Acta Mathematica et Informatica Universitatis Ostraviensis

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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 6 (1998), No. 1, 183--190

Persistent URL: http://dml.cz/dmlcz/120532

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Abstract: Harrison n-equivalence between two fields containing a primitive n–th root of unity is an isomorphism between the groups of n–th power classes of these fields which preserves the norm subgroups of cyclic extensions of degree n. Hilbert n-equivalence between two number fields containing a primitive n–th root of unity is an isomorphism between the groups of n–th power classes of these fields preserving Hilbert symbols of degree n. Both notions have been invented and used in the case $n = 2$ within the theory of quadratic forms in discussion of the structure of Witt rings. In the paper we use Harrison n-equivalence and Hilbert n-equivalence in discussion of the structure of the factor ring $\mathsf{K}_{/n}$ of the Milnor K-ring K by its ideal $n\,\mathsf{K}$. We introduce Milnor n-equivalence and classify number fields with respect to it. At the end of the paper certain 'going up' property of Milnor n-equivalence is discussed.

Key Words: Harrison equivalence, Hilbert equivalence, Milnor equivalence, Milnor K-ring, number field

Mathematics Subject Classification: 11R21, 12F05

1. Introduction

The notions of Harrison equivalence and Hilbert equivalence were invented in connection with quadratic forms and were used in discussion of the structure of Witt rings and so called Witt equivalence. Quite recently both equivalences were generalized to higher degree Harrison equivalence and higher degree Hilbert equivalence, respectively (see [CS1] [CS2]). Although this generalization seems to be quite natural, there is no generalization of the Witt ring which could be a natural motivation for this procedure. The aim of this paper is threefold. First we want to point out that Milnor K–ring (in fact its quotient) is a quite nice object for discussion in connection with higher degree Harrison equivalence and higher degree Hilbert equivalence. The notion of Milnor equivalence that we introduce in section 1 is a counterpart of the Witt equivalence. Secondly we want to use the main results of [CS1] [CS2] on higher degree Harrison equivalence and higher degree Hilbert equivalence to discuss Milnor equivalence over number fields. It is done in section 2. Thirdly we formulate 'going up' property of Milnor equivalence that we verify in the case of number fields.

Supported by KBN (State Commitee for Scientific Research) under Grant PB Nr 2 P03A 02412

2. Harrison equivalence and Milnor equivalence

For a field F containing a primitive n-th root of unity ξ_n and for $a \in \dot{F}$ we write $\mathsf{N}^n_F(a)$ for the norm group of the F-algebra $F[X]/(X^n - a)$. By our assumption on the roots of unity $\mathsf{N}^n_F(a)$ equals the norm group of the cyclic field extension $F(\sqrt[n]{a})/F$. Since $\dot{F}^n \subseteq \mathsf{N}^n_F(a)$, it is possible, and sometimes more convenient, to consider the group $\mathsf{N}^n_F(a)$ as a subroup of the group $\dot{F}/\dot{F}^n$. We also often use the same character to denote an element a in the multiplicative group $\dot{F}$ of the field F as well as the coset $a\dot{F}^n$ in the group $\dot{F}/\dot{F}^n$.
Let K and L be fields containing ξ_n. *Harrison n-map* between the fields K and L is an isomorphism

$$t : \dot{K}/\dot{K}^n \to \dot{L}/\dot{L}^n$$

of the groups of n-th power classes of K and L satisfying the following conditions:
$t(-1) = -1$,
$a \in \mathsf{N}^n_K(b) \iff ta \in \mathsf{N}^n_L(tb)$ for all $a, b \in \dot{K}/\dot{K}^n$.

We write $K \overset{n}{\simeq} L$ when the fields K and L are Harrison n-equivalent, that is, when there exists a Harrison n-map between these fields. In [CS1] and [CS2] Harrison n-equivalence is called Harrison equivalence of degree n.

Remark 2.1. *When n is odd the condition $t(-1) = -1$ follows from the fact that t is a group isomorphism, because then $-1 = 1$ in groups $\dot{K}/\dot{K}^n$ and $\dot{L}/\dot{L}^n$.* Let

$$\mathsf{K}_{/n}(F) = \mathsf{K}_{0/n}(F) \oplus \mathsf{K}_{1/n}(F) \oplus \ldots$$

be the factor ring of the graded Milnor ring

$$\mathsf{K}(F) = \mathsf{K}_0(F) \oplus \mathsf{K}_1(F) \oplus \ldots$$

by its ideal $n\,\mathsf{K}(F)$. The ring $\mathsf{K}_{/n}(F)$ is a tensor algebra of $\mathsf{K}_{1/n}(F) \cong \dot{F}/\dot{F}^n$ reduced modulo the ideal $\mathcal{I}_n$ generated by the elements

$$a\dot{F}^n \otimes (1-a)\dot{F}^n, \ a \in \dot{F}.$$

For $a_1, \ldots, a_k \in \dot{F}$ denote

$$\{a_1, \ldots, a_k\}_n := (a_1\dot{F}^n \otimes \ldots \otimes a_k\dot{F}^n) + \mathcal{I}_n.$$

If $\xi_n \in F$ or n is square-free and coprime with the characteristic of F then (see [M, Corollary 15.11, Theorem 15.12]):

$$\{a, b\}_n = 0 \iff a \in \mathsf{N}^n_F(b). \tag{1}$$

As the result we get the following

Proposition 2.2. *Assume the fields K and L contain ξ_n and*

$$t : \dot{K}/\dot{K}^n \to \dot{L}/\dot{L}^n$$

is a group isomorphism such that $t(-1) = -1$. *The map* t *is a Harrison* n*-map if and only if there exists a group isomorphism*

$$\mathcal{T} : \mathsf{K}_{2/n}(K) \longrightarrow \mathsf{K}_{2/n}(L)$$

such that $\mathcal{T}(\{a,b\}_n) = \{ta, tb\}_n$ *for* $a, b \in \dot{K}$. Let K and L be fields containing ξ_n. *Milnor* n*-map* between the fields K and L is a pair $(t, \mathcal{T})$ where

$$t : \dot{K}/\dot{K}^n \to \dot{L}/\dot{L}^n$$

is a group isomorphism such that $t(-1) = -1$ (vacuosly satisfied when n is odd), and

$$\mathcal{T} : \mathsf{K}_{2/n}(K) \longrightarrow \mathsf{K}_{2/n}(L)$$

is a group isomorphism such that

$$\mathcal{T}(\{a,b\}_n) = \{ta, tb\}_n \text{ for } a, b \in \dot{K}.$$

We write $K \overset{n}{\equiv} L$ when the fields K and L are Milnor n-equivalent, that is, when there exists a Milnor n-map between these fields. From the previous Proposition we have

Corollary 2.3.

$$K \overset{n}{\cong} L \iff K \overset{n}{\equiv} L.$$

The group isomorphisms t *in Milnor and Harrison* n*-map between* K *and* L *may be chosen to be the same.* Just from the definition of $\mathsf{K}_{/n}$ we obtain the following

Proposition 2.4. *The fields* K *and* L *are Milnor* n*-equivalent if and only if there exists an isomorphism* $\Phi : \mathsf{K}_{/n}(K) \longrightarrow \mathsf{K}_{/n}(L)$ *of graded rings such that* $\Phi(\{-1\}_n) = \{-1\}_n$. The concept of Harrison map comes from the theory of quadratic forms. For $n = 2$ the group $\mathsf{N}^n_K(a)$ is just the group $D_K(1, -a)$ of elements represented by the form $\langle 1, -a \rangle$ and it turns out that $K \overset{2}{\simeq} L$ if and only if the fields K and L are Witt equivalent, that is, their Witt rings $W(K)$ and $W(L)$ are isomorphic (see [Sz3] for more detailed comments). For n greater than 2 there is no counterpart of the Witt ring. However, there is the graded Milnor ring $\mathsf{K}_{/n}$ and it seems natural to consider fields with respect to Milnor n-equivalence as it is done for Witt equivalence in the case $n = 2$.

It turns out that we can reduce the problem of Milnor n-equivalvence to Milnor l-equivalence for l runnig through all coprime factors of n thanks to the following.

Lemma 2.5. *([CS1, Lemma 2.3]) Suppose* $n = k_1 \cdot \ldots \cdot k_r$ *and* $GCD(k_i, k_j) = 1$ *for* $i, j = 1, \ldots, r$, $i \neq j$. *Then the map*

$$\varphi_F : \dot{F}/\dot{F}^n \longrightarrow \dot{F}/\dot{F}^{k_1} \times \ldots \times \dot{F}/\dot{F}^{k_r}, \ \varphi_F(a\dot{F}^n) = (a\dot{F}^{k_1}, \ldots, a\dot{F}^{k_r}) \ \text{for } a \in \dot{F},$$

is a group isomorphism and $\varphi_F(\mathsf{N}^n_F(a)) = \mathsf{N}^{k_1}_F(a) \times \ldots \times \mathsf{N}^{k_r}_F(a)$.

By this lemma the map

$$\varphi_F : \mathsf{K}_{1/n}(F) \longrightarrow \mathsf{K}_{1/k_1}(F) \times \ldots \times \mathsf{K}_{1/k_r}(F)\ ,\ \varphi_F(\{a\}_n) = (\{a\}_{k_1}, \ldots , \{a\}_{k_r})$$

for $a \in \dot{F}$, is a group isomorphism that induces a group isomorphism

$$\varphi'_F : \mathsf{K}_{2/n}(F) \longrightarrow \mathsf{K}_{2/k_1}(F) \times \ldots \times \mathsf{K}_{2/k_r}(F)$$

such that

$$\varphi'_F(\{a,b\}_n) = (\{a,b\}_{k_1}, \ldots , \{a,b\}_{k_r}) \text{ for } a, b \in \dot{F}.$$

As a consequence we have the following theorem.

Theorem 2.6. ([CS1, Th.2.2]) *Under the assumption of the above lemma we have*

$$K \overset{n}{\equiv} L \quad \Longleftrightarrow \quad K \overset{k_i}{\equiv} L \ \text{for } i = 1, \ldots , r.$$

Moreover, any Milnor n-map $(t, \mathcal{T})$ between K and L determines uniquely Milnor k_i-maps $(t_i, \mathcal{T}_i)$ between K and L for $i = 1, ..., r$ such that

$$\varphi_L \circ t = (t_1 \times \ldots \times t_r) \circ \varphi_K.$$

3. Milnor equivalence for number fields

For number fields Harrison n-equivalence (thus also Milnor n-equivalence) can be described via so called Hilbert n-equivalence called also Hilbert equivalence of degree n in [CS1]. Recall the definition. Let n be a natural number > 1. Let K and L be number fields containing a primitive $n-$th root of unity. Hilbert n-equivalence between global fields K and L is defined to be a triple of maps

$$f : \mu_n(K) \to \mu_n(L), \quad t : \dot{K}/\dot{K}^n \to \dot{L}/\dot{L}^n, \quad T : \Omega(K) \to \Omega(L),$$

where f is an isomorphism between the groups of $n-$th roots of unity, t is an isomorphism between the class groups of $n-$th powers of the two fields and T is a bijective map between the sets of all primes of K and L, with (f, t, T) preserving Hilbert symbols of $n-$th degree in the sense that

$$(a, b)_P^f = (ta, tb)_{TP} \text{ for all } a, b \in \dot{K}/\dot{K}^n,\ P \in \Omega(K).$$

We write $K \overset{n}{\sim} L$ when K and L are Hilbert n-equivalent. The main result of [CS1, Corollary 4.3] says the following.

Theorem 3.1. *Let l be a prime number and let K and L be number fields containing ξ_l. Then K and L are Harrison l-equivalent if and only if K and L are Hilbert l-equivalent.*

Remark 3.2. Analysis of the proof of the above Theorem in [CS1] draws a conclusion that the group isomorphisms in the definitions of Harrison and Hilbert l-maps may be chosen to be the same. Following [Sz3] and [CS2] for a number field K containing a primitive l-th root of unity and l being an prime number we can define an object $A_l(K)$ called *Hilbert l-equivalence invariant of K*. We do it separately for $l = 2$ and for $l \neq 2$. In the case $l = 2$ we define

$$A_2(K) = (m,\, r,\, s,\, g_2;\, m_1, \dots, m_{g_2};\, s_1, \dots, s_{g_2})$$

whereas for $l \neq 2$ we put

$$A_l(K) = (m,\, g_l;\, m_1, \dots, m_{g_l})$$

where

m is the degree of K over $\mathbf{Q}$,

r is the number of infinite real primes of K,

s is the level of K (s=0 when K is formally real)

g_l is the number of elements of the set $\Omega_l(K) = \{P_1, \dots, P_{g_l}\}$ of l-adic primes of K,

$m_1, \dots, m_{g_l}$ are the local degrees $[K_{P_i} : \mathbf{Q}_l]$ of l-adic completions $K_{P_1}, \dots, K_{P_{g_l}}$ of K over $\mathbf{Q}_l$,

$s_1, \dots, s_{g_2}$ are the levels of dyadic completions $K_{P_1}, \dots, K_{P_{g_2}}$. It is always supposed that $m_1 \leq \dots \leq m_{g_l}$ and $s_i \leq s_{i+1}$ provided $m_i = m_{i+1}$. We have to define $A_l(K)$ separately for $l = 2$ and odd primes, because if $l \neq 2$ then $r = 0$ and the notion of the level has no sense. Composing main results of [Sz1] for $l = 2$ or [CS1] and [CS2] for $l \neq 2$ with the observation from the previous section we get the following.

Theorem 3.3. *([Sz1, Theorem 1.5], [CS1, Corollary 4.3], [CS2, Corollary 5.2]) If n is a square-free natural number, K and L are number fields containing a primitive n-th root of unity, then the following conditions are equivalent:*

1. *The fields K and L are Milnor n-equivalent.*
2. *The fields K and L are Hilbert l-equivalent for every prime divisor l of n.*
3. *$A_l(K) = A_l(L)$ for every prime divisor l of n.*

Proof. By Theorem 2.6 and Corollary 2.3 the statement (1) is equivalent to 'the fields K and L are Harrison l-equivalent for every prime divisor l of n' which is equivalent to (2) by [CS1, Corollary 4.3]. The proof of equivalence of (2) and (3) the reader can find in [CS2, Corollary 5.2]. □

Remark 3.4. The statement (3) gives a very condensed information on invariants of Milnor n-equivalent fields. We should notice one more consequence of the above Theorem. Namely, if $(t, \mathcal{T})$ is a Milnor n-map between K and L, then for any prime divisor l of n there exists a bijective map

$$T : \Omega_l(K) \to \Omega_l(L)$$

such that the natural homomorphisms

$$t : \dot{K}_p/\dot{K}_p^l \to \dot{L}_{TP}/\dot{L}_{TP}^l, \quad t : \dot{K}_p/\dot{K}_p^n \to \dot{L}_{TP}/\dot{L}_{TP}^n \quad \text{for } P \in \Omega_l(K)$$

as well as the homomorphism $t : \dot{K}/\dot{K}^l \to \dot{L}/\dot{L}^l$ induced by the group isomorphism $t : \dot{K}/\dot{K}^n \to \dot{L}/\dot{L}^n$ are isomorphisms. It is a consequence of Corollary 2.3, Theorem 2.6 and [CS2, Lemma 2.3].

4. 'Going up' property

Let $\mathcal{R}$ be a family of fields. We say that Milnor n-equivalence has the *'going up' property with respect to* $\mathcal{R}$ if for any $K, L \in \mathcal{R}$, any Milnor n-map $(t, \mathcal{T})$ between the fields K and L and any $a \in \dot{K}$ the fields $K(\sqrt[n]{a})$ and $L(\sqrt[n]{ta})$ are Milnor n-equivalent.

The aim of this section is to show that Milnor n-equivalence, n being a square-free natural number, has the 'going up property'with respect to the family of number fields containig a primitive n-th root of unity. We obtain it by examination of the invariant A_l of Kummer extention $K(\sqrt[n]{a})$ of K for any prime l dividing n. For $n = l = 2$ it was done and used for describing Witt equivalence classes of all quadratic extensions of a given number field by K.Szymiczek [Sz2]. Suppose n is a square-free natural number, $l > 2$ is a prime divisor of n, K is a number field containig ξ_n, $E = K(\sqrt[n]{a})$, where $a \in \dot{K}$, and

$$A_l(K) = (m,\, g_l;\, m_1, \dots, m_{g_l}), \quad A_l(E) = (M, G_l; M_1, \dots, M_{G_l})$$

are Hilbert $l-$equivalence invariants of K and E, respectively. If n is even, then $l = 2$ should be taken into account. In this case let

$$A_2(K) = (m,\, r,\, s,\, g_2;\, m_1, \dots, m_{g_2};\, s_1, \dots, s_{g_2})$$

and

$$A_2(E) = (M,\, R,\, S,\, G_2;\, M_1, \dots, M_{G_2};\, S_1, \dots, S_{G_2})$$

be Hilbert $2-$equivalence invariants of K and E, respectively. Denote by $\mathtt{t}_n(a)$ the order of $a\dot{K}^n$ in the group $\dot{K}/\dot{K}^n$ and by $\mathtt{t}_{n,p}(a)$ the order of $a\dot{K}_p^n$ in the group $\dot{K}_p/\dot{K}_p^n$ for $p \in \Omega(K)$.

Theorem 4.1. *The invariant $A_l(E)$ is determined as follows.*

(1) $M = \mathtt{t}_n(a)m$.
(2) $G_l = \mathtt{t}_n(a) \sum_{p \in \Omega_l(K)} \frac{1}{\mathtt{t}_{n,p}(a)}$.
(3) *For $P_j \in \Omega_l(E)$ and $p_i \in \Omega_l(K)$, if P_j lies above p_i, then $M_j = \mathtt{t}_{n,p_i}(a) m_i$.*
(4) *Ifn $= l = 2$ then $R = 2r(a)$, where $r(a) = |\{p \in \Omega_\infty(K) : a \in \dot{K}_p^2\}|$ and Ω_∞ is the set of infinite real primes of K. If $n > l = 2$ then $r = R = 0$.*
(5) *If $l = 2$, then*

$$S = \begin{cases} 0 & \textit{if } R \neq 0, \\ 1 & \textit{if } s = 1 \textit{ or } -a \in \dot{K}^2, \\ 4 & \textit{if } R = 0 \textit{ and } S_j = 4 \textit{ for at least one } j, \\ 2 & \textit{otherwise} \end{cases}$$

(6) *Suppose $l = 2$. If S_j is the level of E_{P_j}, s_i is the level of K_{p_i} and P_j lies over p_i, then*

$$S_j = \begin{cases} s_i & \text{if } s_i = 1 \text{ or } a \in \dot{K}_{p_i}^2, \\ 1 & \text{if } s_i \neq 1,\ a \notin \dot{K}_{p_i}^2 \text{ and } -a \in \dot{K}_{p_i}^2, \\ 2 & \text{if } s_i \neq 1,\ a \notin \dot{K}_{p_i}^2 \text{ and } -a \notin \dot{K}_{p_i}^2. \end{cases}$$

Proof. It follows from Kummer's theory (see for example [Mo, Prop. 9.6]) that if a field F contains ξ_n then $[F(\sqrt[n]{a} : F]$ equals the order of $a\dot{F}^n$ in $\dot{F}/\dot{F}^n$. It explains (1) and (3). To explain (2) it suffices to observe that any $p \in \Omega_l(K)$ extends to $\frac{\mathfrak{t}_n(a)}{\mathfrak{t}_{n,p}(a)}$ primes of E. For explanation (4),(5) and (6) when $n = l = 2$ see [Sz2, Th.1.1]. Now suppose $n > l = 2$. Since $\xi_n \in K$, the field K is non-real and $r = R = 0$. To prove (5) and (6) denote by F the only quadratic extension of K contained in E. Since $[E : F]$ is odd, so the level of E equals the level of F. The same argument one can use for the local levels. Thus it suffices to use (5) and (6) for $n = 2$. □

Theorem 4.2. *If n is a square-free natural number then Milnor n-equivalence has 'going up property' with respect to the family of number fields containig ξ_n.*

Proof. Suppose K is a number field containing ξ_n and $E = K(\sqrt[n]{a})$. The above Theorem says that $A_l(E)$ depend on $A_l(K)$ and orders of a and $-a$ in the groups $\dot{K}/\dot{K}^n$, $\dot{K}/\dot{K}^2$, $\dot{K}_p/\dot{K}_p^n$, $\dot{K}_p/\dot{K}_p^2$. But those, by Remark 3.4, are preserved by Milnor n-equivalence. □

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Received: May 23, 1998