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On the Number of Maximal Theta Pairs in a Finite Group

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Abstract: In [6], Bhattacharya and Mukherjee defined the notion of θ-pair for a maximal subgroup of a finite group. They proved that for any maximal subgroup M of a finite group G, there exists a θ-pair related to M. In [11], Zhao improved this result. He proved that for any maximal subgroup M of a finite group G, there exists a normal maximal θ-pair related to M.

In this paper we introduce the notion of $n\theta$-maximal and primitive $n\theta$-maximal group. We show that for $n = 1, 2$, G is $n\theta$-maximal if and only if G is primitive $n\theta$-maximal. Also, we characterize the 1θ-maximal group and prove some results about 2θ-maximal groups. Finally, we introduce the notion of $n\theta$-pair group and prove that for all $n \neq 2, 3$, there exists $n\theta$-pair groups and for $n = 2, 3$ there is no $n\theta$-pair groups.

Key Words: Maximal θ-pair, $n\theta$-maximal group, primitive $n\theta$-maximal group, $n\theta$-pair group

Mathematics Subject Classification: 1991 Mathematics Subject Classification: $20E34$, $20D10$

1. Introduction

In this paper all groups considered are assumed to be finite groups. For convenience we denote $M < \cdot G$ to indicate that M is a maximal subgroup of a group G. Also, M_G denotes the core of M in G and $\Phi(G)$ is the Frattini subgroup of the group G.

In [6], Mukherjee and Bhattacharya introduced the concept of θ-pairs associated to maximal subgroups of a group, and used this concept to investigate the structure of some groups. In [2], Beidlemen and Smith generalized the concept to the universe of infinite groups. The investigation on θ-pairs are continued in [1], [2], [7], [10], [11], [12], [13] and [14].

Let us recall the definition of θ-pair which is introduced by Mukherjee and Bhattacharya in [6].

Definition 1 [6]. Given a maximal subgroup M of a group G, a θ-pair of M is any pair (A, B) of subgroups satisfying the following conditions:

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(a) *B* < *G, B <A.*
(b) *<M,A>=G* a n d B *<M.*
(c) | has no proper normál subgroup of ^.

In addition, if *A* <3 G, then *{A,B)* is called a normál #-pair. A #-pair (A, *B)* is said to be maximal if there is no #-pair (C, *D)* such that *A* < *C.* The nonempty set of all 0-pairs of *M* in G is denoted by *9(M)* and *6{G)* = U M < G ^ (M) * S i m " ilarly, $Om_{ax}(M)$ denotes the set of all maximal 0-pairs of *M* in *G* and $O_{ma}x\{.G)$ — $\backslash J_{M}$*<G°rnax(M).*

Definition 2. A group *G* is called n#-maximal if $\backslash 0_{max}(G)\backslash$ = *n.* Also, we say that *G* is primitive nfl-maximal, if Af < G and *N* C $(G) implies that $|\#_{ma;r}(G)|$ = |*ma«(^)l = n.

In this páper, all notations are standard and taken mainly from [3], [4], [6] and [9].

2. Groups with exactly *n* Maximal 0-pairs, n =1,2

In this section we obtain the number of maximal #-pairs of some finite groups and prove that for any positive integer n, there exists a finite group G such that *\@max(G)* = n. To do this, suppose G is a finite group and TT(G) denotes the set of all prime factors of *\G\.* In the following simple lemma, we obtain the number of maximal #-pairs in a finite nilpotent group.

Lemma 1. Let G be a nilpotent group with exactly *n* maximal subgroup. Then G is a primitive n#-maximal group.

Proof. We first show that if *M* is a maximal subgroup of G, then $0_{max}(M)$ = {(G, M)}. To do this, suppose *M* is a maximal subgroup of G, then || has a prime order and so (G, *M)* G *9{M).* If *(A, B)* is another maximal 0-pair of *M* in G, then A = G and so (A, J3) is a normál maximal 0-pair. Now, by Theorem 2.5 of [11], *B* = *MQ* = *M* and $6_{ma}x(M)$ = {(G, M)}. Next, we assume that G is a nilpotent group with exactly *n* maximal subgroup, Mi, M_2, • • • , M_n. Therefore, for all i, 1 < *i* < n, *Omax(Mi)* = {(G,Mi)}. This shows that $e_{max}\{G)$ = $\{(G,M_t)|1 < i < n\}$ and G is a n#-maximal group. We now assume that *N* < $(G) is a normál subgroup of G. Set, *S* = *{M\M* < *G}* and *T* = {$|$ < -f}. Therefore, the map from 5 to *T* that sends M to ^ is easily seen to be a one-to-one correspondence. Thus, *\@max{-jšf)* = n and the lemma is proved, o

Corollary. For all positive integer n, there exist a primitive n0-maximal group.

Proof. Let G be a cyclic group with |7r(G)| = n. Then G has exactly n maximal subgroup and by Lemma 1, G is a primitive n0-maximal group. o

Lemma 2. Let G be a finite group and iV be a normál subgroup of G. Then *\0max(jjr)* < *\9max(G)\.*

Proof. By Lemma 2.1 of [6], the map r : $0_{max}(\sim)$ —> $0_{max}(G)$ that sends *(~, ~)* to (C, *D)* is well-defined. Now, it is easy to see that the map *r* is one-to-one. o

Remark 1. In the definition of primitive $n\theta$-maximal group, if we omit the condition $N \leq \Phi(G)$ then there is no primitive $n\theta$-maximal group, for $n > 1$. To see this, we assume that G is an arbitrary $n\theta$-maximal group, for $n > 1$. By Theorem 2.3 of [11] there is a normal maximal θ-pair (A, M_G) of M, in which M is a maximal subgroup. Consider $\frac{G}{A}$, then we can see that the map τ, in the proof of Lemma 2, is not onto. This shows that G is not primitive. $\diamond$

Remark 2. Let G be a finite group. G is 1θ-maximal if and only if G is primitive 1θ-maximal. To see this, it is enough to show that every 1θ-maximal group is primitive. Suppose $N \trianglelefteq G$, then by Lemma 2, $|\theta_{max}(\frac{G}{N})| \leq |\theta_{max}(G)| = 1$. Thus, $|\theta_{max}(\frac{G}{N})| = 1$, proving the result. $\diamond$

In [11], Zhao proved that if M is a maximal subgroup of G and (S, T) is a normal θ-pair of M, then M has a normal maximal θ-pair (A, B) such that $(S, T) \leq (A, B)$ and $\frac{A}{B} \cong \frac{S}{T}$. Furthermore, he proved that if $M < \cdot G$ and (A, B) is a normal maximal θ-pair of $\theta(M)$, then $B = M_G$. We use these results to prove the following theorem:

Theorem 1. G is 1θ-maximal if and only if $\frac{G}{\Phi(G)}$ is a simple group.

Proof. Suppose G is 1θ-maximal, say $\theta_{max}(G) = \{(C, D)\}$. Suppose $C \neq G$. Then there exists a maximal subgroup M of G such that $C \subseteq M$. Since $\theta_{max}(M) \neq \emptyset$, hence $\theta_{max}(M) = \{(C, D)\}$. This implies that $G =< M, C >= M$, a contradiction. Thus $C = G$ and (C, D) is a normal maximal θ-pair. Now, by the mentioned result of Zhao, $D = M_G$. If K is a maximal subgroup of G, then by assumption $K_G = M_G$ and so $D = M_G = \Phi(G)$. This shows that $\frac{G}{\Phi(G)}$ is a simple group.

Conversely, suppose $\frac{G}{\Phi(G)}$ is a simple group and $(C, D) \in \theta_{max}(G)$ is a maximal θ-pair. Since $\frac{G}{\Phi(G)}$ is simple, hence for any maximal subgroup K of G, $(G, \Phi(G))$ is a normal maximal θ-pair of K. Therefore, $C = G$ and $D \leq \Phi(G)$. Now, by the above result of Zhao, for any maximal subgroup K of G, $D = K_G$. Therefore, $(C, D) = (G, \Phi(G))$, proving the theorem. $\diamond$

Theorem 2. G is 1θ-maximal if and only if there exists a maximal subgroup M of G such that $\theta_{max}(M) = \theta_{max}(G)$.

Proof. Suppose M is a maximal subgroup of G such that $\theta_{max}(M) = \theta_{max}(G)$ and $|\theta_{max}(G)| > 1$. Let (A, M_G) be a normal maximal θ-pair of G associated to M. If $A \neq G$ then $\frac{G}{A}$ contains a normal maximal θ-pair $(\frac{R}{A}, \frac{T_G}{A})$ associated to a maximal subgroup $\frac{T}{A}$ of $\frac{G}{A}$. By Lemma 2.1 of [6], (R, T_G) is a normal maximal θ-pair of G and so $(R, T_G) \in \theta_{max}(M)$. But, $A < R$ and $(A, M_G) \in \theta_{max}(M)$, a contradiction. Now for any maxiaml subgroup K of G, $K_G = M_G$ and so $\Phi(G) = M_G$. This shows that $\frac{G}{\Phi(G)}$ is a simple group, which is a contradiction. Therefore, G is a 1θ-maximal group. The converse is obvious. $\diamond$

Lemma 3. If $(C, D) \in \theta(M)$, then for all $g \in G$, $(C^g, D) \in \theta(M^g)$.

Proof. Since, $D \triangleleft G, D < C$ and $C \nsubseteq M$, we have $D < C^g$ and $C^g \nsubseteq M^g$. Assume that $\frac{C^g}{D}$ properly contains a non-trivial normal subgroup $\frac{T}{D}$ of $\frac{G}{D}$. Then we have,

Z - T° (T)9~lD r (C^^D – C
D ~ D ~ VD^ L l D / D'

But, (C,D) G 0(M), a contradiction. Therefore, *(C⁹,D)* G 0(M*) and the lemma is proved, o

Coroilary. Let M be a maximal subgroup of the group *G*. Then, for all *g* G G, |0(M)| = *\0(M')*.

Proof. By Lemma 3, the map r : 0(M) —> 0(M⁵) that sends (C,D) to *{C⁹,D)* is well-defmed. Now, it is easy to see that the map r is a one-to-one correspondence. o

In what follows, we investigate the structure of *29*- maximal and primitive *29*-maximal groups.

Lemma 4. *G* is 20-maximal if and only if *G* is primitive 2#-maximal.

Proof. Suppose *G* is a 20-rnaximal group and *N* is a normál subgroup of *G* such that TV < $(G). By Lemma 2, *\9max{jr)* < 2- If *\0max(%)* = 1 then by Theorem [1]» $ffe] [is] simple. But, A/' C $(G) and so *($) = ^^, this implies that ^y is a simple group. Therefore, *G* is 10-maximal, which is a contradiction. o

Lemma 5. Let *G* be a 2<9-maximal group and *Omax (G)* = *{(A,B),(C,D)}*. Then the following statements hold:

(a) *A* < *G* and *C* < *G*.

(b) *A* = G or G = G.

(c) |{7b | T < -G}| = 2.

Froo/. We can assume that *(A, B)* is a normál maximal 0-pair. Suppose *C* is not normál in G and *g* G G - *NQ{C)*. Then *(C⁹,D)* is a maximal 0~pair diíferent from (.4,13) and *(C,D)*, which is a contradiction. Next, we assume that *A* / G and C # G. Suppose that (§,£) G 0mflíB(§) and (g,£) G 0maa;(g), then (i?,r),(í/,F) *e emax(G)*. Since ^ < *R*, *(A,B)* = *(U,V)* and so *(C,D)* = (fl,r). Therefore, *C* < *U* — *A* < *R* = *C*, *a*, contradiction. Finally, by Theorem 2, there are two maximal subgroups M and *L* such that *(A,B)* G ímoi(^) and *(C,D)* G ^max(^>), so by part (a), *B* = MG and *D* = *LQ*. We now assume that if is another maximal subgroup of G, then #max(^0 contains (-4, 1?) or *(C,D)*. Thus, it'G = MG or *KG* = LG and so *m* = |{rG | T < -G}| < 2. Suppose m = 1 then $(G) = MG = *LQ> **li** A ^ G* then *C* = *G* and -— is a simple group, a contradiction. If 4 = G then (4,£) = *(G,MG)* and **(C,Z?)** = (G,LG) and so (4, *B)* — *(C,D)*, which is a contradiction. Therefore, *m* = 2, as desired. o

Theorem 3. Suppose G satisfies the following conditions,

a) |{MG |M<-G}| = 2,

b) ^TO is a direct product of two simple groups.

Then G is 2#-maximal.

Proof. By condition (a) and Theorem 1, *$j^* is not simple. Hence we can assume that ^y = ^y x ^y, in which $^j and $TGy are two non-trivial simple subgroups of $\frac{G}{\Phi(G)}$. By condition (a) there are two maximal subgroups M and L such that $M_G \neq L_G$ and $\Phi(G) = M_G \cap L_G$. We now assume that T is a maximal subgroup of G such that $K \subseteq T_G$. So, $T_G = L_G$ or M_G. Suppose $K \subseteq M_G$ and $K \nsubseteq L_G$, then $G = KL$, $P \nsubseteq M_G$, $(K, \Phi(G)) \in \theta(L)$ and $(P, \Phi(G)) \in \theta(M)$. Let (U, L_G) be a normal maximal θ-pair of L such that $(K, \Phi(G)) \leq (U, L_G)$. We can see that $U = G$. Using similar argument as in above, if (V, M_G) is a normal maximal θ-pair of M such that $(P, \Phi(G)) \leq (V, M_G)$, then $V = G$. If (C, D) is another maximal θ- pair of G then there exists a maximal subgroup T such that $(C, D) \in \theta(T)$, so $T_G = L_G$ or $T_G = M_G$. Suppose that $T_G = L_G$, then $(G, L_G), (C, D) \in \theta(T)$ and since $C \neq G$ so $(C, D) \leq (G, L_G)$, which is a contradiction. Therefore G is 2θ-maximal.◇

Corollary. If $\frac{G}{\Phi(G)}$ is a direct product of two simple groups with co-prime orders, then G is 2θ-maximal.

Proof. By Theorem 3, it is enough to show that $|\{M_G \mid M < \cdot G\}| = 2$. To do this, we prove that if $G = A \times B$, where A and B are normal simple subgroups of G with co-prime orders, then G has exactly four normal subgroups. Suppose N is a normal subgroup of G different from A and B. We can assume that $N \cap A = N \cap B = 1$ and so $A \cong \frac{G}{N} \cong B$, a contradiction. Therefore, $|\{X_G \mid X < \cdot G\}| = 2$ and the proof is complete. ◇.

3. Groups with exactly n θ-pair

In this section we introduce the notion of $n\theta$-pair group and prove that there is no 2θ- and 3θ-pair group. Finally, we construct a groups with exactly n θ-pairs, for $n \neq 2, 3$. To do this, we need the structure of groups with exactly one or two maximal subgroups. It is well known that if a finite group G has exactly one maximal subgroup, then $|G|$ is divisible by exactly one prime number and G is cyclic. It has been proved [5] that if G has exactly two maximal subgroups then $|G|$ is indeed divisible by two primes and G is cyclic. Throughout this section $m(G)$ denotes the number of maximal subgroups of G.

Definition 3. A group G is called $n\theta$-pair, if and only if $|\theta(G)| = n$.

Lemma 6. A group G is 1θ-pair if and only if G is a cyclic group of prime power order.

Proof. Suppose G is 1θ-pair. Then by Theorem 1, $\frac{G}{\Phi(G)}$ is a simple group and $\theta(G) = \{(G, \Phi(G))\}$. Suppose $m(G) > 1$. Then $\Phi(G)$ is not maximal in G and for any maximal subgroup M of G, $(M, \Phi(G))$ is a θ-pair of L, in which L is a maximal subgroup of G distinct from M, a contradiction. This shows that $m(G) = 1$ and so G is a cyclic group of prime power order. ◇

Lemma 7. If there exists a maximal subgroup M of G such that $\theta(M) = \theta(G)$, then G is 1θ-pair.

Proof. By Theorem 2, G is 1θ-maximal and so $\frac{G}{\Phi(G)}$ is a simple group. If $m(G) > 1$ then $(M, \Phi(G)) \in \theta(L)$ and $(L, \Phi(G)) \in \theta(M)$, for two distinct maximal subgroups

M and L of G, which is a contradiction. Therefore, $m(G) = 1$ and by Lemma 6, G is 1θ-pair. ◇

Lemma 8. There is no $n\theta$-pair cyclic group of order $p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}$, $p_1 < p_2 < \cdots < p_n$, in which $n > 1$.

Proof. Suppose $\{M_1, M_2, \cdots, M_n\}$ is the set of all maximal subgroups of G. Then $(G, M_i), 1 \le i \le n$, are n maximal θ-pairs for G and so G has at least n θ-pair. Assume that M is a maximal subgroup of index p_1, A is a maximal subgroup of M of index p_2 and L is a maximal subgroup of G of index p_2. Then $(M, A) \in \theta(L)$, a contradiction. ◇

Theorem 4. There is no 2θ-pair group.

Proof. Let G be a 2θ-pair group. By Lemma 7, there is no maximal subgroup M of G such that $\theta(M) = \theta(G)$ and so G is 2θ-maximal. Thus, $|\{X_G \mid X < \cdot G\}| = 2$. Suppose that (C, L_G) and (G, M_G) are two distinct maximal θ-pairs of G associated to maximal subgroups L and M, respectively. We claim that G has exactly two maximal subgroups. To do this, we assume that T is a maximal subgroup of G distinct from M and L. If $C \neq G$ then $\Phi(G) = L_G$ and $(L, \Phi(G)) \in \theta(T)$, which is a contradiction. We now assume that $C = G$, then $\frac{G}{M_G}$ and $\frac{G}{L_G}$ are simple groups. Therefore, $T_G = L_G$ or $T_G = M_G$. Suppose $T_G = L_G$ then $(L, L_G) \in \theta(T)$, a contradiction. Also, if $T_G = M_G$ then $(M, M_G) \in \theta(T)$ and so $M_G = L_G$. This implies that $\frac{G}{\Phi(G)}$ is a simple group, which is a contradiction. Therefore, G has exactly two maximal subgroups and by a theorem of Khazal, mentioned above, $|G|$ is indeed divisible by two primes. Now by Lemma 8, the proof is complete. ◇

Lemma 9. Let G be a finite group such that all of maximal θ-pairs of G are normal and $\{M_G \mid M < \cdot G\} = \{L_{1G}, \cdots, L_{rG}\}$. Then $\theta_{max}(G) = \theta_{max}(L_1) \cup \cdots \cup \theta_{max}(L_r)$.

Proof. Suppose (C, D) is an arbitrary maximal θ-pair of G. Then $D = L_{iG}$, for some $1 \le i \le r$. If $C \subseteq L_i$ then $C \subseteq D$, a contradiction. Thus $(C, D) \in \theta(L_i)$. Now we assume that (E, F) is a maximal θ-pair of $\theta(L_i)$ such that $(C, D) \le (E, F)$. Therefore, $C \le E$, $D = F$, $\frac{C}{D} \le \frac{E}{D}$ and $\frac{C}{D} \trianglelefteq \frac{G}{D}$. This shows that (C, D) is a maximal θ-pair of $\theta(L_i)$ and the proof is complete. ◇

Theorem 5. There is no 3θ-pair group.

Proof. Let G be a 3θ-pair group. By Lemma 7, there is no maximal subgroup M of G such that $\theta(M) = \theta(G)$. Our main proof will consider a number of cases.

Case 1. *There are two maximal subgroups M and L of G such that $|\theta(M)| = 2$ and $|\theta(L)| = 1$.* Assume that $(B, M_G), (C, D) \in \theta(M)$ and $(A, L_G) \in \theta(L)$. We can see that $C \trianglelefteq G$ and $C \neq G$. We claim that G has at least three maximal subgroups. By lemma 6, G has at least two maximal subgroups. Assume that G has exactly two maximal subgroups, say M and L. Thus, by a theorem of Khazal, mentioned above, G is cyclic and so $(A, L_G) = (G, L)$, $(B, M_G) = (G, M)$. Since $\frac{G}{L}$ is a simple group, we have $(M, \Phi(G)) \in \theta(L)$, a contradiction. Therefore G has at least three maximal subgroups. We now see that $M_G \neq L_G$. Thus, for any maximal subgroup X of G, $X_G = L_G$ or $X_G \le M_G$. Suppose $A = G$. If L is non-normal and $g \in G - N_G(L)$, then $(L^g, L_G) \in \theta(L)$, which is impossible. So $L \trianglelefteq G$ and we can see that $(M_G, L \cap M_G) \in \theta(L)$, a contradiction. Thus $A \neq G$ and so $A \leq M_G$. Also, $C \leq L_G$ and hence $C \leq L_G \leq A \leq M_G$, which is a contradiction.

Case 2. *G is 3θ-maximal and there are maximal subgroups M, L and K of G such that $(A, L_G) \in \theta(L)$, $(B, K_G) \in \theta(K)$ and $(C, M_G) \in \theta(M)$.* By Lemma 9 and Case 1, $|\{M_G \mid M < \cdot G\}| = 3$. We claim that one of the subgroups A, B and C is equal to G and the other two are proper. To do this, suppose $A = C = G$. Then $M, L \lhd G$ and $(L, M \cap L) \in \theta(M)$, which is impossible. Therefore, we can assume that $A \neq G, B \neq G$ and $|\theta(\frac{G}{A})| = |\theta(\frac{G}{B})| = 1$. Suppose $\frac{R}{A}$ and $\frac{S}{B}$ are the unique maximal subgroups of $\frac{G}{A}$ and $\frac{G}{B}$, respectively. Thus, $(\frac{G}{A}, \frac{R}{A}) \in \theta(\frac{G}{A})$ and $(\frac{G}{B}, \frac{S}{B}) \in \theta(\frac{G}{B})$. This shows that (G, R) and (G, S) are θ-pairs of G and so $R = S$. We can assume that $M \lhd G$ and $A, B \leq M$. Now $(\frac{A}{L_G}, \frac{L_G}{L_G}), (\frac{G}{L_G}, \frac{M}{L_G}) \in \theta(\frac{G}{L_G})$ and $|\theta_{max}(\frac{G}{L_G})| \leq 3$. Therefore, $|\theta_{max}(\frac{G}{L_G})| = 3$ and there exists another θ-pair $(\frac{R_1}{L_G}, \frac{U_1}{L_G}) \in \theta(\frac{G}{L_G})$. It is easy to see that $L_G \subseteq K_G$. Using similar argument as in above, $K_G \subseteq L_G$ and so $L_G = K_G$, which is a contradiction. ◇

Theorem 6. There exists a group with exactly n θ-pair, for $n \neq 2, 3$.

Proof. For $n = 1$, a cyclic group of prime power order has exactly one θ-pair. Suppose $n \geq 4$ and $G = Z_{p^n q}$. Then G has exactly two maximal subgroups M and N of orders p^n and $p^{n-1}q$, respectively. Suppose A_i and B_i, $0 \leq i \leq n$, are subgroups of G of order p^i and $p^i q$. Now it is easy to see that $\theta(M) = \{(B_i, A_i) \mid 0 \leq i \leq n\}$ and $\theta(N) = \{(A_n, A_{n-1}), (B_n, B_{n-1})\}$. Therefore G has exactly $n+3$, θ-pair, proving the result.◇

We conclude this paper with the following open question:
Question: Is there a non-abelian finite group with exactly n θ-pairs, for a given positive integer $n \neq 2, 3$?

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