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Multi Regional State Space Systems — Controller Design and Stability

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Abstract: The paper is concerned with a construction method for control functions for systems given by an algebraic description of the type $\dot{x} = f(x,u)$. The resulting control functions guaranty stability under weak conditions on the system. An approximation of the system by affine linear subsystems combined to an overall system is suggested and a feedback which transforms each affine linear subsystem into a linear one is constructed. The method is illustrated by a simulation of a highly nonlinear coupled system of three tank.

Key Words: control systems, piecewise linear approximation, blending, controller design

Mathematics Subject Classification: 93C35, 65D17, 93C15, 93D15

1. Introduction

It is a difficult task to design stable controllers for nonlinear systems. For this reason, simple but effective fuzzy controllers were used quite successfully in a number of applications, see e.g. Buckley, Hayashi (1993), Kruse, Gebhardt, Klawonn (1995). However, the analysis and especially the proof of stability for these fuzzy controllers turns out to be difficult. Work in this direction was done by Tanaka, Sugeno (1992), Cao, Rees, Feng (1996/1997), Domanski, Brdys, Tatjewski (1997), Möllers (1997).

The approach in the present paper is different. The design of the related nonlinear controllers is as simple as that of fuzzy controllers and it applies to a vast number of complex system while at the same time the stability of the resulting control system can be guaranteed. The control function can be derived directly from the algebraic description

$$\dot{x} = f(x,u) \tag{1}$$

of the system's dynamic where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$. We suggest an approximation of the system (1) by *affine linear* subsystems combined to an overall system. Statements concerning stability of systems consisting of *linear* subsystems are known Cao, Rees, Feng (1997). Therefore we construct, with methods from linear control theory, a feedback which transforms each affine linear subsystem into a linear one. The stability behavior of the approximating system is proved and the relationship to the stability

behavior of the original nonlinear system is pointed out. The obtained control function can be regarded as a Sugeno-Takagi type fuzzy controller.

The outline of the paper is as follows: Section 2 gives the main results of the paper; first we define an approximating system, then a design procedure for local state space systems is presented. Section 2.3 considers the stability of controlled local state space systems and Section 2.4 shows the relation between the stability behavior of the local state space system and the original system. Section 2.5 concludes with an adequate stability statement concerning smooth control functions. Section 3 gives an explicit example. In the Section 4 some concluding remarks are drawn.

2. Theoretical Results

We start with a short outline of the method. Assume that a system is given by

$$\dot{x} = f(x,u) \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$. We derive local, affine linear subsystems $g_\nu : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\nu = 0, \dots, N$, and combine these subsystems to obtain a global approximation g of f, such that $||g - f|| < \varepsilon$. Then feedback controllers for the subsystems are determined and combined to a smooth and global control function by means of blending methods as mentioned in Section 2.5.

2.1. Multi regional state space systems

Suppose we are given system (2) with $x = x(t) \in \mathbb{R}^n$ and $u = u(x(t)) \in \mathbb{R}^m$, $t \in [0, \infty)$ with $f(0,0) = 0$. For this we consider some area $U \subseteq \mathbb{R}^n$ which contains 0 (without loss of generality). We assume that U is partitioned into $N+1$ subareas U_ν, $\nu = 0, \dots, N$, with

$$U = \bigcup_{\nu=0}^{N} U_\nu \qquad \text{and} \qquad |U_\nu \cap U_\mu| = 0, \mu, \nu = 0, \dots, N, \nu \neq \mu \ ,$$

where $0 \in U_0$ and $|\cdot|$ denotes the usual Lebesgue-measure in $\mathbb{R}^n$. For each subarea a linear approximation $g_\nu : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ of f is to be determined. We assume that for $\nu = 0, \dots, N$ the approximation g_ν is of the form

$$f(x,u) \approx g_\nu(x,u) = A_\nu(x - x_\nu) + B_\nu(u - u_\nu) + f(x_\nu, u_\nu) \qquad \text{for } x \in U_\nu,$$

whereas x_ν is some point of the interior of U_ν, $u_\nu \in \mathbb{R}^m$, $A_\nu \in \mathbb{R}^{n\times n}$, $B_\nu \in \mathbb{R}^{n\times m}$. Without loss of generality, we will assume $x_0 = 0$ and $u_0 = 0$ for the rest of the paper.

The linear approximation may be provided, e.g., in the classical way by means of some Taylor expansion, or alternatively, by means of some linear regression method applied to identification data. However, we assume that for each $\nu = 0, \dots, N$ there is a $\varepsilon_\nu > 0$ such that

$$||g_\nu - f|| < \varepsilon_\nu \ \text{ in } U_\nu. \tag{3}$$

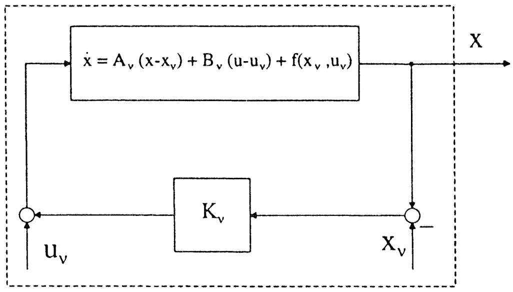

Figure 1: Local state space system with feedback (7)

The systems

$$\dot{x} = g_\nu(x, u) \tag{4}$$

are called *local state space systems*, while the resulting overall system

$$\dot{x} = g(x, u) \ , \tag{5}$$

with

$$g : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \qquad (x, u) \longmapsto g_\nu(x, u) \quad \text{for } x \in U_\nu$$

is called a *multi regional state space system*. Note, that there is

$$||g - f|| < \max_{\nu=0,\dots,N} \varepsilon_\nu \tag{6}$$

by the supposition (3). Thus, the mapping g is an approximation of f.

Moreover g has a simple structure and we will show that it is possible to derive a successful control function for such systems (5). If the approximation of f is good, i.e. $\max_{\nu=0,\dots,N} \varepsilon_\nu$ is small, then a successful control function of the multi regional state space system g is a successful control function of the original system f (see 2.4).

We start with the design process by constructing appropriate control functions for the local state space systems g_ν (4).

2.2. Control of local state space systems

We are trying to find a control function for each local state space system g_ν (4), such that the corresponding closed loop system has a stable equilibrium 0. Keep in mind all local state space systems should have the same equilibrium 0 (in global coordinates). We consider control functions of the form

$$u(x) = K_\nu(x - x_\nu) + u_\nu, \tag{7}$$

$K_\nu \in \mathbb{R}^{m \times n}$. For the resulting control loop see Figure 1. Control functions of this form are characterized by the parameter $K_\nu \in \mathbb{R}^{n \times m}$. Thus the quality of the control is only related to an appropriate choice of K_ν. Conditions for this choice are derived in the sequel.

Substituting the control function (7) in the local state space systems (4) we obtain for all $\nu = 0, \dots, N$

$$\begin{aligned} \dot{x} &= g_\nu(x, K_\nu(x - x_\nu) + u_\nu) = \\ &= (A_\nu + B_\nu K_\nu)(x - x_\nu) + f(x_\nu, u_\nu) \ , \qquad x \in U_\nu \ , \quad \nu = 0, \dots, N \ . \end{aligned}$$

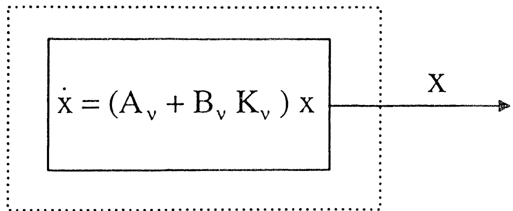

Figure 2: Resulting closed loop local state space system (9) with property (8)

In general, systems of this kind do not have the equilibrium 0. In order to derive an appropriate control function we choose the feedback matrices K_ν such that

$$(A_\nu + B_\nu K_\nu)x_\nu = f(x_\nu, u_\nu), \qquad \nu = 0, \dots, N. \tag{8}$$

We get

$$\begin{aligned} \dot{x} &= g_\nu(x, K_\nu(x - x_\nu) + u_\nu) = \\ &= \mathcal{A}_\nu x \qquad \text{for } x \in U_\nu \end{aligned} \tag{9}$$

where $\mathcal{A}_\nu = (A_\nu + B_\nu K_\nu)$. If the matrices $\mathcal{A}_\nu$ are stable, the resulting control function is successful. The corresponding closed loop local state space system is shown in Figure 2.

In the following we give sufficient conditions for the existence of a matrix K_ν, such that condition (8) is fulfilled and the closed loop matrix $\mathcal{A}_\nu$ is stable. The corresponding local control functions are used in Section 2.3 for the design of the global control function.

For sake of simplicity we omit the index ν for the remainder of this subsection.

But keep in mind that the following construction procedure is done for all subsystems.

Lemma 2.1. *Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_i) < 0$, $i = 1, 2$, and $\lambda_1 + \lambda_2 \in \mathbb{R}$, and choose $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$. Let*

$$\begin{aligned} X &:= (x, f(x,u)) \in \mathbb{R}^{n \times 2} \\ Y &:= (f(x,u) - Ax, (\lambda_2 + \lambda_1) f(x,u) - \lambda_2 \lambda_1 x - A f(x,u)) \in \mathbb{R}^{n \times 2} \end{aligned}$$

Let the following assumptions be fulfilled:

(i) There exists a matrix $\widetilde{K} \in \mathbb{R}^{m \times n}$, such that $B\widetilde{K}X = Y$.

(ii) Let $e_i \in \mathbb{R}^m$ be the i-th unit vector and let $b := Be_i \in \mathbb{R}^n$ be non zero, such that the pair $(A + B\widetilde{K}, b)$ is controllable.

(iii) The vector $k \in \mathbb{R}^n$ is such that the spectrum of $A + B\widetilde{K} + bk^T$ is $\lambda_1, \dots, \lambda_n$ with $\mathrm{Re}(\lambda_i) < 0$, $i = 3, \dots, n$.

Set $K := \widetilde{K} + e_i k^T$. Then $A + BK$ is stable and the equation $(A + BK)x = f(x,u)$ is fulfilled.

Proof. For single-input systems one way to determine a feedback is Ackermann's formula (13). We can reduce the problem of this Lemma in the multi-input case to the single-input case in the following way, see SONTAG (1990) p. 136.

Let b a non zero column of B. Choose $\widetilde{K} \in \mathbb{R}^{m\times n}$ such that $(A + B\widetilde{K}, b)$ is controllable. For randomly chosen $\widetilde{K}$ this is fulfilled with probability 1. Now apply Ackermann's formula to the system $(A + B\widetilde{K}, b)$ and compute a feedback vector k. The whole feedback matrix is given by

$$K = \widetilde{K} + e_i k^T,$$

where e_i is the i-th unity vector with $b = Be_i$. With Equation (8) we get

$$(A + B(\widetilde{K} + e_i k^T))x = f(x, u). \tag{10}$$

Sufficient for (10) are the following two conditions:

$$(A + B\widetilde{K})\,x = f(x, u) \tag{11}$$

$$B\,e_i\,k^T\,x = 0 \tag{12}$$

Equation (11) is a condition for the choice of $\widetilde{K}$. In Equation (12) $\widetilde{K}$ is implicitly contained via Ackermann's formula, to express this fact we write $k(\widetilde{K})$. Now we want to answer the question, whether it is possible to find a $\widetilde{K}$ which solves Equations (11) and (12) simultaneously. Let's have a closer look at Ackermann's formula. We denote the controllability matrix by $\mathcal{R}$ and the prescribed roots of the characteristic polynomial χ with λ_i.

$$\begin{aligned} k^T(\widetilde{K})x &= -(0\ldots01)\mathcal{R}(A + B\widetilde{K}, b)\chi(A + B\widetilde{K})x = \\ &= -(0\ldots01)\mathcal{R}(A + B\widetilde{K}, b)\prod_{i=1}^{n}(A + B\widetilde{K} - \lambda_i I)\,x = \\ &= -(0\ldots01)\mathcal{R}(A + B\widetilde{K}, b)\prod_{i=1}^{n-2}(A + B\widetilde{K} - \lambda_i I)\cdot \\ &\quad \cdot (A + B\widetilde{K} - \lambda_{n-1}I)(A + B\widetilde{K} - \lambda_n I)\,x \end{aligned} \tag{13}$$

This shows that for Equation (12) the following is sufficient

$$0 = (A + B\widetilde{K} - \lambda_{n-1}I)(A + B\widetilde{K} - \lambda_n I)\,x. \tag{14}$$

Since we are only interested in real solutions we assume $\lambda_n, \lambda_{n-1} \in \mathbb{R}$ or $\lambda_n = = \overline{\lambda}_{n-1} \in \mathbb{C}$.

If $\widetilde{K}$ is a solution of (11), by expanding Equation (14) we get another linear equation for $\widetilde{K}$

$$B\widetilde{K}f(x, u) = (\lambda_n + \lambda_{n-1})f(x, u) - \lambda_n\lambda_{n-1}x - Af(x, u). \tag{15}$$

Putting Equations (11) and (15) together we get the matrix equation of assumption (i). Eq. (15) implies Eq. (12) and therefore together with (11) yields (10). □

2.3. Stability of multi regional state space systems

In the present paragraph we give sufficient conditions for the stability of a closed loop multi regional state space systems (9). For a composite system of differential equations of this type, we state the following result:

Theorem 2.2. *Let* $U \subseteq \mathbb{R}^n$ *with*

$$U = \bigcup_{\nu=0}^{N} U_\nu \qquad \text{and} \qquad |U_\nu \cap U_\mu| = 0, \mu, \nu = 0, \dots, N, \nu \neq \mu \ . \tag{16}$$

Let

$$\dot{x} = \mathcal{A}_\nu x \qquad \text{for } x \in U_\nu \tag{17}$$

be given with $\mathcal{A}_\nu \in \mathbb{R}^{n \times m}$ *stable and normal for* $\nu = 0, \dots, N$. *Moreover, we assume that the solution* $x(t)$ *changes the subareas at most finitely many times. Then the system (17) is asymptotically stable.*

Proof. Let $t_0 = 0$ be the time where $x(t_0)$ equals the starting point $x_0 \in U$. Let $i(t)$ denote the index of the subarea that satisfies $x(t) \in U_{i(t)}$. Observe that by assumption, i is well–defined on $[0, \infty]$ with exception of finitely many points $0 < < t_1 < \dots < t_N < \infty$ which are exactly the points where x changes the subarea. Then i is a piecewise constant function with the jumps $t_1, t_2, \dots$. We have the following representation for the starting point $x_0 \in U_{i(t_0)}$ in terms of eigenvectors of $\mathcal{A}_{i(t_0)}$ in the region $U_{i(t_0)}$:

$$x_0 = \sum_{q=1}^{n} \alpha_q a_{q,i(0)} \ .$$

Moreover, we have

$$x(t) = \sum_{q=1}^{n} \alpha_q a_{q,i(t)} e^{\lambda_{q,i(t)} t} \ , \quad t \in [0, t_1) \ . \tag{18}$$

Now we introduce the terminology $\Phi_{i(t)} := (a_{1,i(t)}, \dots a_{n,i(t)})$ for the matrix of eigenvectors and $\Lambda_{i(t)} := \operatorname{diag}(e^{\lambda_{1,i(t)}(t-t_k)}, \dots, e^{\lambda_{n,i(t)}(t-t_k)})$ for the diagonal matrix of eigenfunctions of $\mathcal{A}_{i(t)}$ in $U_{i(t)} = U_{i(t_k)}$, where $t_k = \max\{t_\nu \leq t \mid \nu \in \in \mathbb{N}_0\}$. Then (18) can be rewritten as

$$x(t) = \Phi_{i(t)} \cdot \Lambda_{i(t)} \cdot \Phi_{i(t)}^{-1} \cdot x_0 \ , \quad t \in [t_0, t_1) \ . \tag{19}$$

At the point $t = t_1$, the piecewise linear approximation of the control function has a jump (and so does i), whereas for $t \in (t_1, t_2)$ the control system is linear again. Therefore, in this situation we express $x(t)$ in terms of the eigenvector basis $a_{1,i(t)}, \dots, a_{n,i(t)}$, $t \in (t_1, t_2)$, and obtain

$$x(t) = (\Phi_{i(t)} \Lambda_{i(t)} \Phi_{i(t)}^{-1}) \cdot (\Phi_{i(t_0)} \Lambda_{i(t_0)} \Phi_{i(t_0)}^{-1}) \cdot x_0, \ t \in [t_1, t_2) \ .$$

In general, we have (with the notation $i(t_\nu -) := \lim_{t\to t_\nu -} i(t)$ for the left-sided limit of i at some breakpoint t_ν)

$$x(t) = (\Phi_{i(t)}\Lambda_{i(t)}\Phi_{i(t)}^{-1}) \prod_{\nu=1}^{k} (\Phi_{i(t_\nu -)}\Lambda_{i(t_\nu -)}\Phi_{i(t_\nu -)}^{-1})x_0 \ , \quad t \in (t_k, t_{k+1}) \ ,$$

thus

$$\|x(t)\|_2 \le \|\Lambda_{i(t)}\|_2 \left(\prod_{\nu=1}^{k} \|\Lambda_{i(t_\nu -)}\|_2 \right) \|x_0\|_2 \quad t \in (t_k, t_{k+1}) \ , \tag{20}$$

since $\Phi_{i(t)}$ is unitary for $t \ge 0$. We observe

$$\|\Lambda_{i(t_\nu)}\|_2 = \max_{\mu=1}^{n} |e^{\lambda_{\mu,i(t)}(t-t_\nu)}| = e^{\mathrm{Re}\lambda_{1,i(t)}(t-t_\nu)} \ , t \in (t_\nu, t_{\nu+1}).$$

Here, we assume $0 < \mathrm{Re}\lambda_1 \le \mathrm{Re}\lambda_2 \le \ldots \le \mathrm{Re}\lambda_n$ Thus with $M := \max_{\mu=0}^{N} \mathrm{Re}\lambda_{1,\mu}$, we can estimate

$$\|x(t)\|_2 \le e^{M(t-t_k)} \prod_{\mu=1}^{k} e^{M(t_k - t_{k-1})} \|x_0\|_2 = e^{M(t-t_0)} \|x_0\|_2 \ , \quad t \in (t_k, t_{k+1}) \ ,$$

which holds for any t and k, thus $\lim_{t\to\infty} x(t) = 0$, and we are done. □

Assume we are given a control function that is defined piecewise on each subarea U_ν by means of the construction method of Section 2.2. As a result of Theorem 2.2 we get, that if the closed loop matrices $\mathcal{A}_\nu$ are normal then this control function is a successful control function for the multi regional state space system g (see (5)).

From now on we assume that such a control function can be found and denote this specific control function by

$$\hat{u}(x) = K_\nu(x - x_\nu) + u_\nu \qquad \text{for } x_\nu \in U_\nu \ .$$

2.4. Stability of approximating systems

As we have shown above it is possible to approximate a given system f (2) by a local state space system (5) and to find an appropriate control function $\hat{u}$ such that the local state space system (5) is stable. In this section we are going to show that the control function $\hat{u}$ is an appropriate control function for the original system (2), too. First we state the following Lemma. It gives an estimation for the difference of the solutions of two systems, if we assume that the systems do not vary too much.

Lemma 2.3. *Let*

$$\dot{x} = f(x), \tag{21}$$

$$\dot{\tilde{x}} = g(\tilde{x}) \tag{22}$$

be two closed loop systems with $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and let* f *fulfill the Lipschitz condition* $\|f(x) - f(y)\| \le L\|x - y\|$ *for all* $x, y \in U \subseteq \mathbb{R}^n$*, where* $L > 0$*. Let further be*

$$\|f - g\| < \varepsilon \, .$$

Then for the solutions x and $\tilde{x}$ of (21) and (22) the following statement holds

$$\|x(t,x_0) - \tilde{x}(t,x_0)\| \le \varepsilon \, \frac{e^{Lt}-1}{L}\,, \tag{23}$$

where x_0 denotes the initial value of (21) and (22).

Proof. Let us have a closer look at the distance between the two solutions

$$\begin{aligned}
\|x(t,x_0) - \tilde{x}(t,x_0)\| &= \| \int_0^t \dot{x}(\tau,x_0) - \dot{\tilde{x}}(\tau,x_0)\, d\tau \| \le \\
&\le \int_0^t \|\dot{x}(\tau,x_0) - \dot{\tilde{x}}(\tau,x_0)\|\, d\tau = \\
&= \int_0^t \|f(x(\tau,x_0)) - g(\tilde{x}(\tau,x_0))\|\, d\tau \le \\
&\le \int_0^t \|f(x(\tau,x_0)) - f(\tilde{x}(\tau,x_0))\| + \\
&\quad + \|f(\tilde{x}(\tau,x_0)) - g(\tilde{x}(\tau,x_0))\|\, d\tau \le \\
&\le \int_0^t L\, \|x(t,x_0) - \tilde{x}(t,x_0)\|\, d\tau + \varepsilon\, t
\end{aligned}$$

This is a Bellman-Gronwall inequality, see CALLIER, DESOER (1991) p. 475. From there it follows

$$\begin{aligned}
\|x(t,x_0) - \tilde{x}(t,x_0)\| &\le \varepsilon\, t + \left| \int_0^t \varepsilon\, L\, \tau\, \mathrm{e}^{\left|\int_\tau^t L\, ds\right|}\, d\tau \right| = \\
&= \varepsilon\, t + \varepsilon\, L \int_0^t \tau\, \mathrm{e}^{L(t-\tau)}\, d\tau = \\
&= \varepsilon\, t + \varepsilon\, L\, \mathrm{e}^{Lt} \int_0^t \tau\, \mathrm{e}^{-L\tau}\, d\tau = \\
&= \varepsilon\, t + \varepsilon\, L\, \mathrm{e}^{Lt} \left(-\frac{1}{L} t\, \mathrm{e}^{-Lt} - \frac{1}{L^2}(\mathrm{e}^{-Lt} - 1) \right) = \\
&= \varepsilon\, \frac{\mathrm{e}^{Lt} - 1}{L}.
\end{aligned}$$

□

From inequality (23) we can derive a sufficient condition to decide whether the control function $\hat{u}$ for the multi regional state space system g (5) is successful for the original system f (2).

Theorem 2.4. *Consider the closed loop original system*

$$\dot{x} = f(x) \tag{24}$$

with $f(0) = 0$ and the multi regional state space system

$$\dot{\tilde{x}} = g(\tilde{x}) \tag{25}$$

where both control loops are closed by the feedback $\hat{u}$ *and let the local state space systems of (25) be stable. Moreover, let* $\|f-g\| < \varepsilon$ *and let* $M = \max_{\mu=0}^{N} \mathrm{Re}\lambda_{1,\mu}$ *be as in the proof of Theorem 2.2. Let* $\delta_0 > 0$ *be such that the ball* $B_{\delta_0}(0)$ *is contained in the domain of attraction (*HAHN *(1967) p. 108) of the system* $\dot{x} = \mathcal{A}_0 x$ *and let*

$$\delta(t) = \varepsilon\, \frac{e^{Lt}-1}{L} + e^{Mt}\|x_0\| \,.$$

(i) If δ *is monotone increasing, then* $\|x_0\| = \delta(0) \le \delta_0$ *is sufficient for the asymptotic stability of the equilibrium 0 of the original system (24).*

(ii) Otherwise

$$\delta\left(\frac{1}{M-L}\ln\left(\frac{-\varepsilon}{M\,\|x_0\|}\right)\right) \le \delta_0$$

is sufficient for the asymptotic stability of the equilibrium 0 of the original system (24).

Proof. From Lemma 2.3 we have

$$\Big|\|x(t,x_0)\| - \|\tilde{x}(t,x_0)\|\Big| \le \|x(t,x_0) - \tilde{x}(t,x_0)\| \le \varepsilon\, \frac{e^{Lt}-1}{L} \,.$$

First case: $\|x(t,x_0)\| - \|\tilde{x}(t,x_0)\| < 0$. Then the stability of (24) follows from the asymptotic stability of (25).

Second case: $\|x(t,x_0)\| - \|\tilde{x}(t,x_0)\| \ge 0$. Then

$$\begin{aligned}\|x(t,x_0)\| &\le \|x(t,x_0) - \tilde{x}(t,x_0)\| + \|\tilde{x}(t,x_0)\| \le \\ &\le \varepsilon\, \frac{e^{Lt}-1}{L} + e^{Mt}\|x_0\| = \\ &= \delta(t)\end{aligned} \tag{26}$$

If the function δ is increasing, then the best value of the estimation (26) is at $\delta(0)$. If $\delta(0) \le \delta_0$, then the initial value $x_0 = \delta(0)$ is in the domain of attraction and the stability follows.

Otherwise δ has a relative minimum at

$$t_{min} = \frac{1}{M-L}\ln\left(\frac{-\varepsilon}{M\,\|x_0\|}\right) .$$

If $\delta(t_{min}) \le \delta_0$, then we have

$$\|x(t,x_0)\| \le \delta_0$$

that means the solution of the original system (24) has reached the domain of attraction of 0 and the asymptotic stability follows. □

2.5. Smooth control functions

In the previous sections of this chapter we have described a construction method for a special class of control functions (see Sec. 2.1 and Sec. 2.2) of nonlinear systems and we have given sufficient conditions for the success of those functions (see Sec. 2.3 and Sec. 2.4). Though the construction method is very simple and the conditions for the stability of the control functions are weak, there might be one disadvantage of the obtained control: in general the control function is not continuous. To elude that problem we consider the following Lemma, which is closely related to Lemma 2.3 and Theorem 2.4. Note, that in Lemma 2.5 different control functions but the same system description are considered, while the above mentioned statements are concerned with different systems but the same control.

Lemma 2.5. *Let there be given a system*

$$\dot{x} = f(x,u)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, *and let* f *fulfill the Lipschitz condition* $\|f(x,\cdot) - f(y,\cdot)\| \leq L\|x-y\|$ *for all* $x, y \in \hat{U} \subseteq \mathbb{R}^n \times \mathbb{R}^m$. *Let be* $\hat{u} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *a successful state space controller and let be* $\tilde{u} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *a continuous mapping such that*

$$\|\hat{u} - \tilde{u}\| < \varepsilon \ .$$

Define the closed loop systems

$$\dot{\hat{x}} = \hat{f}(\hat{x}) = f(\hat{x}, \hat{u}(\hat{x})), \tag{27}$$

$$\dot{\tilde{x}} = \tilde{f}(\hat{x}) = f(\tilde{x}, \tilde{u}(\tilde{x})). \tag{28}$$

Then the following statements hold:

1. $\tilde{f}$ *fulfills a Lipschitz condition, i.e.* $\|\tilde{f}(x) - \tilde{f}(y)\| \leq L'\|x-y\|$ *for all* $x, y \in \hat{U} \subseteq \subseteq \mathbb{R}^n$.
2. *There is an* $\varepsilon' > 0$ *such that for each initial value* x_0 *of (27) and (28) there is*
$$\|\hat{x}(t,x_0) - \tilde{x}(t,x_0)\| \leq \varepsilon' \frac{e^{L't} - 1}{L'}.$$
3. *Assume that the equilibrium 0 of the system (27) is exponentially stable, i.e.* $\|\hat{x}(t,x_0)\| \leq e^{M'(t-t_0)}\|x_0\|$ *and that the ball* $B_{\delta_0}(0)$ *is contained in the domain of attraction of the system* $\dot{x} = \mathcal{A}_0\, x$. *Let be* $\tilde{u}|_{B_{\delta_0}(0)} = \hat{u}|_{B_{\delta_0}(0)}$ *and define*
$$\delta(t) = \varepsilon' \ \frac{e^{L't} - 1}{L'} + e^{Mt}\|x_0\| \, .$$

(i) If δ *is monotone increasing, then* $\|x_0\| = \delta(0) \leq \delta_0$ *is sufficient for the asymptotic stability of the original system (28).*

(ii) Otherwise

$$\delta\left(\frac{1}{M' - L'}\ln\left(\frac{-\varepsilon'}{M'\,\|x_0\|}\right)\right) \leq \delta_0$$

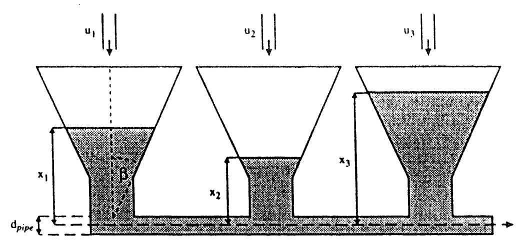

Figure 3: System of three conical tanks

is sufficient for the asymptotic stability of the original system (28).

A proof can be obtained by applying the corresponding statements from the previous sections.

The problem of finding a smooth control is reduced to the problem of determining a continuous approximation of a non continuous function. Various techniques can be applied here. One possible technique is blending of various local control functions as it can be found in Möllers, van Laak (1998). This method has been used in the following example.

3. Numerical Experiments

Let us consider a system of 3 conical tanks according to Figure 3. They are joined together by pipes with diameter d_{pipe}. Tank 3 has an outlet of diameter d_{pipe} . Every tank has an input which can be controlled.

The states of the system are the heights of the filling x_1, x_2, x_3. The control variables are the inputs of the tanks u_1, u_2, u_3, they are used to control the heights x_1, x_2, x_3 in order to converge to given set heights H_1, H_2, H_3. The system equation

$$\dot{x} = f(x, u)$$

is explicitly given by

$$\begin{aligned}
f_1(x,u) &= \frac{1}{F_1(x_1)}\left(-\mathrm{sgn}(x_1 - x_2)\, F\sqrt{2g|x_1 - x_2|} + u_1\right)\\
f_2(x,u) &= \frac{1}{F_2(x_2)}\Big(\mathrm{sgn}(x_1 - x_2)\, F\sqrt{2g|x_1 - x_2|} -\\
&\quad - \mathrm{sgn}(x_2 - x_3)\, F\sqrt{2g|x_2 - x_3|} + u_2\Big)\\
f_3(x,u) &= \frac{1}{F_3(x_3)}\left(\mathrm{sgn}(x_2 - x_3)\, F\sqrt{2g|x_2 - x_3|} - F\sqrt{2g\, x_3} + u_3\right)
\end{aligned}$$

where $F = \pi(d_{pipe}/2)^2$ is the surface of the pipes and $F_1(x_1)$, $F_2(x_2)$, $F_3(x_3)$ is the surface of the water depending on the height in each tank, g is the gravitation constant. In this example the surfaces depend on the shape of the cones. We have

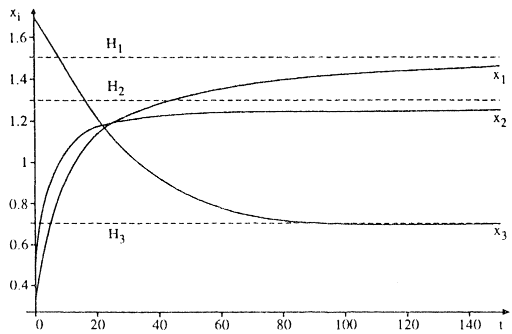

Figure 4: Fuzzy state space controller

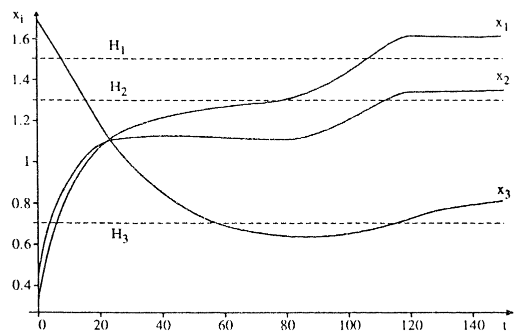

Figure 5: Linear controller

$$F_i(x_i) = \pi(x_i \tan\beta)^2$$

for $i = 1, 2, 3$, For the plant constants we choose $d_{pipe} = 0.1$ m and $\beta = 30°$. Suppose the tanks have a maximal heights of 2 m.

We construct a global controller which guarantees stability with the above mentioned procedure. We use 18 linearization points h_1, h_2, h_3 whereas for either h_1 and h_3 the heights are 0.5 m, 1 m, 1.5 m and for h_2 they are 0.75 m, and 1.25 m. In each linearization point we determine the corresponding affine linear system and the local linear feedback law which makes the subsystems linear.

For every local system the feedback law is chosen, such that its eigenvalues are $\lambda_1 = -0.05$, $\lambda_2 = -0.08$, $\lambda_3 = -0.1$.

In Figure 4 a simulation of the system is shown, where the set heights are $H_1 :\; := 1.5$ m, $H_2 := 1.25$ m and $H_3 := 0.7$ m. Figure 5 shows the simulation of the system controlled by a linear feedback controller, which was design for the linearization point (1.5 m, 1.25 m, 0.5 m). It turns out that the system behavior is much faster and more accurate, if the constructed multi regional state space controller is used.

4. Concluding Remarks

We have shown that it is possible to find a appropriate control function for each system that can be described by (2) if the following conditions hold:

(a) the function f fulfills a Lipschitz condition on the considered area; (b) the multi regional system (9) can be controlled by a control function $\hat{u}$. Both conditions are very weak in applications hence we assume that in many applications successful control functions can be found by means of the method described above.

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