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Abstract: In the paper it is described a construction of a Lebesgue measure zero set of real numbers which can be decomposed into continuum many disjoint subsets each of which has intersection of power of continuum with every open interval in real numbers.

Key Words: Lebesgue measure, sets of measure zero, density

Mathematics Subject Classification: 26A, 28A

Introduction

In mathematics there are several possibilities how to evaluate the size of sets. Some of them are the subjects of individual mathematical branches, for instance measure theory or dimension theory. In this paper we will consider sets of Lebesgue measure zero in the set $\mathbb{R}$ of all real numbers with respect to their cardinalities. First, let us recall the basic notions and relations used in the sequel.

Cardinal number or *cardinality* of a subset of reals means the number of its elements. It can be either a nonnegative integer (for finite sets), or $\aleph_0$ (for infinite countable sets) or an uncountable cardinal number (for infinite uncountable sets). In each case it can not overreach the cardinality of all real numbers which we will denote by c and it is called the ***cardinality of continuum.*** We will also use the phrase "continuum many" for the number of elements of such a set.[1] Cardinality is the most general measure of the size of sets. It needs no additional structure on the set itself and by its means it is possible to determine the size of any set. The cardinal number of a set A we will denote by $\operatorname{card} A$.

Another, more specific, widely used measure of size of sets of real numbers is *Lebesgue measure.* It is a nonnegative real number attached to each measurable subset of $\mathbb{R}$.[2] The least sets with respect to this classification are the ***sets of Lebesgue***

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[1] In the standard axiomatics of set theory, i.e. Zermelo-Fraenkel axiomatics, it is not possible to decide whether there exist subsets of reals whose cardinality is strictly between $\aleph_0$ and $c = 2^{\aleph_0}$. This question is a subject of well known Continuum hypothesis (CH) which says that there are no such sets. The rejection of (CH) admits such sets and without additional set-theoretical axioms there is no possibility to determine all possible cardinalities between $\aleph_0$ and c.

[2] Notice that there is a possibility of existence of sets of reals which are not measurable in the sense of Lebesgue measure (by use of the Axiom of Choice).

measure zero (or equivalently Lebesgue measure zero sets). These sets can be briefly characterised by the following condition

> A set $Z \subset \mathbb{R}$ is of Lebesgue measure zero if for all $\varepsilon > 0$ there exists a countable system of closed intervals $\{I_n\}_{n=1}^{\infty}$ covering the set Z (i.e. $Z \subset \bigcup_{n=1}^{\infty} I_n$) such that $\sum_{n=1}^{\infty} |I_n| < \varepsilon$, where $|I|$ denotes the length of the interval I.

There are some relations between the cardinality of a set and its Lebesgue measure. For instance it is easy to show that "small" sets in the sense of cardinality (i.e. at most countable sets) are of Lebesgue measure zero. The most known example that the opposite implication does not hold provides the famous Cantor set. This is the set of all real numbers in the interval $\langle 0, 1\rangle$ in whose ternary development does not occur the digit 1. Geometrically this set can be characterised as the set produced from the set $M_0 = \langle 0, 1\rangle$ by the (infinite) inductive application of the following rule

> If the set M_n is a union of a finite number of disjoint closed intervals then M_{n+1} is formed from M_n by removing all the middle open subintervals of lengths one third of the length of the original intervals.

Cantor set is of Lebesgue measure zero and cardinality of continuum. So it is the least possible set with respect to Lebesgue measure and the greatest possible set with respect to the cardinality. To evaluate its size let us involve the third "measure of the size", the topological *density.*

> A set $A \subset \mathbb{R}$ is *dense* if its intersection with each open interval is nonempty. A set $A \subset \mathbb{R}$ is *dense in the interval* (a, b) if its intersection with each nonempty subinterval of (a, b) is nonempty. A set $A \subset \mathbb{R}$ is *nowhere dense* if it is dense in no interval in $\mathbb{R}$.

With respect to this qualitative "measure of size" the Cantor set belongs among the least sets, it is nowhere dense.

Remark. Obviously, adding the set of all rationals Q to the Cantor set C, the set $Q \cup C$ is dense, has cardinality c and Lebesgue measure zero. Therefore, we strenghthen the density property of a real subsets. For the purposes of this paper, let us call a set $A \subset \mathbb{R}$ superdense if it intersects each open interval in continuum many points.

Thus a natural question arises whether there exist superdense subsets of Lebesgue measure zero in $\mathbb{R}$?

As the following theorem says we can say even more.

Theorem. *There exists a set $Z \subset \mathbb{R}$ of Lebesgue measure zero which can be decomposed into continuum many pairwise disjoint superdense subsets.*

Remark. The statement of the theorem can be simply derived from results by Gütig [G] who extended the known results by Jarník [J] and Besicovitch [B] on Hausdorff dimension of sets of irrational numbers classified by their approximability by rational numbers. However, because of the complexity of these sets, there is no simple insight to the structure of the resulting decomposition. The main aim of this contribution is to present a simple description of such a construction.

Construction

Proof of Theorem. Denote by $\mathbb{B}$ the set of all sequences of zeros and ones in which zero occurs infinitely many times and by $\mathbb{F}$ the set of all finite sequences of zeros and ones. In the sequel, for members b of the set $\mathbb{B}$ we will suppose $b = \{b_1, b_2, b_3, \dots\}$. For each $f = \{f_1, f_2, \dots, f_n\} \in \mathbb{F}$ let us denote

$$\mathbb{B}_f = \{b = \{b_1, b_2, b_3, \dots\} \in \mathbb{B};\ b_1 = f_1,\ b_2 = f_2, \dots, b_n = f_n\}.$$

We will consider the terms of $\mathbb{F}$ as finite sequences of independent realizations of a binary random variable X for which $p(X = 0) = p(X = 1) = \frac{1}{2}$ and similarly we will consider the terms of the set $\mathbb{B}$ as infinite sequences of independent realiztions of the same random variable. Doing so, for each $f = \{f_1, f_2, \dots, f_n\} \in \mathbb{F}$ we have

$$p(\mathbb{B}_f) = \frac{1}{2^n}.$$

The transformation $\beta\colon \mathbb{B} \to \langle 0, 1)$ defined by relation

$$\beta(b) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$$

is bijective and moreover for each $f = \{f_1, f_2, \dots, f_n\} \in \mathbb{F}$ we have

$$\beta^{-1}(\mathbb{B}_f) = \left\langle \sum_{i=1}^{n} \frac{f_i}{2^i}\, ,\ \sum_{i=1}^{n} \frac{f_i}{2^i} + \frac{1}{2^n} \right).$$

Consequently

$$p(\mathbb{B}_f) = \lambda(\beta^{-1}(\mathbb{B}_f)) = \frac{1}{2^n},$$

where $\lambda(S)$ denotes the Lebesgue measure of a set S.
Further for each $b \in \mathbb{B}$ let us denote

$$S_n(b) = \sum_{i=1}^{n} (2b_i - 1),$$

i.e. we add a unit at indices where it is $b_i = 1$ and we substract a unit at indices where it is $b_i = 0$. In the sequel we will use this notation also in cases when only a part $\{b_1, b_2, \dots b_m\};\ m \geq n$ of a sequence $b \in \mathbb{B}$ has been already defined.
Evidently, the mean value of each random value in the previous sum is equal to 0 and so the following equality holds by Borel's Strong law of large numbers [Z–S] (Theorem 9.5 p. 130)

$$p\left(\left\{b \in \mathbb{B};\ \lim_{n\to\infty} \frac{S_n(b)}{n} = 0\right\}\right) = 1.$$

For each $\alpha \in \langle 0, 1\rangle$ denote

$$M_\alpha = \left\{x \in \langle 0, 1);\ \limsup_{n\to\infty} \frac{|S_n(\beta^{-1}(x))|}{n} = \alpha\right\}.$$

The uniqueness of Lebesgue measure as the extension of Jordan content defined on intervals implies

$$\lambda(M_0) = p\left(\left\{b \in \mathbb{B};\ \lim_{n\to\infty} \frac{S_n(b)}{n} = 0\right\}\right) = 1.$$

For each $\alpha \in (0,1\rangle$ let us denote $Z_\alpha = \bigcup_{n\in\mathbb{Z}} (M_\alpha + n)$ where $M_\alpha + n = \{x+n;\ x \in M_\alpha\}$ and $\mathbb{Z}$ denotes the set of all integers. The above facts, σ-aditivity of Lebesgue measure and its invariance under translation imply that for $Z = \bigcup_{\alpha\in(0,1\rangle} Z_\alpha$ we have

$$\lambda(Z) = \sum_{n\in\mathbb{Z}} \lambda\Big(\bigcup_{\alpha\in(0,1\rangle} (M_\alpha + n)\Big) = \sum_{n\in\mathbb{Z}} (\lambda(\langle 0,1\rangle) - \lambda(M_0)) = 0.$$

By the above definitions $Z_{\alpha_1} \bigcap Z_{\alpha_2} = \emptyset$ for all $\alpha_1 \neq \alpha_2$ from the interval $(0,1\rangle$. So, to finish the proof it suffices to show that $\operatorname{card}(Z_\alpha \bigcap (a,b)) = c$ for each $\alpha \in (0,1\rangle$ and each open interval $(a,b) \subset \mathbb{R}$.
First we will show that $\operatorname{card} M_\alpha = c$ for each $\alpha \in (0,1\rangle$. Denote by

$$\mathbb{S} = \{\{s_1, s_2, s_3, \dots\};\ s_n \in \{+,-\}\ \forall n \in \mathbb{N}\}.$$

Evidently, $\operatorname{card}\mathbb{S} = c$. Fix an $\alpha \in (0,1\rangle$. By means of induction we will attach to each $s \in \mathbb{S}$ a number $x_s \in M_\alpha$ such that for $s_1 \neq s_2$ we will have $x_{s_1} \neq x_{s_2}$.
Step $k = 1$. Put $b_1 = 1$ and $b_2 = 0$ if $s_1 = +$ otherwise put $b_1 = 0$ and $b_2 = 1$. Further put $m_1 = 1$ and $n_1 = 2$.
Induction step. Let us suppose that for each $i = 1,2,\dots,k$ the positive integers $m_1 < n_1 < m_2 < n_2 < \dots < m_k < n_k$ already has been defined together with the values b_i for all $i = 1,2,\dots,n_k$ such that

$$\frac{|S_{m_j}(b)|}{m_j} > \alpha,\ S_{n_j}(b) = 0,\ \forall j = 1,2,\dots,k$$

and

$$\frac{|S_i(b)|}{i} \le \alpha + \frac{1}{i},\ \forall i = 1,2,\dots,n_k.$$

Denote by m_{k+1} the least positive integer greater than n_k such that $\frac{m_{k+1}-n_k}{m_{k+1}} > \alpha$ and put $n_{k+1} = m_{k+1} + (m_{k+1} - n_k) = 2m_{k+1} - n_k$. If $s_{k+1} = +$ then put $b_i = 1,\ \forall i = n_k+1, n_k+2, \dots, m_{k+1}$ a $b_i = 0,\ \forall i = m_{k+1}+1, m_{k+1}+2, \dots, n_{k+1}$, otherwise, if $s_{k+1} = -$, put $b_i = 0,\ \forall i = n_k+1, n_k+2, \dots, m_{k+1}$ a $b_i = 1,\ \forall i = m_{k+1}+1, m_{k+1}+2, \dots, n_{k+1}$.
In the both cases we have

$$\frac{|S_{m_j}(b)|}{m_j} > \alpha,\ S_{n_j}(b) = 0,\ \forall j = 1,2,\dots,k+1$$

and

$$\frac{|S_i(b)|}{i} \le \alpha + \frac{1}{i},\ \forall i = 1,2,\dots,n_{k+1}.$$

Set $x_s = \beta^{-1}(b)$. The construction implies that the values x_s differ for different values of $s \in \mathbb{S}$, so consequently $\mathrm{card}(M_\alpha) = c$.
Now let $(a,b) \subset \langle 0,1)$ be an arbitrary open interval. Then there exists such a $f = \{f_1, f_2, \dots, f_n\} \in \mathbb{F}$ that $\beta^{-1}(\mathbb{B}_f) \subset (a,b)$. For each $b \in \mathbb{B}$ let us define $\varphi_f(b) \in \mathbb{B}$ the unique element such that $\varphi_f(b)_i = f_i$, $i = 1,2,\dots,n$ and $\varphi_f(b)_{i+n} = b_i$, $i = 1,2,\dots$.

Because the value of lim sup does not depend on finite number of terms, we have $\beta(b) \in M_\alpha$ if and only if $\beta(\varphi_f(b)) \in M_\alpha$. So, as the correspondence $b \to \varphi_f(b)$ is injective and $\beta(\varphi_f(b)) \in (a,b)$, we have $\mathrm{card}(M_\alpha \bigcap (a,b)) = c$. Finally, the sets Z_α are invariant with respect to the integer valued translations, so we have $\mathrm{card}(Z_\alpha \bigcap (a,b)) = c$ for an arbitrary open interval of real numbers. □

Remark. Using the above mentioned results by Gütig [G] it is possible to prove also an extension of the statement proved in Theorem. This extension concerns the topological density criterion and mainly the introduction of another kind of measure of size of sets, the Hausdorff dimension. As the Hausdorff dimension of each set of reals belongs to the interval $\langle 0,1\rangle$ and also the Hausdorff dimension of each set of positive Lebesgue measure is equal to one, it is a "natural measure of size" of sets of Lebesgue measure zero. In fact, in the paper [M] it is proved.

Theorem A. *There exists a subset of Lebesgue measure zero which can be decomposed into continuum many pairwise disjoint subsets each of which has the Hausdorff dimension* 1 *in each open interval and one of the sets in decomposition is residual in each open interval in* $\mathbb{R}$.

Theorem B. *There exists a subset of Lebesgue measure zero which can be decomposed into continuum many pairwise disjoint subsets each of which is of the second category and the Hausdorff dimension* 1 *in each open interval.*

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