The theory of quasi-divisors on cartesian products

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Abstract: In this paper we study, using r-ideal systems, how some properties of the directed groups $G_i, i = 1, 2$, can be transferred into their cartesian product G, and vice-versa. In particular, beginning from the structures (G_1, r_1) and (G_2, r_2), we construct the $r_1 \otimes r_2$-ideal system in G and we prove that (G_i, r_i) are r_i-Prüfer groups, $i = 1, 2$, if and only if $(G, r_1 \otimes r_2)$ is an $r_1 \otimes r_2$-Prüfer group. Our main result is that the group G admits a theory of divisors, theory of quasi-divisors or strong theory of quasi-divisors, if and only if the groups $G_i, i = 1, 2$, admit such a theory, respectively. Finally, when the groups G_1, G_2 and G admit theories of quasi-divisors, we investigate the relation between their corresponding Lorenzen t-groups and divisor class groups.

Key Words: Theory of divisors, quasi-divisors, cartesian products of groups.

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1. Introduction and Preliminaries

In the investigation of arithmetical properties of *po*-groups, the notions of *r-ideal system* and *theory of divisors* play an important role. A historical source of this study is the work of Borevič and Šafarevič [2], who defined the notion of a theory of divisors as a map h, from the group of divisibility G of an integral domain A into a free abelian group $Z^{(P)}$ (considered as an l-group with pointwise ordering), satisfying the following conditions :

(1) For every $a, b \in G$, $a \leq b$ if and only if $h(a) \leq h(b)$ in $Z^{(P)}$.
(2) If $h(a) \geq \omega, h(b) \geq \omega$, then $h(a + b) \geq \omega$.
(3) If $\alpha \in Z^{(P)}$ then there exist $g_1, \ldots, g_n \in G$ such that $\alpha = h(g_1) \wedge \ldots \wedge h(g_n)$.

The disadvantage of this approach is the presence of the additive operation which is irrelevant for the general treatment of divisors in groups and, as Skula proved [11], is redundant even in the case of rings, so it can be discarted. Thus, a theory of divisors of a directed group G, is defined as a map h from G into a free abelian group $Z^{(P)}$ (considered again as an l-group with pointwise ordering) satisfying the previous conditions (1) and (3). A further generalization of a divisor theory was done by Aubert [1] who introduced the notion of a quasi-divisor theory. We recall that, if G and Γ are *po*-groups, a group homomorphism $f: G \to \Gamma$ is called

(i) *o*-homomorphism, if $f(G^+) \subseteq \Gamma^+$
(ii) *o*-monomorphism, if it is injective and $f(G^+) = \Gamma^+ \cap f(G)$
(iii) *o*-epimorphism, if it is surjective and $f(G^+) = \Gamma^+$
(iv) *o*-isomorphism, if it is an injective o-epimorphism.

An *o*-homomorphism $h : G \to \Gamma$, from a directed group G into an l-group Γ, is called a *quasi-divisor theory* if the following two conditions are satisfied :

(1) h is an *o*-monomorphism,
(2) for all $a \in \Gamma^+$ there exist $g_1, \ldots, g_n \in G^+$, such that $a = h(g_1) \wedge \ldots \wedge h(g_n)$.

We say that an *o*-monomorphism $h : G \to \Gamma$, from a directed group G into an l-group Γ, is a *strong theory of quasi-divisors*, if for every $\alpha, \beta \in \Gamma^+$ there exists $\gamma \in \Gamma^+$, such that $\alpha \cdot \gamma \in h(G)$ and $\beta \wedge \gamma = 1$. It is known [10] that a strong theory of quasi-divisors is a quasi-divisor theory as well.

Skula has also introduced the notion of the divisor class group as the generalization of a class group from the theory of Krull domains. For an *o*-monomorphism $h : G \to \Gamma$ of a directed group G into another directed group Γ, the factor group $\Gamma/h(G)$ is called a divisor class group of h and it is denoted by C_h.

In this paper we prove that the cartesian product G of the directed groups G_1 and G_2 admits a theory of divisors, theory of quasi-divisors or strong theory of quasi-divisors, if and only if the groups $G_i, i = 1, 2$, admit such a theory, respectively and we study properties of the corresponding Lorenzen t-groups and divisor class groups. In this investigation, we mainly use systems of ideals defined on groups. It is necessary to mention that Jaffard in [6],using ideals, had proved many results concerning divisors, which have been later rediscovered and published under modern terminology.

By an r-system of ideals in a directed *po*-group G we mean a map $X \mapsto X_r$ (X_r is called the r-ideal generated by X) from the set $B(G)$ of all lower bounded subsets X of G into the power set of G, which satisfies the following conditions :

(1) $X \subseteq X_r$
(2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$
(3) $\{a\}_r = a \cdot G^+ = (a)$, for all $a \in G$
(4) $a \cdot X_r = (a \cdot X)_r$, for all $a \in G$.

The set $\mathcal{I}_r(G)$, of the r-ideals of G, endowed with the multiplication

$$X_r \times {}_r Y_r = (X \cdot Y)_r = (X_r \cdot Y_r)_r$$

is a commutative monoid, which contains the structure $(\mathcal{I}_r^f(G), \times_r)$, where $\mathcal{I}_r^f(G)$ is the set of finitely generated r-ideals, as a submonoid. The set $R(G)$ of all r-systems defined on G is partially ordered by the relation

$$r \leq s \text{ if and only if } X_s \subseteq X_r, \text{ for each } X \in B(G).$$

An r-system is said to be of finite character if, for any $X \in B(G), X_r = \bigcup_{\substack{K \subseteq X \\ K \text{ finite}}} K_r$. Among all r-systems in $R(G)$, there exists one, called the v-system, which is the coarsest one and is defined by $X_v = \bigcap_{X\subseteq(x)}(x)$ and among the systems of finite character there exists a special one, called the t-system, where $X_t = \bigcup_{\substack{Y\subseteq X \\ Y \text{ finite}}} Y_v$. A group G is said to be an r-Prüfer group if the monoid $(\mathcal{I}_r^f(G), \times_r)$ is a group.

We briefly describe the definition of the Lorenzen r-group of G, whose role is to provide the g.c.d.'s which may be missing in G. The group G, endowed with an r-system of ideals, is said to be r-closed if and only if $X_r : X_r \subseteq G^+$, for every finite $X \in B(G)$, where $X_r : X_r = \{a \in G : a \cdot X_r \subseteq X_r\}$. In this case, a system of finite character, denoted by r_a, can be defined on G by

$$X_{r_a} = \{g \in G : g \cdot N_r \subseteq X_r \times_r N_r \text{ for some finite } N \subseteq G\},$$

whenever X is a finite subset of G. The r_a-ideal generated by a lower bounded subset A of G is equal to the set-theoretical union of all r_a-ideals generated by finite subsets of A. The main property we derive out of this construction is that the monoid $(\mathcal{I}_{r_a}^f(G), \times_{r_a})$ satisfies the cancellation law, so it possesses a group of quotients, denoted by $\Lambda_r(G)$, and called the Lorenzen r-group of G. Among various results proved for the Lorenzen r-group, mainly cited in [6], we remind that it is an l-group under the ordering

$$X_{r_a}/Y_{r_a} \in \Lambda_r(G)^+ \text{ if and only if } X_{r_a} \subseteq Y_{r_a}.$$

Moreover, the group G can be considered as one of its subgroups, embedded in $\Lambda_r(G)$ by the map $h : G \to \Lambda_r(G)$, $g \mapsto (g)_{r_a}$.

2. Properties of ideals in cartesian products

In this section, we study how some properties of two groups G_1, G_2 can be transferred into their cartesian product $G = G_1 \times G_2$, using ideal systems defined on them and vice-versa. We denote by p_i the usual projection maps from G to G_i, $i = 1, 2$. In [7] we have described how, beginning from the structures (G_1, r_1) and (G_2, r_2), the directed group G can be endowed with a system of ideals, denoted by $r_1 \otimes r_2$, where

$$X_{r_1\otimes r_2} = (p_1(X))_{r_1} \times (p_2(X))_{r_2},$$

for every $X \in B(G)$. The following proposition shows tnat this constuction transfers the structure of Prüfer groups into their cartesian product.

Proposition 2.1. *Consider the structures $(G_1, r_1), (G_2, r_2)$ and $(G, r_1 \otimes r_2)$, where $G = G_1 \times G_2$. Then,*

$$(\mathcal{I}_{r_1}(G_1), \times_{r_1}) \times (\mathcal{I}_{r_2}(G_2), \times_{r_2}) \cong (\mathcal{I}_{r_1\otimes r_2}(G), \times_{r_1\otimes r_2}).$$

Proof. Obviously, the cartesian product of the monoids $A_1 = (\mathcal{I}_{r_1}(G_1), \times_{r_1})$ and $A_2 = (\mathcal{I}_{r_2}(G_2), \times_{r_2})$ is also a monoid.

We denote by A the monoid $(\mathcal{I}_{r_1\otimes r_2}(G), \times_{r_1\otimes r_2})$ and we consider the map

$$f : A_1 \times A_2 \to A,\ ((X_1)_{r_1}, (X_2)_{r_2}) \mapsto (X_1 \times X_2)_{r_1\otimes r_2}.$$

Let $x = ((X_1)_{r_1}, (X_2)_{r_2})$, $y = ((Y_1)_{r_1}, (Y_2)_{r_2}) \in A_1 \times A_2$. Obviously, $X_1 \times X_2$, $Y_1 \times Y_2 \in B(G)$ and if $x = y$, then

$$(X_1)_{r_1} \times (X_2)_{r_2} = (Y_1)_{r_1} \times (Y_2)_{r_2}$$

thus, the map f is well defined. In order to prove that this map is a homomorphism, it is enough to observe that

$$f(x \cdot y) = f((X_1 \cdot Y_1)_{r_1}, (X_2 \cdot Y_2)_{r_2}) = (X_1 \cdot Y_1 \times X_2 \cdot Y_2)_{r_1 \otimes r_2} = \\ = (X_1 \times X_2)_{r_1 \otimes r_2} \times {}_{r_1 \otimes r_2} (Y_1 \times Y_2)_{r_1 \otimes r_2} = f(x) \times {}_{r_1 \otimes r_2} f(y),$$

since $X_1 \cdot Y_1 \times X_2 \cdot Y_2 = (X_1 \times X_2) \cdot (Y_1 \times Y_2)$. The map f is a monomorphism, because, from the equality $(X_1 \times X_2)_{r_1 \otimes r_2} = (Y_1 \times Y_2)_{r_1 \otimes r_2}$, it follows $(X_1)_{r_1} = (Y_1)_{r_1}$ and $(X_2)_{r_2} = (Y_2)_{r_2}$. Moreover, if $X_{r_1 \otimes r_2} \in A$, it is clear that $(p_1(X))_{r_1} \in A_1$, $(p_2(X))_{r_2} \in A_2$ and

$$f((p_1(X))_{r_1}, (p_2(X))_{r_2}) = (p_1(X) \times p_2(X))_{r_1 \otimes r_2} = \\ = (p_1(X))_{r_1} \times (p_2(X))_{r_2} = X_{r_1 \otimes r_2}.$$

Thus the monoid A is isomorphic to $A_1 \times A_2$. □

Corollary 2.2. *Consider the structures $(G_1, r_1), (G_2, r_2)$ and $(G, r_1 \otimes r_2)$, where $G = G_1 \times G_2$. Then,*

$$(\mathcal{I}^f_{r_1}(G_1), \times_{r_1}) \times (\mathcal{I}^f_{r_2}(G_2), \times_{r_2}) \cong (\mathcal{I}^f_{r_1 \otimes r_2}(G), \times_{r_1 \otimes r_2}).$$

Proof. Following the same procedure as in the proof of proposition 2.1, we consider the map $\bar{f} = f|_{\mathcal{I}^f_{r_1}(G_1) \times \mathcal{I}^f_{r_2}(G_2)}$. Obviously, $\bar{f} : \mathcal{I}^f_{r_1}(G_1) \times \mathcal{I}^f_{r_2}(G_2) \to \mathcal{I}^f_{r_1 \otimes r_2}(G)$ is well defined , since the cartesian product of two finite sets is a finite set and it is a monomorphism as a restriction of the monomorphism f. In order to prove that it is an isomorphism it is enough to observe that if $X_{r_1 \otimes r_2} \in \mathcal{I}^f_{r_1 \otimes r_2}(G)$, then $(p_i(X))_{r_i} \in \mathcal{I}^f_{r_i}(G_i)$ for $i = 1, 2$, and $\bar{f}((p_1(X))_{r_1}, (p_2(X))_{r_2}) = X_{r_1 \otimes r_2}$. □

Corollary 2.3. *Consider the structures $(G_1, r_1), (G_2, r_2)$ and $(G, r_1 \otimes r_2)$, where $G = G_1 \times G_2$. The following statements are equivalent :*

1. *(G_i, r_i) are r_i-Prüfer groups, for $i = 1, 2$.*
2. *$(G, r_1 \otimes r_2)$ is an $r_1 \otimes r_2$-Prüfer group.*

Proof. It results easily from corollary 2.2. □

Corollary 2.4. *Consider the structures (G_1, r_1), (G_2, r_2) and (G, r), where $G = G_1 \times G_2$ and $r = r_1 \otimes r_2$. If B_r is the inverse of an invertible r-ideal A_r of G, then the r_i-ideals $(p_i(A))_{r_i}$ of G_i are invertible with inverses $(p_i(B))_{r_i}$, $i = 1, 2$.*

Proof. Since $A_r \times {}_r B_r = (1_G)$, it follows from proposition 2.1 that

$$f^{-1}((A \cdot B)_{r_1 \otimes r_2}) = ((p_1(A \cdot B))_{r_1}, (p_2(A \cdot B))_{r_2}) = ((1_{G_1}), (1_{G_2})).$$

Hence, $(p_i(A))_{r_i} \times {}_{r_i} (p_i(B))_{r_i} = (1_{G_i})$, for $i = 1, 2$. □

In the sequel we denote by v_i, t_i, the v, t systems defined on $G_i, i = 1, 2$, respectively.

Proposition 2.5. *Consider the structures* (G_1, v_1) *and* (G_2, v_2). *On the group* $G = G_1 \times G_2$ *the* v*-system and the* $v_1 \otimes v_2$*-system coincide.*

Proof. Since the v-system is the coarsest one in G, it follows that $v \leq v_1 \otimes v_2$. Conversely, let $X \in B(G), x = (x_1, x_2) \in X_v$. For any $y_1 \in G_1, y_2 \in G_2$ such that $X_1 = p_1(X) \subseteq (y_1), X_2 = p_2(X) \subseteq (y_2)$, it is $X \subseteq X_1 \times X_2 \subseteq (y)$, where $y = (y_1, y_2)$. Therefore $X_v \subseteq (y)$, which means that $x \in (y)$. Thus, $x_1 \in (y_1), x_2 \in (y_2)$ and finally $x \in (X_1)_{v_1} \times (X_2)_{v_2} = X_{v_1 \otimes v_2}$, that is, $v_1 \otimes v_2 \leq v$. □

Proposition 2.6. *Consider the structures* (G_1, t_1) *and* (G_2, t_2). *On the group* $G = G_1 \times G_2$ *the* t*-system and the* $t_1 \otimes t_2$*-system coincide.*

Proof. Consider $X \in B(G)$ and put $X_i = p_i(X)$, $i = 1, 2$. Let $x = (x_1, x_2) \in X_t$. There exists a finite subset K of X such that

$$x \in K_v = (p_1(K))_{v_1} \times (p_2(K))_{v_2}.$$

Since $p_1(K)$ is a finite subset of X_1, it follows that $x_1 \in (X_1)_{t_1}$ and similarly $x_2 \in (X_2)_{t_2}$. Thus $x \in X_{t_1 \otimes t_2}$. Conversely, let $x = (x_1, x_2) \in X_{t_1 \otimes t_2} = (X_1)_{t_1} \times (X_2)_{t_2}$. There exist K_i finite subsets of $X_i, i = 1, 2$, respectively, such that $x_i \in (K_i)_{v_i}, i = 1, 2$. For any $a_j \in K_1 = \{a_1, a_2, \ldots, a_n\}$ there exists a $y_j \in G_2$ such that $(a_j, y_j) \in X$ and similarly, for any $b_k \in K_2 = \{b_1, b_2, \ldots, b_m\}$, there exists a $w_k \in G_1$ such that $(w_k, b_k) \in X$. Put

$$K = \{(a_1, y_1), \ldots, (a_n, y_n), (w_1, b_1), \ldots, (w_m, b_m)\}.$$

Obviously, K is a finite subset of X and $K_i \subseteq p_i(K), i = 1, 2$. Hence,

$$x \in (K_1)_{v_1} \times (K_2)_{v_2} \subseteq K_{v_1 \otimes v_2} = K_v,$$

which means that $x \in X_t$. Thus, $t = t_1 \otimes t_2$. □

3. Cartesian products and quasi-divisors

In this section, we deal with special o-homomorphisms from directed groups into l-groups, which are related to the notions of t-Prüfer group and Lorenzen t-group. We propose three known results on this topic, which we use in the sequel.

Proposition 3.1. (C.f. [1]). *Let* G *be a directed group. Then, the following statements are equivalent :*

(1) *G admits a divisor theory.*
(2) *The structure* $(\mathcal{I}_t(G), \times_t)$ *is a group.*

Proposition 3.2. (C.f. [1]). *The directed group* G *has a theory of quasi-divisors if and only if* G *is a* t*-Prüfer group. The group of quasi-divisors of* G *is then uniquely determined as the Lorenzen* t*-group of* G *which, in this case, is isomorphic to the group of finitely generated* t*-ideals of* G.

Proposition 3.3. (C.f. [3]). *Let G be a directed group. Then, the following statements are equivalent :*

(1) *G admits a quasi-divisor theory.*
(2) *The embedding $h : G \to \Lambda_t(G)$, given by $h(g) = (g)_t$, is a quasi-divisor theory.*

In the following, we investigate how the concepts of theory of divisors, theory of quasi- divisors and strong theory of quasi-divisors of two directed groups G_1 and G_2, can be tranferred in their cartesian product and vice-versa.

Proposition 3.4. *Consider the groups G_1, G_2 and $G = G_1 \times G_2$. The following statements are equivalent :*

(1) *The groups G_i admit a theory of divisors $h_i : G_i \to \Gamma_i$, for $i = 1, 2$.*
(2) *The group G admits a theory of divisors $h : G \to \Gamma$.*

Proof. It results easily from propositions 2.1 and 3.1. □

Theorem 3.5. *Consider the groups G_1, G_2 and $G = G_1 \times G_2$. The following statements are equivalent :*

(1) *The groups G_i admit a theory of quasi-divisors $h_i : G_i \to \Gamma_i$, for $i = 1, 2$.*
(2) *The group G admits a theory of quasi-divisors $h : G \to \Gamma$.*

Moreover, whenever one of the previous statements holds, then

(i) $\Lambda_t(G) \cong \Lambda_{t_1}(G_1) \times \Lambda_{t_2}(G_2)$.
(ii) $\Gamma \cong \Gamma_1 \times \Gamma_2$.
(iii) $h(G) \cong h_1(G_1) \times h_2(G_2)$.
(iv) *There exist two epimorphisms $f : C_h \to C_{h_1}$ and $g : C_h \to C_{h_2}$, such that the following diagram commutes*

$$\begin{array}{ccccc}
G_2 & \xleftarrow{p_2} & G & \xrightarrow{p_1} & G_1 \\
\downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_1} \\
\Gamma_2 & \xleftarrow{q_2} & \Gamma & \xrightarrow{q_1} & \Gamma_1 \\
\downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_1} \\
C_{h_2} & \xleftarrow{g} & C_h & \xrightarrow{f} & C_{h_1}
\end{array}$$

where C_h, C_{h_1}, C_{h_2} are the divisor class groups of h, h_1, h_2 respectively, q_1, q_2 are the projection maps and ϕ, ϕ_1, ϕ_2 are the canonical epimorphisms.
(v) $C_h \cong C_{h_1} \times C_{h_2}$.

Proof. We observe that in the groups G_1, G_2 and G there are always defined the t-systems denoted by t_1, t_2 and t respectively, for which the equality $t = t_1 \otimes t_2$ holds. From proposition 3.2 and corollary 2.3, it results the equivalence of the statements (1) and (2). In the following we suppose that one of these statements holds.

(i) From proposition 3.2 it results that $\Lambda_t(G) \cong \mathcal{I}_t^f(G)$ and $\Lambda_{t_i}(G_i) \cong \mathcal{I}_{t_i}^f(G_i)$, for $i = 1, 2$. Using the isomorphism proved in corollary 2.2 we conclude the statement (i).

(ii) Obvious, from proposition 3.2 and the above statement.

(iii) By the uniqueness of a quasi-divisor theory (c.f.[3]), we can consider the morphisms h, h_1, h_2 as the embeddings of G, G_1, G_2 into their t-Lorenzen groups, respectively. Since $\Gamma \cong \Gamma_1 \times \Gamma_2$, which means that there exists an isomorphism $k : \Gamma \to \Gamma_1 \times \Gamma_2$, we denote by $\bar{k}$ the restriction $k\,|_{h(G)}$. Obviously, the map k is defined by $k(X_t) = ((p_1(X))_{t_1}, (p_2(X))_{t_2})$. Thus, the map

$$\bar{k} : h(G) \to h_1(G_1) \times h_2(G_2)\,,\ ((x_1, x_2)_t) \mapsto ((x_1)_{t_1}, (x_2)_{t_2})$$

is an isomorphism.

(iv) Put $f : C_h \to C_{h_1}\,,\ A_t h(G) \mapsto (p_1(A))_{t_1} h_1(G_1)$. The set $p_1(A)$ is a finite subset of G_1, thus $f(A_t h(G)) \in C_{h_1}$, for every $A_t h(G) \in C_h$. Let $A_t h(G), B_t h(G) \in C_h$. If $A_t h(G) = B_t h(G)$, then $A_t \times {}_t K_t \in h(G)$, where K_t is the inverse of B_t into Γ, that is, $A_t \times {}_t K_t = (y)_t$, $y = (y_1, y_2) \in G$. Thus,

$$(y_1)_{t_1} \times (y_2)_{t_2} = (A \cdot K)_t = (p_1(A \cdot K))_{t_1} \times (p_2(A \cdot K))_{t_2}.$$

Hence, from corollary 2.4, it results that

$$\begin{aligned}(p_1(A))_{t_1} \times {}_{t_1}((p_1(B))_{t_1})^{-1} &= (p_1(A))_{t_1} \times {}_{t_1}(p_1(K))_{t_1} = \\ &= (p_1(A \cdot K))_{t_1} = (y_1)_{t_1} \in h_1(G_1).\end{aligned}$$

Then $f(A_t h(G)) = f(B_t h(G))$, which means that the map f is well defined. Moreover,

$$\begin{aligned}f(A_t h(G) \cdot B_t h(G)) &= f((A \cdot B)_t h(G)) = (p_1(A \cdot B))_{t_1} h_1(G_1),\\ f(A_t h(G)) \cdot f(B_t h(G)) &= ((p_1(A))_{t_1} \times {}_{t_1}(p_1(B))_{t_1}) h_1(G_1) = \\ &= (p_1(A) \cdot p_1(B))_{t_1} h_1(G_1) = (p_1(A \cdot B))_{t_1} h_1(G_1),\end{aligned}$$

thus the map f is a homomorphism. Let $(A_1)_{t_1} h_1(G_1) \in C_{h_1}$, which means that A_1 is a finite subset of G_1. Consider a finite subset A of G, such that $p_1(A) = A_1$. Then $\phi(h(A)) \in C_h$ and

$$f(\phi(h(A))) = f(A_t h(G)) = (p_1(A))_{t_1} h_1(G_1) = (A_1)_{t_1} h_1(G_1),$$

thus, the map f is an epimorphism.

Put now $g : C \to C_2\,,\ A_t h(G) \mapsto (p_2(A))_{t_2} h_2(G_2)$. Similarly, we can prove that this map is an epimorphism. The commutativity of the diagram follows easily from the definitions of the maps and the relation $t = t_1 \otimes t_2$.

(v) The map $u : C_h \to C_{h_1} \times C_{h_2}\,,\ A_t h(G) \mapsto (f(A_t h(G)), g(A_t h(G)))$ is a well defined group homomorphism. Let $A_t h(G) \in Ker u$, which means that

$$(p_i(A))_{t_i} h_i(G_i) = h_i(G_i)\,,\ \text{ for } i = 1, 2.$$

Then, there exist $x_i \in G_i$, such that $(p_i(A))_{t_i} = (x_i)_{t_i}$, for $i = 1, 2$. Put $x = (x_1, x_2)$. Then

$$A_t = A_{t_1 \otimes t_2} = (p_1(A))_{t_1} \times (p_2(A))_{t_2} = (x)_t,$$

thus $A_t h(G) = h(G)$ and therefore the map u is a monomorphism.

Let $((A_1)_{t_1} h_1(G_1), (A_2)_{t_2} h_2(G_2)) \in C_{h_1} \times C_{h_2}$. Then, the set $A = A_1 \times A_2$ is a finite subset of G, thus $A_t h(G) \in C_h$ and it is obvious that

$$u(A_t h(G)) = ((A_1)_{t_1} h_1(G_1), (A_2)_{t_2} h_2(G_2)).$$

Hence $C_h \cong C_{h_1} \times C_{h_2}$. □

Proposition 3.6. *Consider the groups G_1, G_2 and $G = G_1 \times G_2$. The following statements are equivalent :*

(1) *The groups G_i admit a strong theory of quasi-divisors $h_i : G_i \to \Gamma_i$, for $i = 1, 2$.*

(2) *The group G admits a stong theory of quasi-divisors $h : G \to \Gamma$.*

Proof. (1) $\to$ (2). Since the homomorphisms $h_i, i = 1, 2$, are theories of quasi-divisors, it follows from theorem 3.5 that the group G admits a theory of quasi-divisors $h : G \to \Gamma$, where $\Gamma \cong \Gamma_1 \times \Gamma_2$ and $h(G) \cong h_1(G_1) \times h_2(G_2)$. Let $\alpha, \beta \in \Gamma^+$, that is, $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, with $\alpha_1, \beta_1 \in \Gamma_1^+$, $\alpha_2, \beta_2 \in \Gamma_2^+$. Then, for $i = 1, 2$, there exist $\gamma_i \in \Gamma_i^+$ such that

$$\alpha_i \cdot \gamma_i \in h_i(G_i) \quad \text{and} \quad \beta_i \wedge \gamma_i = 1_{\Gamma_i}.$$

Put $\gamma = (\gamma_1, \gamma_2)$. Obviously, $\gamma \in \Gamma^+$, $\alpha \cdot \gamma \in h_1(G_1) \times h_2(G_2)$ and $\beta \wedge \gamma = 1_\Gamma$, which means that the monomorphism h is a strong theory of quasi-divisors.

(2) $\to$ (1). Arguing as above, we can prove that, for $i = 1, 2$, the groups G_i admit theories of quasi-divisors $h_i : G_i \to \Gamma_i$, where $\Gamma \cong \Gamma_1 \times \Gamma_2$ and $h(G) \cong \cong h_1(G_1) \times h_2(G_2)$. Let $\alpha, \beta \in \Gamma_1^+$. Then, $(\alpha, 1_{\Gamma_2})$ and $(\beta, 1_{\Gamma_2})$ are elements of Γ^+, thus there exists a $(\gamma_1, \gamma_2) \in \Gamma^+$, such that

$$(\alpha \cdot \gamma_1, \gamma_2) \in h(G) \quad \text{and} \quad (\beta, 1_{\Gamma_2}) \wedge (\gamma_1, \gamma_2) = 1_\Gamma.$$

It is clear now that $\alpha \cdot \gamma_1 \in h_1(G_1)$ and $\beta \wedge \gamma_1 = 1_{\Gamma_1}$, which means that the monomorphism h_1 is a strong theory of quasi-divisors. Similarly, we can prove that the monomorphism h_2 is also a strong theory of quasi-divisors and the proof is over. □

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