Acta Mathematica et Informatica Universitatis Ostraviensis 9 (2001) 75-87

Some remarks on iterative methods for systems of linear equations

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Abstract: The aim of this paper is to study the periodicity in iterative methods for solving systems of linear equations. The problem of the periodicity in an iterative process for a nonlinear equation has been investigated for example in [6], [9] and many others.

We briefly describe the behaviour of an iterative sequence from the point of view of convergence and explore in detail the periodicity in this sequence. Necessary and sufficient conditions for occurrence of the periodicity will be given. The periodic sequences generated by an iterative process possess an interesting geometric property. This geometric representation will be also given.

Key Words: A convergent matrix, a cycle, a diagonable matrix, an ellipsoid

Mathematics Subject Classification: 65F10

Consider a system of linear equations of the form

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where A is an $n \times n$ real matrix and $\boldsymbol{b}$ is a real column vector of order n. We shall assume that A is nonsingular, so that (1) has the unique solution

$$\widehat{\boldsymbol{x}} = A^{-1}\boldsymbol{b}. \tag{2}$$

We will be dealing with iterative methods to approximate $\widehat{\boldsymbol{x}}$. Consider the system

$$\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{g} \tag{3}$$

which is equivalent to the system (1) and T is an $n \times n$ real matrix, $\boldsymbol{g}$ is a real n vector, i.e., $\widehat{\boldsymbol{x}} = T\widehat{\boldsymbol{x}} + \boldsymbol{g}$, $\widehat{\boldsymbol{x}} = (E-T)^{-1}\boldsymbol{g}$, under assumption $(E-T)^{-1}$ exists.

Let $\boldsymbol{x}^0$ be an initial approximation and let a sequence $\{\boldsymbol{x}^k\}, k = 1, 2, \dots,$ be generated by the formula

$$\boldsymbol{x}^{k+1} = T\boldsymbol{x}^k + \boldsymbol{g}; \tag{4}$$

the matrix T is often called an *iterative matrix*. Let us summarize well-known results. The main convergence theorem reads as follows:

Theorem. *The iterative method* (4) *converges to the solution* $\widehat{\boldsymbol{x}} = A^{-1}\boldsymbol{b}$ *of* (1) *for each* $\boldsymbol{x}^0$ *if and only if* $\rho(T) < 1$ *(*$\rho(T)$ *denotes a spectral radius of* T*).*

The proof can be find in any basic book on numerical analysis. It is clear that the iterative sequence is divergent whenever $\rho(T) > 1$.

Now, we focus our attention to the case $\rho(T) = 1$ and we will give a geometric representation of the corresponding iterative sequence. Consider the system (1) and let (4) be an associated iterative process.

Definition. *Let* $\{\boldsymbol{x}^k\}_{k=0}^{\infty}$ *be a sequence generated by* (4) *with an initial approximation* $\boldsymbol{x}^0 \neq \widehat{\boldsymbol{x}}$. *The vector* $\boldsymbol{x}^0$ *is said to generate a cycle of order* p, $p \in \mathbb{N}$, $p \geqslant 2$, *if* $\boldsymbol{x}^p = \boldsymbol{x}^0$ *provided that* $\boldsymbol{x}^k \neq \boldsymbol{x}^0$, $k = 1, 2, \dots, p-1$ *(i.e., p is the least number satisfying this relation).*

Theorem 1. *Each initial approximation* $\boldsymbol{x}^0 \neq \widehat{\boldsymbol{x}}$ *generates a cycle of order* p *if and only if*

$$T^p = E \tag{5}$$

where E *is the* $n \times n$ *identity matrix.*

Proof. Let $\boldsymbol{x}^0$ be an initial approximation. The application of the formula (4) gives

$$\boldsymbol{x}^k = T^k\boldsymbol{x}^0 + (T^{k-1} + \ldots + T + E)\,\boldsymbol{g}.$$

The systems $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{g}$ are equivalent so that $(T - E)\,\widehat{\boldsymbol{x}} = -\boldsymbol{g}$ and hence for $k = p$

$$\boldsymbol{x}^p = T^p\boldsymbol{x}^0 - (T^{p-1} + \ldots + T + E)\,(T - E)\,\widehat{\boldsymbol{x}},$$

which gives

$$\boldsymbol{x}^p = T^p\boldsymbol{x}^0 - (T^p - E)\,\widehat{\boldsymbol{x}} \tag{6}$$

I. Let (5) hold. Then (6) implies $\boldsymbol{x}^p = T^p\boldsymbol{x}^0 \;\Rightarrow\; \boldsymbol{x}^p = \boldsymbol{x}^0$ and $\boldsymbol{x}^k \neq \boldsymbol{x}^0$ for $1 \leqslant k \leqslant p-1$.

II. Let an arbitrary initial approximation generate a cycle of order p. From (6) it follows

$$(T^p - E)(\boldsymbol{x}^0 - \widehat{\boldsymbol{x}}) = \boldsymbol{o}. \tag{7}$$

Now the initial approximation $\boldsymbol{x}^0$ can be chosen in such a way that $\boldsymbol{x}^0 = \widehat{\boldsymbol{x}} + \boldsymbol{x}$, where $\boldsymbol{x}$ is an arbitrary vector. It means that

$$(T^p - E)\,\boldsymbol{x} = \boldsymbol{o}$$

for any vector $\boldsymbol{x}$, i.e., the matrix $T^p - E$ has to be a null matrix. □

Diagonable matrices will play an important role in our further considerations. Now, we introduce the definition and some properties.

Definition. *A square matrix A of order n is called diagonable, if there exist a nonsingular matrix $\mathcal{U}$ and a diagonal matrix J of order n such that*

$$A = \mathcal{U} J \mathcal{U}^{-1}, \tag{8}$$

$J = diag(\lambda_1, \ldots, \lambda_n)$ and λ_i are eigenvalues of A.

Some other characterizations of such matrices are known.

Theorem 2([7]). *Let A be a square matrix of order n. The following statements are equivalent:*

(i) *A is diagonable.*
(ii) *All Jordan blocks are matrices of order 1.*
(iii) *Rank of $A - \lambda E$ equals $n - m$ for each eigenvalue λ of A with multiplicity m.*
(iv) *There exists a system of n linearly independent vectors each of which is an eigenvector of A.*

In terms of diagonable matrices the main theorem reads as follows:

Theorem 3. *The identity $T^p = E$ holds if and only if T is diagonable and all eigenvalues λ_i, $i = 1, \ldots, n$, of T satisfy*

$$\lambda_i^p = 1$$

(p is the least natural number satisfying these conditions).

Proof. Consider the Jordan canonical form of T: $T = \mathcal{U} J \mathcal{U}^{-1}$. Then $T^p = \mathcal{U} J^p \mathcal{U}^{-1}$ and $T^p = E$ if and only if $J^p = E$ and thus this identity can be only investigated.

I. Let T be diagonable and $\lambda_i^p = 1$, $i = 1, \ldots, n$, for a natural p. According to the theorem 2 Jordan blocks are matrices of order 1, i.e.,

$$J = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & .. & \lambda_n \end{pmatrix}.$$

Hence

$$J^p = diag(\lambda_1^p, \lambda_2^p, \ldots, \lambda_n^p) = E$$

for $\lambda_i^p = 1$, $i = 1, \ldots, n$ by our assumption. This implies $T^p = \mathcal{U} J^p \mathcal{U}^{-1} = E$.

II. Now let $T^p = E$ and let the Jordan canonical matrix J

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & 0 & \ldots & 0 \\ 0 & J_{n_2}(\lambda_2) & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & J_{n_s}(\lambda_s) \end{pmatrix},$$

where

$$J_{n_j}(\lambda_j) = \begin{pmatrix} \lambda_j & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_j & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda_j & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda_j \end{pmatrix},$$

$\sum_{j=1}^{s} n_j = n,\ n_j \geqslant 1$ for $j = 1, \dots, s$. Then

$$J^p = \begin{pmatrix} J^p_{n_1}(\lambda_1) & 0 & 0 & \dots & 0 \\ 0 & J^p_{n_2}(\lambda_2) & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & J^p_{n_s}(\lambda_s) \end{pmatrix}.$$

Put $J_{n_j}(\lambda_j) = N_{n_j} + \lambda_j E_{n_j}$, where E_{n_j} is the $n_j \times n_j$ identity matrix and $N_{n_j} = J_{n_j}(0)$, $j = 1, \dots, s$. Then

$$J^p_{n_j}(\lambda_j) = \sum_{i=0}^{p} \binom{p}{i} \lambda_j^{p-i} N^i_{n_j},$$

and all terms with $i \geqslant n_j$ are null matrices because $N^i_{n_j} = O$ for $i \geqslant n_j$. Applying this expression to the p-th power of $J_{n_j}(\lambda_j)$ we obtain

$$J^p_{n_j}(\lambda_j) = \begin{pmatrix} \binom{p}{0}\lambda_j^p & 0 & \dots & 0 & 0 \\ \binom{p}{1}\lambda_j^{p-1} & \binom{p}{0}\lambda_j^p & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \binom{p}{n_j-2}\lambda_j^{p-n_j+2} & \binom{p}{n_j-3}\lambda_j^{p-n_j+3} & \dots & \binom{p}{0}\lambda_j^p & 0 \\ \binom{p}{n_j-1}\lambda_j^{p-n_j+1} & \binom{p}{n_j-2}\lambda_j^{p-n_j+2} & \dots & \binom{p}{1}\lambda_j^{p-1} & \binom{p}{0}\lambda_j^p \end{pmatrix}.$$

According to our assumption $T^p = E$ which is equivalent to $J^p = E$ and thus

$$J^p_{n_j}(\lambda_j) = E_{n_j},\ j = 1, 2, \dots, s.$$

Whenever $n_j > 1$ for a block $J_{n_j}(\lambda_j)$ it would imply

$$\lambda_j^p = 1 \text{ and } \lambda_j^{p-1} = \lambda_j^{p-2} = \dots = \lambda_j^{p-n_j+1} = 0,$$

which leads to the contradiction. This concludes the proof, i.e., T is diagonable and $\lambda_i^p = 1,\ i = 1, 2, \dots, n.$ $\square$

We are now in a position to give necessary and sufficient conditions for existence of a cycle in an iterative method. The following theorem summarizes the preceding results.

Theorem 4. *Let $\{x^k\}_0^\infty$ be a sequence generated by an iterative process $x^{k+1} = Tx^k + g$, $k = 0, 1, \dots$. The cycle of order p is generated by any initial approximation if and only if*

(i) T is diagonable

(ii) for each eigenvalue λ_j of T there exists a natural number p_j such that

$$\lambda_j^{p_j} = 1.$$

The order of the cycle is defined as the least common multiple of all p_j's.

Remark. Here p_j is the least exponent satisfying the given equality.

The existence of a cycle of the Jacobi iterative method is analysed in the following example.

Example. Consider the system

$$\begin{aligned} x_1 - x_2 - x_3 &= -4, \\ x_1 + x_2 + x_3 &= 6, \\ x_1 - x_3 &= -1. \end{aligned}$$

The true solution $\hat{x} = (1, 3, 2)^T$. The Jacobi iterative process takes the form

$$x^{k+1} = Tx^k + g,$$

where

$$T = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} -4 \\ 6 \\ 1 \end{pmatrix}.$$

Let $x^0 = (2, 4, 3)^T$. Simple calculation gives

$$x^1 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \quad \dots, \quad x^6 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

There is a hypothesis: Does any initial approximation generate a cycle of order 6? Let us check up whether the assumption of the theorem 4 are satisfied. The characteristic equation of T takes the form

$$\lambda^3 + 1 = 0,$$

i.e., the eigenvalues of T are $\lambda_1 = -1$, $\lambda_{2,3} = e^{\pm i \frac{\pi}{3}}$. The Jordan canonical matrix

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{i\frac{\pi}{3}} & 0 \\ 0 & 0 & e^{-i\frac{\pi}{3}} \end{pmatrix}$$

is a diagonal matrix. Further $\lambda_1^2 = 1$, $\lambda_{2,3}^6 = 1$. The least common multiple of $p_1 = 2$ and $p_{2,3} = 6$ is $p = 6$. According to the theorem 4 there is a cycle of order 6 for any initial approximation. Our hypothesis was correct.

Remark. With respect to the fact that $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{g}$ is equivalent to the system $A\boldsymbol{x} = \boldsymbol{b}$ with A nonsingular, it is evident $\lambda = 1$ cannot be an eigenvalue of T.

Corollary 1. *Let the iterative formula $\boldsymbol{x}^{k+1} = T\boldsymbol{x}^k + \boldsymbol{g}$, $k = 1, 2, \dots$, generate a cycle for any initial approximation $\boldsymbol{x}^0$. Then $\rho(T) = 1$.*

Remark. The fact $\rho(T) = 1$ does not mean that there exists the cycle in the given iterative process. This condition does not guarantee a diagonality of the Jordan canonical form. This fact is illustrated by the following counter example.

Example. Consider the system

$$\begin{aligned} x_1 + 2x_2 &= 3, \\ 2x_1 + 3x_2 &= 5 \end{aligned}$$

and let the equivalent system take the form

$$\begin{aligned} x_1 &= 3x_1 + 4x_2 - 6, \\ x_2 &= -4x_1 - 5x_2 + 10. \end{aligned}$$

The eigenvalues of T are $\lambda_{1,2} = -1$, which means $\rho(T) = 1$. But the Jordan canonical matrix

$$J = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

is not a diagonal matrix.

Remark. Let T be the Gauss-Seidel iterative matrix. It is clear this matrix is singular (all entries of the first column are zeros). Then there is the eigenvalue $\lambda = 0$ and any cycle cannot occur for an arbitrary initial approximation. But, for example, for a matrix of order 2 a cycle of order 2 can appear but with the initial approximation $T\boldsymbol{x}^0$ instead of $\boldsymbol{x}^0$.

Now we will draw our attention to a geometric representation of cycles. In order to describe the behaviour of the periodic sequence let us extend our considerations to the vector space V_n^C of complex n vectors. As usual a vector $\boldsymbol{x} \in V_n^C$ can be expressed by means of the natural basis $\mathfrak{B}_0$. This basis is formed by vectors $\boldsymbol{e}_1 = (1, 0, \dots, 0)^T, \dots, \boldsymbol{e}_n = (0, \dots, 0, 1)^T$ and then

$$x = x_1 e_1 + \cdots + x_n e_n,$$

where $x_i \in \mathbb{C}$, $i = 1, \ldots, n$, are coordinates of x with respect to $\mathfrak{B}_0 = \{e_1, \cdots, e_n\}$. It is evident that in our preceding considerations these representations of real vectors , $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, have been used. A scalar product of vectors x , $y \in \mathsf{V}_n^C$ is defined as usual by

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y}_i = y^* x \,, \; y^* = \overline{y}^T$$

and the corresponding norm

$$\| x \| = \sqrt{(x, x)}.$$

Let us assume the existence of a cycle of order p for the iterative method $x^{k+1} = T x^k + g$. Thus T is diagonable and according to the theorem 2 there exists a system of n linearly independent vectors each of which is an eigenvector of T. This system forms the so-called a canonical basis in V_n^C and is denoted by $\mathfrak{B}_v$. If $T = \mathcal{U} J \mathcal{U}^{-1}$ then the columns of $\mathcal{U}$ represent the $\mathfrak{B}_v$ basis. Let $\mathfrak{B}_v = \{v_1, \ldots, v_n\}$ and $x_{\mathfrak{B}_v}$ denote the vector expressed with respect to the $\mathfrak{B}_v$ basis.

Remark. For the sake of simplicity we omit the index $\mathfrak{B}_0$ denoting a vector expressed with respect to the $\mathfrak{B}_0$ basis.

Let us recall an important result from linear algebra ([2]): *If the matrix A defines a linear map φ with respect to the $\mathfrak{B}_A$ basis, then an arbitrary matrix C similar to A, $C = Q A Q^{-1}$, defines the same linear map in the $\mathfrak{B}_C$ basis. The matrix Q is called a transition matrix between the $\mathfrak{B}_A$ basis and the $\mathfrak{B}_C$ basis.*

Let us come back to our problem. If $T = \mathcal{U} J \mathcal{U}^{-1}$ then $\mathcal{U}$ is the transition matrix between the $\mathfrak{B}_0$ basis and the $\mathfrak{B}_v$ basis. This means

$$x = \mathcal{U} x_{\mathfrak{B}_v}. \tag{9}$$

The system (3) can be rewritten in the form

$$x - \widehat{x} = T(x - \widehat{x}) \tag{10}$$

and then the corresponding iterative process takes the form

$$x^{k+1} - \widehat{x} = T(x^k - \widehat{x}). \tag{11}$$

With respect to (9) the iterative process (11) can be written as

$$\mathcal{U}(x_{\mathfrak{B}_v}^{k+1} - \widehat{x}_{\mathfrak{B}_v}) = T\mathcal{U}(x_{\mathfrak{B}_v}^{k} - \widehat{x}_{\mathfrak{B}_v}) \tag{12}$$

and hence

$$x_{\mathfrak{B}_v}^{k+1} - \widehat{x}_{\mathfrak{B}_v} = J(x_{\mathfrak{B}_v}^{k} - \widehat{x}_{\mathfrak{B}_v})$$

or

$$x_{\mathfrak{B}_v}^{k} - \widehat{x}_{\mathfrak{B}_v} = J^k(x_{\mathfrak{B}_v}^{0} - \widehat{x}_{\mathfrak{B}_v}). \tag{13}$$

Lemma. *Let an iterative process* $x^{k+1} = Tx^k + g$, $k = 0, 1, \ldots,$ *be given and the assumptions of the theorem 4 are satisfied. Let* $x^0 \in V_n^C$ *generate a cycle of order* p. *Then all iterates* $\{x^k_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v}\}$, $0 \leqslant k \leqslant p$, *have a constant norm equal to* $\| x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v} \|$.

Proof. The straightforward calculation gives

$$\| x^k_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v} \|^2 = (x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v})^* (J^k)^* J^k (x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v}) =$$

$$= (x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v})^* \mathrm{diag}\{|\lambda_1|^k, \ldots, |\lambda_n|^k\} (x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v}) =$$

$$= \| x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}} \|^2, \; k = 0, 1, \ldots, p.$$

for $|\lambda_i| = 1$, $i = 1, \ldots, n$. $\square$

Now we are able to describe the geometric properties of vectors x^k, $k = 0, \ldots, p$. With respect to (9) the identity

$$\| x^k_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v} \| = \| x^0_{\mathfrak{B}_v} - \widehat{x}_{\mathfrak{B}_v} \|, \; k = 0, \ldots, p$$

means that

$$\| \mathcal{U}^{-1}(x^k - \widehat{x}) \| = \| \mathcal{U}^{-1}(x^0 - \widehat{x}) \|.$$

Now let us investigate the set of vectors $x \in V_n^C$ satisfying

$$\| \mathcal{U}^{-1}(x - \widehat{x}) \| = \alpha, \; \alpha = \| \mathcal{U}^{-1}(x^0 - \widehat{x}) \|. \tag{14}$$

Putting $\mathcal{U}^{-1} = V, V^* = \overline{V}^T$ and taking the definition of the vector norm into account we obtain from (14)

$$(x - \widehat{x})^* \; V^* V (x - \widehat{x}) = \alpha^2. \tag{15}$$

The matrix W is a positive definite matrix and the left side of (15) is a positive definite quadratic form. The sequence $\{x^k - \widehat{x}\}_{k=0}^p$ is generated by a real initial approximation x^0 and all vectors $\{x^k - \widehat{x}\}_{k=0}^p$ are real vectors. Then it is sufficient to investigate (15) only for real vectors $x - \widehat{x}$. Hence

$$(x - \widehat{x})^T \; W (x - \widehat{x}) = \alpha^2, \; \alpha^2 = (x^0 - \widehat{x})^T \; W (x^0 - \widehat{x}). \tag{16}$$

With respect to the fact W is the positive definite matrix the equality (16) can be considered as the equation of the n-dimensional ellipsoid with the center at $\widehat{x}$.

The above considerations and results can be summarized in the theorem

Theorem 5. *Let the iterative process* $x^{k+1} — Tx^k$ -f $g_{}$ $k = 0, 1, \ldots$ *satisfy the assumption of the theorem 4- Then all iterates* $\{x^k\}^{\wedge}_{—0}$ *belonging to the cycle of order p lie on the ellipsoid surface* (16).

Remarh. If a systém Ax — o is given, then the ellipsoid (16) is a centrál ellipsoid.

Detailed geometrie considerations from the point of view of the analytic geometry are given in [10].

In conclusion we will give an example illustrating eyeles generated by a relaxation method. Consider a systém

$$Xi + x_2 = 1,$$
$$qx\backslash + x_2 = 1.$$

The relaxation method for this systém can be described by the matrix T_u and the vector b_u:

$$T_u = \begin{pmatrix} 1-\omega & -\omega \\ -q\omega(1-\omega) & q\omega^2 + (1-\omega) \end{pmatrix}, \quad b_\omega = \begin{pmatrix} \omega \\ \omega - q\omega^2 \end{pmatrix}$$

where o; is a reál parameter. The necessary condition for the convergence of the iterative process

$$x^{k+1} = T_u x^k + g» \tag{17}$$

is given by the requirement $LÚ$ € (0, 2).

For UJ — 1 the relaxation method coincides with the Gauss-Seidel iterative method.

It can be proved ([10]) that there is a cycle of order $p > 3$ generated by relaxation method if and only if

i) $u = 2$,
H) 9 6(0,1),
iii) the eigenvalues of T_u are $X\backslash^{\wedge} = e^{\pm lifi}$, $<P \sim 27T5$, $0 < s < \sim$.

Let these assumptions are satisfied and let us compute the equation of the ellipse. For $UJ = 2$ the matrix T_w takés the form

$$T_« = (t \wedge - 1)' \tag{18}$$

The eigenvalues of T_u are given by the relations

$$Ai_{f2} = -l + 2q \pm 2y/q(q-l).$$

With respect to the fact that q G (0,1) the eigenvalues $Ai^{\wedge}$ are complex conjugate numbers, Le.,

$$Ai_{,2} = -l + 2q \pm 2iy/q(l-q) \tag{19}$$

and $|\lambda_{1,2}| = 1$. The eigenvectors of T_ω are the columns of the nonsingular matrix $\mathcal{U}$ in the decomposition

$$T_\omega = \mathcal{U} J \mathcal{U}^{-1}.$$

It can be easy shown that the vectors

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ -q - i\sqrt{q(1-q)} \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ -q + i\sqrt{q(1-q)} \end{pmatrix}$$

are the eigenvectors corresponding the eigenvalues $\lambda_1 = -1 + 2q + 2i\sqrt{q(1-q)}$ and $\lambda_2 = -1 + 2q - 2i\sqrt{q(1-q)}$ respectively. Hence the matrix

$$\mathcal{U} = \begin{pmatrix} 1 & 1 \\ -q - i\sqrt{q(1-q)} & -q + i\sqrt{q(1-q)} \end{pmatrix}$$

and

$$\mathcal{U}^{-1} = -\frac{i}{2\sqrt{q(1-q)}} \begin{pmatrix} -q + i\sqrt{q(1-q)} & -1 \\ -q + i\sqrt{q(1-q)} & 1 \end{pmatrix}.$$

If $\mathcal{U}^{-1} = V$, $V^* = \overline{V}^{-T}$ then $W = V^* V$ takes the form

$$W = \begin{pmatrix} 2q & 2q \\ 2q & 2 \end{pmatrix}.$$

Now the equation of the ellipse is given by

$$(\boldsymbol{x} - \widehat{\boldsymbol{x}})^T W (\boldsymbol{x} - \widehat{\boldsymbol{x}}) = (\boldsymbol{x}^0 - \widehat{\boldsymbol{x}})^T W (\boldsymbol{x}^0 - \widehat{\boldsymbol{x}}).$$

In our case the true solution $\widehat{\boldsymbol{x}} = (0,1)^T$ and then

$$(\boldsymbol{x} - \widehat{\boldsymbol{x}})^T \; W (\boldsymbol{x} - \widehat{\boldsymbol{x}}) = \sum_{i,j=1}^{2} w_{i,j}(x_i - \widehat{x}_i)(x_j - \widehat{x}_j) =$$

$$= 2qx_1^2 + 2x_2^2 + 4qx_1x_2 - 4x_2 - 4qx_1 - 2.$$

Thus the equation of the ellipse is of the form

$$2qx_1^2 + 2x_2^2 + 4qx_1x_2 - 4x_2 - 4qx_1 - 2 = 2q(x_1^0)^2 + 2(x_2^0)^2 + 4qx_1^0x_2^0 - 4x_2^0 - 4qx_1^0 - 2$$

and hence

$$x_1^2 + \frac{x_2^2}{q} + 2x_1x_2 - \frac{2x_2}{q} - 2x_1 = (x_1^0)^2 + \frac{(x_2^0)^2}{q} + 2x_1^0x_2^0 - \frac{2x_2^0}{q} - 2x_1^0. \qquad (20)$$

Remark. The eigenvalues of T_ω given by the relation (18) can be also expressed in the form

$$\lambda_{1,2} = \cos\varphi \pm i\sin\varphi.$$

The comparision with (19) yields

$$q = \frac{1}{2}(1+\cos\varphi). \qquad (21)$$

It is clear that the cycle of order p exists if and only if $\varphi = 2\pi l/p$, $0 < l < p/2$. Now, the relation (21) enables us to find a system generating the cycle of the given order $p > 2$ by a suitable choice p, l.

Let $\omega = 2$ and $x_0 = (0,0)^T$, then

- for $q = 0.5(1+\cos 2\pi\frac{4}{11}) \doteq 0.17257$ there exists the cycle of order 11,
- for $q = 0.5(1+\cos 2\pi\frac{19}{40}) \doteq 0.578217$ there exists the cycle of order 40.

These cycles are illustrated in Fig.1.

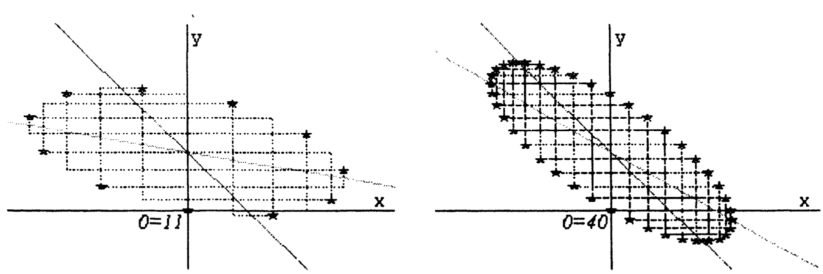

FIG.1. The cycles of orders 11, 40.

For various choice of an initial approximation a set of concentric ellipses is obtained (see Fig.2, where $q = 0.5(1+\cos 2\pi\frac{4}{11}) \doteq 0.17257$ and initial approximation are $(0,0), (3,3), (4.5,4.5), (6.5,6.5)$).

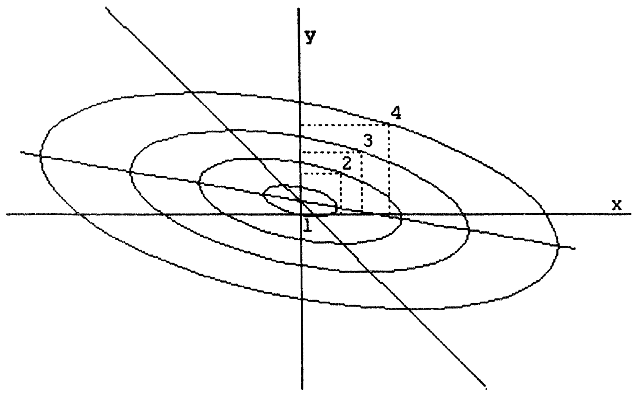

FIG.2. The set of concentric ellipses for various initial approximations.

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Received: April 20, 2001