal{J}_n(x) = \sum_{i=0}^{n-2} \binom{n}{i} x^{n-i} , \; n \geqq 2 , \; \mathcal{J}_0(x) = \mathcal{J}_1(x) = 0 ,$$

therefore

$$\mathcal{J}_0(x) = 0 , \qquad \mathcal{J}_n(x) = (1 + x)^n - 1 - n\,x , \; n \in \mathbb{N} .$$

In this paper we will deal with the numbers

$${}^k\mathfrak{J}_n = \frac{(k+1)^n - nk - 1}{k^2} , \tag{1}$$

where k is any positive integer and n is any nonnegative integer. It is evident that the numbers ${}^1\mathfrak{J}_n$ are identical to $\mathfrak{J}_n$ from [1] and all the numbers ${}^k\mathfrak{J}_n$ are integers as

$$ {}^k\mathfrak{J}_n = \mathcal{J}_n(k)/k^2 = \sum_{i=0}^{n-2} \binom{n}{i} k^{n-2-i} . \tag{2}$$

Because ${}^k\mathfrak{J}_0 = {}^k\mathfrak{J}_1 = 0$ and ${}^k\mathfrak{J}_2 = 1$ for any k we will mostly assume $n > 2$ in the following text.

Further, for instance, we get the following recurrences for the numbers ${}^k\mathfrak{J}_n$ from the recurrences for the polynomials $\mathcal{J}_n(x)$ in [1]:

$$ {}^k\mathfrak{J}_{n+1} - (k+1)\,{}^k\mathfrak{J}_n = n , \quad {}^k\mathfrak{J}_0 = 0 ,$$

$$ {}^k\mathfrak{J}_{n+2} - (k+2)\,{}^k\mathfrak{J}_{n+1} + (k+1)\,{}^k\mathfrak{J}_n = 1 , \quad {}^k\mathfrak{J}_0 = {}^k\mathfrak{J}_1 = 0 ,$$

$$ {}^k\mathfrak{J}_{n+1} = k\sum_{i=0}^{n} {}^k\mathfrak{J}_i + \binom{n+1}{2} , {}^k\mathfrak{J}_0 = 0 ,$$

$$ k^3 \sum_{i=0}^{n} {}^k\mathfrak{J}_{n-i}\, {}^k\mathfrak{J}_i = k(n+1)\,{}^k\mathfrak{J}_n - 4\,{}^k\mathfrak{J}_{n+1} + (k-2)(n+1) + $$
$$ + 2\binom{n+1}{2} - 2(k-1)\binom{n+2}{2} + k\binom{n+3}{3} $$

2. The main results

The main results established in this paper concern divisibility of the numbers ${}^k\mathfrak{J}_n$. They are expressed in the following theorems.

Theorem 1. *Let k, $n \geqq 2$ be any positive integers.*

$$ \begin{aligned} {}^k\mathfrak{J}_n \equiv 0 \pmod 2 \Longleftrightarrow & (k \equiv 0 \pmod 2 \wedge n \equiv 0,1 \pmod 4) \vee \\ & (k \equiv 1 \pmod 2 \wedge n \equiv 1 \pmod 2) , \\ {}^k\mathfrak{J}_n \equiv 1 \pmod 2 \Longleftrightarrow & (k \equiv 0 \pmod 2 \wedge n \equiv 2,3 \pmod 4) \vee \\ & (k \equiv 1 \pmod 2 \wedge n \equiv 0 \pmod 2) , \end{aligned} $$

Theorem 2. *Let k, $n \geqq 2$ be any positive integers.*

$$ \begin{aligned} {}^k\mathfrak{J}_n \equiv 0 \pmod 3 \Longleftrightarrow & (k \equiv 0 \pmod 3 \wedge n \equiv 0,1 \pmod 3) \vee \\ & (k \equiv 1 \pmod 3 \wedge n \equiv 0,1 \pmod 6) \vee \\ & (k \equiv 2 \pmod 3 \wedge n \equiv 1 \pmod 3) , \\ {}^k\mathfrak{J}_n \equiv 1 \pmod 3 \Longleftrightarrow & (k \equiv 0 \pmod 3 \wedge n \equiv 2 \pmod 3) \vee \\ & (k \equiv 1 \pmod 3 \wedge n \equiv 2,3 \pmod 6) \vee \\ & (k \equiv 2 \pmod 3 \wedge n \equiv 2 \pmod 3) , \\ {}^k\mathfrak{J}_n \equiv 2 \pmod 3 \Longleftrightarrow & (k \equiv 1 \pmod 3 \wedge n \equiv 4,5 \pmod 6) \vee \\ & (k \equiv 2 \pmod 3 \wedge n \equiv 0 \pmod 3) , \end{aligned} $$

3. Some lemmas and preliminary results

At first we give some results showing the relation of numbers ${}^k\mathfrak{J}_n$ to other mathematical expressions. For the most part they are elemantary consequences of the binomial theorem.

Lemma 1. *Let k, $n > 2$ be any positive integers. Then*

$${}^k\mathfrak{J}_n \equiv \binom{n}{2} \pmod{k} .$$

Proof. The proved congruence follows from the relation (2) because we can write

$${}^k\mathfrak{J}_n = \sum_{i=0}^{n-2} \binom{n}{i} k^{n-2-i} = k \sum_{i=0}^{n-3} \binom{n}{i} k^{n-3-i} + \binom{n}{2} . \quad \square$$

Lemma 2. *Let k be any positive integer and p be any prime. Then*

$${}^k\mathfrak{J}_p \equiv k^{p-2} \pmod{p} .$$

Proof. The identity

$${}^k\mathfrak{J}_p = \sum_{i=0}^{p-2} \binom{p}{i} k^{p-2-i} = k^{p-2} + \sum_{i=1}^{p-2} \binom{p}{i} k^{p-2-i}$$

and the well-known fact that $p \mid \binom{p}{i}$ for $i = 1, 2, \ldots, p-1$ imply the assertion. $\square$

Corollary of Lemma 2. *Let k be any positive integer, p be any prime and k, p be relatively prime. Then*

$$p \mid k\,{}^k\mathfrak{J}_p - 1 .$$

Proof. Using Fermat's Little Theorem we obtain

$$k\,{}^k\mathfrak{J}_p - 1 \equiv k^{p-1} - 1 \equiv 0 \pmod{p}. \quad \square$$

Before we will give other divisibility properties of the numbers ${}^k\mathfrak{J}_n$ we prove two lemmas on binomial coefficients.

Lemma 3. *Let $a > 1$ be any positive integer and b, m be any nonnegative integers, $c \equiv b \pmod{a}$, where $0 \leqq c \leqq a - 1$. Then*

(i)

$$\binom{am+b}{2} \equiv \binom{b}{2} \pmod{a}$$

iff a is odd or a, m are even.

(ii)

$$\binom{b}{2} \equiv \binom{c}{2} \pmod{a}$$

iff a is odd or a, $\left[\frac{b}{a}\right]$ *are even.*

Proof.

(i) The following obvious identity on binomial coefficients

$$\binom{am+b}{2} = \binom{am}{2} + abm + \binom{ab}{2}$$

implies the assertion,

(ii) as $c \equiv b \pmod{a}$ means $b = aq + c$, where q is a nonnegative integer, the assertion results from the case (i). □

Theorem 3. *Let* $a > 1$, l *be any positive integers,* b, m *be any nonnegative integers,* $c \equiv b \pmod{a}$, *where* $0 \leqq c \leqq a-1$. *Then*

$${}^{al}\mathfrak{J}_{am+b} \equiv \binom{c}{2} \pmod{a}$$

iff *a is odd* *or* *a, m,* $\left[\frac{b}{a}\right]$ *are even.*

Proof. With the use of (2), this becomes

$${}^{al}\mathfrak{J}_{am+b} = a \sum_{i=0}^{am+b-2} \binom{am+b}{i} a^{am+b-3-i} l^{am+b-2-i} + \binom{am+b}{2}.$$

The congruence

$${}^{al}\mathfrak{J}_{am+b} \equiv \binom{am+b}{2} \pmod{a}$$

is true and the assertion is valid with respect to Lemma 3. □

Corollary 1 of Theorem 3. *Let* $a > 1$, l *be any positive integers and* b *be any nonnegative integer. Then*

$${}^{al}\mathfrak{J}_{a(m+d)} \equiv {}^{al}\mathfrak{J}_{am} \equiv 0 \pmod{a}$$

iff *a is odd* *or* *a, b are even.*

Proof. The assertion is a direct consequence of Theorem 3 where we put $b = ad$. □

We can also formulate another special case of Theorem 3 when a is a prime.

Corollary 2 of Theorem 3. *Let* l, m *be any positive integers,* b *be any nonnegative integer and* p *be any odd prime. Then*

$$p \mid {}^{pl}\mathfrak{J}_{pm+b}$$

iff

$$b \equiv 0 \pmod{p} \qquad or \qquad b \equiv 1 \pmod{p}\ .$$

Theorem 4. *Let $a > 1$, $n > 2$ be any integers, l be any nonnegative integer. Then*
(i)

$$a \mid {}^{al+a-1}\mathfrak{J}_n \quad \textit{iff} \quad a \mid n-1$$

(ii)

$$a \mid {}^{al+1}\mathfrak{J}_n \quad \textit{iff} \quad a \mid {}^{1}\mathfrak{J}_n\ .$$

Proof.
(i) As

$${}^{al+a-1}\mathfrak{J}_n = \frac{a^n(l+1)^n - na(l+1) + n - 1}{(al+a-1)^2}$$

and $(a(l+1)-1)^2 \equiv 1 \pmod a$ the assertion is valid.
(ii) We can write

$$\begin{aligned} {}^{al+1}\mathfrak{J}_n &= \frac{(al+2)^n - n(al+1) - 1}{(al+1)^2} = \\ &= \frac{1}{(al+1)^2}\left(a\sum_{i=0}^{n-1}\binom{n}{i}a^{n-i-1}l^{n-i}2^i - anl + 2^n - n - 1\right)\ , \end{aligned}$$

where $(al+1)^2 \equiv 1 \pmod a$. Therefore the assertion holds. □

4. The proof of Theorems 1 and 2

Proof of Theorem 1. Let us consider these four cases.
If $k \equiv 0 \pmod 2$ and $n \equiv 0, 1 \pmod 4$ then Theorem 3, the part (ii), implies ${}^{k}\mathfrak{J}_n \equiv 0 \pmod 2$.
If $k \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 2$ then Theorem 4 implies ${}^{k}\mathfrak{J}_n \equiv 0 \pmod 2$.
If $k \equiv 0 \pmod 2$ and $n \equiv 2, 3 \pmod 4$ then Theorem 3, the part (ii), implies ${}^{k}\mathfrak{J}_n \equiv 1 \pmod 2$.
If $k \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$ then Theorem 4 implies ${}^{k}\mathfrak{J}_n \equiv 1 \pmod 2$.

Because the previous cases include all possibilities for the values of the numbers k and n the inverse implications are true. □

Proof of Theorem 2. All cases can be proved in a similar way. If $k \equiv 0 \pmod 3$ then the assertion is a conclusion of Theorem 3. If $k \equiv 1 \pmod 3$ then

$$\begin{aligned} {}^{3l+1}\mathfrak{J}_n &= \frac{(3l+2)^n - (3l+1)n - 1}{(3l+1)^2} = \\ &= \frac{1}{(3l+1)^2}\left(3\sum_{i=0}^{n-1}\binom{n}{i}3^{n-i-1}l^{n-i}2^i - 3nl + 2^n - n - 1\right)\ . \end{aligned}$$

As $(3l+1)^2 \equiv 1 \pmod 3$ we obtain ${}^{3l+1}\mathfrak{J}_n \equiv {}^{1}\mathfrak{J}_n \pmod 3$ and the assertion follows from Theorem 5 of [1].

If $k \equiv 2 \pmod 3$ then the basic idea of the proof is the same as in the previous case. It leads to the congruence ${}^{3l+2}\mathfrak{J}_n \equiv n - 1 \pmod 3$ from which the remaining cases are obtained. □

4. Remark on primality of ${}^k\mathfrak{J}_n$ numbers

The following theorem is the basis for our computer testing of the primality of the numbers ${}^k\mathfrak{J}_n$.

Theorem 5. *Let k, $n \geqq 2$ be any positive integers.*

$$\begin{aligned} 2 \mid {}^k\mathfrak{J}_n \Longleftrightarrow & (k \equiv 0 \pmod 2) \wedge n \equiv 0,1 \pmod 4) \vee \\ & (k \equiv 1 \pmod 2) \wedge n \equiv 1 \pmod 2) \ , \\ 3 \mid {}^k\mathfrak{J}_n \Longleftrightarrow & (k \equiv 0 \pmod 3) \wedge n \equiv 0,1 \pmod 3) \vee \\ & (k \equiv 1 \pmod 3) \wedge n \equiv 0,1 \pmod 6) \vee \\ & (k \equiv 2 \pmod 3) \wedge n \equiv 1 \pmod 3) \ , \\ 5 \mid {}^k\mathfrak{J}_n \Longleftrightarrow & (k \equiv 0 \pmod 5) \wedge n \equiv 0,1 \pmod 5) \vee \\ & (k \equiv 1 \pmod 5) \wedge n \equiv 0,1,7,18 \pmod{20}) \\ & (k \equiv 2 \pmod 5) \wedge n \equiv 0,1,3,14 \pmod{20}) \\ & (k \equiv 3 \pmod 5) \wedge n \equiv 0,1 \pmod{10}) \\ & (k \equiv 4 \pmod 5) \wedge n \equiv 1 \pmod 5) \ . \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 1 and Theorem 2. □

The computer testing of the primality of the numbers ${}^k\mathfrak{J}_n$ became more effective using Theorem 8. For example, if $k = 7$ the conditions of the divisibility by the numbers 2, 3 and 5 lead to the fact that every prime in the form ${}^7\mathfrak{J}_n$ must be in the form $60q + r$, where q is nonnegative integer and r = 2, 4, 8, 10, 16, 22, 26, 28, 32, 38, 44, 46, 50, 52, 56, 58.

Table 1. The list of indices of the primes ${}^k\mathfrak{J}_n$ for $k = 1, 2, \dots, 13$ and $n < 4500$:

k	n									
1	4	10	14	16	26	50	56	70	116	2072
2	3	6	11	30	167	626	1842			
3	14	458	3794							
4	3	38	47	118	130	3075				
5	18	528	3102	4254						
6	26									
7	4	10	278	452						
8	3	95	359							
9	2498	3302								
10	3									
11	6									
13	4	16	256	374						

References

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