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A remark on the ideal extension property

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Abstract: In [1], the authors proved that for algebras of subtractive varieties, the principal ideal extension property implies the ideal extension property. We give another proof which avoids the use of Zorn Lemma.

Key Words: subtractive variety, ideal extension property

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In a series of papers [4, 1, 2] the authors study subtractive varieties of universal algebras, mainly from the point of view of ideal theory. Let $\mathcal{V}$ denote a subtractive variety, i.e. there is a binary term s such that $s(x,x) = 0$ and $s(x,0) = 0$ hold in $\mathcal{V}$ where 0 is an equationally defined constatnt of $\mathcal{V}$. A subset I of A where A is a support of some $A \in \mathcal{V}$ is called an ideal if for every ideal term $p(x_1,\dots,x_m,y_1,\dots,y_n)$ of $\mathcal{V}$ and for every $a_1,\dots,a_m \in A$, $i_1,\dots,i_n \in I$ it holds that $p(a_1,\dots,a_m,i_1,\dots,i_n) \in I$. A term p is called an ideal term of $\mathcal{V}$ if $p(x_1,\dots,x_m,0,\dots,0) = 0$ holds in $\mathcal{V}$. The set of all ideals of A is denoted by $I(A)$. For $X \subseteq A$, $(X)^A$ denotes the least ideal of $I(A)$ containing X. An ideal I is called principal if $I = (i)^A$ for some $i \in A$. An algebra $A \in \mathcal{V}$ satisfies the ideal extension property (IEP) if for each subalgebra B of A and each $I \in I(B)$ there is $J \in I(A)$ such that $J \cap B = I$. A satisfies the principal IEP if for each subalgebra B of A and any $b \in B$, $(b)^A \cap B = (b)^B$. For $\theta \subseteq A \times A$ and $X \subseteq A$ put $(X)_\theta = \{a \in A \mid \langle x,a\rangle \in \theta \text{ for some } x \in X\}$. The following proposition has been proved in [1] using Zorn Lemma.

Proposition. *Let $A \in \mathcal{V}$, where $\mathcal{V}$ is subtractive. If A has the principal IEP, then A has the IEP.*

We give simple proof of Proposition without the use of Zorn Lemma.

Proof. Let B be a subalgebra of A, $I \in I(B)$. Obviously, we have to show $(I)^A \cap B \subseteq I$. $a \in (I)^A$ holds iff $a \in (i_1)^A \vee \cdots \vee (i_n)^A$ (since $()^A$ is an algebraic closure system [4, p. 205]) which is by [1, 1.14] equivalent to $s_n(a,c_n,\dots,c_2) \in (i_1)^A$ for some $c_k \in (i_k)^A$, $k = 2,\dots,n$ (where s_n is defined inductively by $s_1(x) = x$ and

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$s_{n+1}(x, y_1, \ldots, y_n) = s(s_n(x, y_2, \ldots, y_n), y_1))$. We prove by induction over n the following claim: for each $C \in Sub\,A$ such that $B \subseteq C$, and $a \in C$, $i_1, \ldots, i_n \in I \in I(C)$, if there are $c_k \in (i_k)^A$, $k = 2, \ldots, n$, such that $s_n(a, c_2, \ldots, c_n) \in (i_1)^A$, then $a \in I$. For $C = B$ the claim obivously yields the required inclusion $(I)^A \cap B \subseteq I$. The claim follows directly by the principal IEP for $n = 1$. Suppose the claim holds for $n - 1$. Take $D = (C)_{\theta_{(i_n)^A}} \in Sub\,A$ where $\theta_{(i_n)^A}$ is some congruence [4, Propositon 1.4] with the class $(i_1)^A$. Since, by the principal IEP, $(i_n)^A \cap C = (i_n)^C \subseteq I$, it follows from [1, Lemma] that $J = (I)_{\theta_{(i_n)^A}}$ is an ideal in D such that $J \cap C = I$. Clearly, $s(a, c_n) \in D$, $i_1, \ldots, i_{n-1} \in J \in I(D)$, $c_2, \ldots, c_{n-1}$, satisfy the assumptions for $n - 1$, hence $s(a, c_n) \in J$. Morevover, by $a \in D$, $c_n \in J \in I(D)$, we get $a \in J$. Hence $a \in J \cap C = I$, completing the proof. ◇

Remark. [3] shows a general result: for quasivarieties, so-called principal relative congruence extension property implies relative congruence extension property (with the notions appropriately defined). The proof does not use Zorn lemma. For varieties, their result gives a proof of the well-known Day's result avoiding the use of Zorn lemma. Our result concerns extension of ideals, not congruences, and is valid for subtractive varieties. Note that subtractivity is havily used in the proof.

References

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