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Abstract: The main result of this paper is a criterion for irrational series which consist of rational numbers where the denominators are special numbers and numerators are not so much high. If we little increase the numerator then the example for rational series is included also.

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1. Introduction

Erdös and Straus in [3] proved that if A is an integer greater then one and $\{c_n\}_{n=1}^{\infty}$ is a sequence of integers such that $\sum_{n=1}^{\infty} |c_n| A^{-2^n} < \infty$ then the number

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{A^{2^n} + c_n}$$

is irrational.

Mahler introduced the most general method of proving the irrationality and transcendence of sums of infinite series. A survey of these results can be found in Nishioka's book [4]. In 1987 in [1] Bundschuh and Pethö proved the following theorem.

Theorem 1. *Let $\{R_n\}_{n=1}^{\infty}$ be the second order linear recurrence with characteristic polynomial $x^2 - A_1 x - 1$ where A_1 is a positive integer and with $R_0 = 0$ and $R_1 = 1$. Assume that $\{b_n\}_{n=1}^{\infty}$ is a sequence of integers such that $|b_n|$ is not a constant for large indices and there is a positive real number ϵ such that for every sufficiently large n*

$$|b_n| \leq R_{2^n-1}^{1-\epsilon}.$$

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Then the number

$$\sum_{n=1}^{\infty} \frac{b_n}{R_{2^n}}$$

is transcendental.

In 1999 in [2] Duverney proved the following theorem.

Theorem 2. *Let β be a positive rational number and γ be a real number with $0 \leq \gamma < 2$. Assume that $\{u_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of nonzero integers such that*

$$\lim_{n\to\infty} u_n = \infty,$$

$$u_{n+1} = \beta u_n^2 + O(u_n^{\gamma}),$$

$$\log |a_n| = o(2^n)$$

and

$$\log |b_n| = o(2^n).$$

Then $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n u_n}$ is a rational number if and only if there is n_0 such that for every $n > n_0$

$$u_{n+1} = \beta u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}}.$$

The main results of this paper are Theorems 3 and 4. Theorem 3 deals with a criterion for the irrationality of rapidly convergent series. The terms of this series consist of special rational numbers which do not depend on arithmetical properties like divisibility. Theorem 4 presents several special rational numbers.

2. Irrational series

Theorem 3. *Let A and B be two algebraic numbers with $1 \leq |B| < |A|$ and such that A^2 and B^2 are positive integers. Assume that $\{b_n\}_{n=1}^{\infty}$ is a sequence of integers such that*

$$\liminf_{n\to\infty} \frac{b_{n+1} - 2B^{2^n} b_n}{A^{2^n}} = 0 \tag{1}$$

and for every sufficiently large n

$$|b_{n+1} - 2B^{2^n} b_n| < A^{2^n} + B^{2^n}. \tag{2}$$

Then the number

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{A^{2^n} + B^{2^n}}$$

is rational iff there exists a rational number D such that for every sufficiently large number n

$$b_n = D 2^n B^{2^n}. \tag{3}$$

Proof 1. Sufficient condition. Let (3) holds for every n with $n > n_0$. Then we have

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{A^{2^n} + B^{2^n}} = \sum_{n=1}^{\infty} \frac{b_n(A^{2^n} - B^{2^n})}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\sum_{n=1}^{\infty} \frac{b_n(A^{2^n} + B^{2^n} - 2B^{2^n})}{A^{2^{n+1}} - B^{2^{n+1}}} = \sum_{n=1}^{\infty}\left(\frac{b_n}{A^{2^n} - B^{2^n}} - \frac{2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}}\right) =$$

$$\frac{b_1}{A^2 - B^2} + \sum_{n=1}^{\infty} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\frac{b_1}{A^2 - B^2} + \sum_{n=1}^{n_0} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} + \sum_{n=n_0+1}^{\infty} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\frac{b_1}{A^2 - B^2} + \sum_{n=1}^{n_0} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} + \sum_{n=n_0}^{\infty} \frac{D2^{n+1}B^{2^{n+1}} - 2D2^nB^{2^n}B^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\frac{b_1}{A^2 - B^2} + \sum_{n=1}^{n_0} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}}.$$

Thus the number α is rational.

2. Necessary condition. Assume that for every rational number D there exist infinitely many n such that $b_n \neq D2^nB^{2^n}$. From this we obtain that for infinitely many N

$$b_{N+1} - 2b_NB^{2^N} \neq 0. \tag{4}$$

Let α be a rational number. Then there exist two integers p and q with $q > 0$ such that $\alpha = \frac{p}{q}$. Now we have

$$\alpha = \frac{p}{q} = \sum_{n=1}^{\infty} \frac{b_n}{A^{2^n} + B^{2^n}} = \sum_{n=1}^{\infty} \frac{b_n\prod_{j=1}^{n-1}(A^{2^j} + B^{2^j})}{\prod_{j=1}^{n}(A^{2^j} + B^{2^j})} =$$

$$\frac{1}{A^2 - B^2}\sum_{n=1}^{\infty} \frac{b_n(A^{2^n} - B^{2^n})}{\prod_{j=1}^{n}(A^{2^j} + B^{2^j})} =$$

$$\frac{1}{A^2 - B^2}\sum_{n=1}^{\infty}\left(\frac{b_n}{\prod_{j=1}^{n-1}(A^{2^j} + B^{2^j})} - \frac{2B^{2^n}b_n}{\prod_{j=1}^{n}(A^{2^j} + B^{2^j})}\right) =$$

$$\frac{b_1}{A^2 - B^2} + \frac{1}{A^2 - B^2}\sum_{n=1}^{\infty} \frac{b_{n+1} - 2B^{2^n}b_n}{\prod_{j=1}^{n}(A^{2^j} + B^{2^j})}.$$

From this we obtain that for every positive integer M the number

$$I_M = \left(\prod_{j=1}^{M}(A^{2^j} + B^{2^j})\right)\left(p(A^2 - B^2) - qb_1 - q\sum_{n=1}^{M} \frac{b_{n+1} - 2B^{2^n}b_n}{\prod_{j=1}^{n}(A^{2^j} + B^{2^j})}\right) =$$

$$q \sum_{n=M+1}^{\infty} \frac{b_{n+1} - 2B^{2^n} b_n}{\prod_{j=M+1}^{n} (A^{2^j} + B^{2^j})}. \tag{5}$$

is an integer. Equation (1) implies that there is a positive integer S such that

$$\frac{|b_{S+1} - 2B^{2^S} b_S| + 1}{A^{2^S} + B^{2^S}} < \frac{1}{q}. \tag{6}$$

From (2), (4), (5) and (6) we obtain that

$$|I_{S-1}| = q| \sum_{n=S}^{\infty} \frac{b_{n+1} - 2B^{2^n} b_n}{\prod_{j=S}^{n} (A^{2^j} + B^{2^j})}| \leq$$

$$q \frac{|b_{S+1} - 2B^{2^n} b_n|}{A^{2^S} + B^{2^S}} + q \sum_{n=S+1}^{\infty} \frac{|b_{n+1} - 2B^{2^n}|}{\prod_{j=S}^{n} (A^{2^j} + B^{2^j})} <$$

$$q \frac{|b_{S+1} - 2B^{2^S}|}{A^{2^S} + B^{2^S}} + q \sum_{n=S+1}^{\infty} \frac{A^{2^n} + B^{2^n} - 1}{\prod_{j=S+1}^{n} (A^{2^n} + B^{2^n})} \leq$$

$$q \frac{|b_{S+1} - 2B^{2^S}| + 1}{A^{2^S} + B^{2^S}} < 1. \tag{7}$$

Condition (4) implies that there is a least integer P with $P \geq S$ such that

$$b_{P+1} - 2B^{2^P} b_P \neq 0.$$

From this, (2), (4) and (5) we obtain that

$$|I_{S-1}| = q| \sum_{n=S}^{\infty} \frac{b_{n+1} - 2B^{2^n} b_n}{\prod_{j=S}^{n} (A^{2^j} + B^{2^j})}| = q| \sum_{n=P}^{\infty} \frac{b_{n+1} - 2B^{2^n} b_n}{\prod_{j=S}^{n} (A^{2^j} + B^{2^j})}| \geq$$

$$\frac{q|b_{P+1} - 2B^{2^P} b_P|}{\prod_{j=S}^{P} (A^{2^j} + B^{2^j})} - q \sum_{n=P+1}^{\infty} \frac{|b_{n+1} - 2B^{2^n} b_n|}{\prod_{j=S}^{n} (A^{2^j} + B^{2^j})} >$$

$$\frac{q|b_{P+1} - 2B^{2^P} b_P|}{\prod_{j=S}^{P} (A^{2^j} + B^{2^j})} - q \sum_{n=P+1}^{\infty} \frac{A^{2^n} + B^{2^n} - 1}{\prod_{j=S}^{n} (A^{2^j} + B^{2^j})} =$$

$$\frac{q(|b_{P+1} - 2B^{2^P} b_P| - 1)}{\prod_{j=S}^{n} (A^{2^J} + B^{2^J})} \geq 0. \tag{8}$$

Inequalities (7) and (8) contradicting the fact that the number I_{S-1} is integer. ◇

Example 1. *Let A and B be two positive integers with $B < A$ and $\pi(n)$ be the number of primes less then or equal to n. As an immediate consequence of Theorem 3 the numbers*

$$\sum_{n=1}^{\infty} \frac{A^{2^n - \pi(n)} + A^{2^{\pi(n)}} + n^n}{A^{2^{n+1}} + B^{2^{n+1}}} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{A^{2^n - \pi(n)} + A^{2^{\pi(n)}} + B^{2^{\pi(n)}}}{A^{2^{n+1}} + B^{2^{n+1}}}$$

are irrational.

3. Rational series

Theorem 4. *Let A and B be two algebraic numbers with $0 \leq |B| < |A|$ such that A^2 and B^2 are integers and n_0 be a positive integer with $n_0 > 1$. Assume that $\{b_n\}_{n=1}^{\infty}$ is a sequence of integers such that for every $n \geq n_0$*

$$b_{n+1} = \sum_{j=n_0}^{n} (A^{2^j} + B^{2^j} - 1)2^{n-j}B^{2^{n+1}-2^{j+1}} + b_{n_0}2^{n-n_0+1}B^{2^{n+1}-2^{n_0}}. \tag{9}$$

Then the number

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{A^{2^n} + B^{2^n}}$$

is rational.

Proof From (9) we obtain that for every positive integer n with $n > n_0$

$$b_{n+1} - 2B^{2^n}b_n = \sum_{j=n_0}^{n} (A^{2^j} + B^{2^j} - 1)2^{n-j}B^{2^{n+1}-2^{j+1}} + b_{n_0}2^{n-n_0+1}B^{2^{n+1}-2^{n_0}} -$$

$$\sum_{j=n_0}^{n-1} (A^{2^j} + B^{2^j} - 1)2^{n-j}B^{2^{n+1}-2^{j+1}} - b_0 2^{n-n_0+1}B^{2^{n+1}-2^{n_0}} =$$

$$A^{2^n} + B^{2^n} - 1.$$

This and the definition of the number α imply

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{A^{2^n} + B^{2^n}} = \sum_{n=1}^{\infty} \frac{b_n(A^{2^n} - B^{2^n})}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\sum_{n=1}^{\infty} \frac{b_n(A^{2^n} + B^{2^n} - 2B^{2^n})}{A^{2^{n+1}} - B^{2^{n+1}}} = \sum_{n=1}^{\infty} \frac{b_n}{A^{2^n} - B^{2^n}} - \frac{2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\frac{b_1}{A^2 - B^2} + \sum_{n=1}^{\infty} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}} =$$

$$\frac{b_1}{A^2 - B^2} + \sum_{n=1}^{n_0} \frac{b_{n+1} - 2b_nB^{2^n}}{A^{2^{n+1}} - B^{2^{n+1}}}.$$

Thus the number α is rational. ◇

Example 2. *As an immediate consequence of Theorem 4 we obtain that the number*

$$\sum_{n=1}^{\infty} \frac{\sum_{j=1}^{n} 5^{2^j}2^{n-j}}{5^{2^{n+1}} + 1}$$

is rational.

Open problem 1. *Let $d(n)$ be the number of divisors of the number n. Is the number*

$$\sum_{n=1}^{\infty} \frac{5^{2^{n-1}+d(n)} + 3^{2^{n-1}+d(n)}}{5^{2^n} + 3^{2^n}}$$

irrational?

References

[1] Bundschuh P., Pethö A.: Zur transzendenz gewisser Reihen, Monatsh. Math. 104, (1987), 199–223.

[2] Duverney D.: Sur les series de nombres rationnels a convergence rapide, C. R. Acad. Sci., ser.I, Math. 328, no.7, (1999), 553–556.

[3] Erdös P., Straus E.G.: On the irrationality of certain Ahmes series, J. Indian. Math. Soc. 27, (1968), 129–133.

[4] Nishioka K.: Mahler Functions and Transcendence, Lecture notes in mathematics 1631, Springer, (1996).

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