On some modifications of two theorems of Erdős

Katalin Kovács

Abstract: If certain sums of two completely additive functions are constant or convergent, then the functions are some constant multiples of the logarithm function.

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In 1946 Erdős [2] proved the following theorems:

Theorem 1 (Erdős). *Let f be a real valued additive function. If $f(n+1)-f(n) \to 0$, then $f(n) = c \log n$ for all $n \in N$.*

Theorem 2 (Erdős). *If a real valued additive function f is monotonically increasing, then $f(n) = c \log n$.*

I. Kátai [3] generalized Theorem 1 for completely additive functions using a result of E. Wirsing [6]:

Theorem 3 (Kátai). *Let f be a completely additive function. If $\sum_{i=1}^{m} c_i f(n + a_i) = o(\log n)$, then $f(n) = c \log n$ for all $n \in N$.*

P.D.T.A. Elliott [1] and the author ([4],[5]) found the following further generalizations:

Theorem 4 (Elliott, [1]). *Let f be an additive function, $A > 0, C > 0, B, D$ integers and $\Delta_1 = AC(AD - BC) \neq 0$. If $f(An + B) - f(Cn + D) \to c$, then $f(n) = c_1 \log n$ for all (n, Δ_1).*

Theorem 5 [4]. *Let f be a completely additive function, $A > 0, C > 0, B, D$ integers and $\Delta_2 = AC(A + 1)(C + 1)(AD - BC) \neq 0$. If $f(An + B) + f(Cn + D) \to c$, then $f(n) = 0$ for all (n, Δ_2).*

Theorem 6 [5]. *Let f be a completely additive function. If*

$$f(2n + A) - f(n)$$

is monotonic from some number on, then $f(n) = c \log n$ with some $c \geq 0$ for all $n \in N$.

In this paper I prove the following generalizations of Theorem 5 and Theorem 6:

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Theorem 7. *Let $A > 0, C > 0, B, D$ be integers and $\epsilon \in \{ 1, -1\}$. If*

(1) $$f_1(An + B) + f_1(Cn + D) + f_2(n) \to c,$$

for the completely additive functions f_1 and f_2, then $f_i(n) = c_i \log n$ or $f_i(n) = 0$ ($i \in 1,2$) for all n coprime to $\Delta_3 = ABCD(C^2B^2 - A^2D^2)(A^2D + 1)(C^2B + 1)$.

Theorem 8. *Let $A > 1, B > 0$ be integers, $\epsilon \in \{ 1, -1\}$ and $\alpha \in C\backslash\{0, -2\}$. If*

(2) $$f(An + B) + f(An - B) + \alpha f(n) \to c$$

for a completely additive function f, then $f(n) = 0$ for all $n \in N$.

Theorem 9. *Let f_1, f_2 denote completely additive arithmetical functions and $\epsilon \in \{1, -1\}$. If one of the conditons*

(3) $$f_1(n + 2k + \epsilon) - f_1(n + 2k) + f_2(n + \epsilon) - f_2(n) = o(\log n),$$

(4) $$f_1(n + 2k + \epsilon) - f_1(n + 2k) + f_2(n) - f_2(n - \epsilon) = c,$$

(5) $$f_1(n + 2) - f_1(n - 1) + f_2(n - 1) - f_2(n) = c,$$

(6) $$f_1(n) - f_1(n - 3) + f_2(n - 1) - f_2(n) = c,$$

(7) $$f_1(n + 3) - f_1(n) + f_2(n - 1) - f_2(n) = c$$

is satisfied, then $f_i(n) = C_i \log n$ ($i = 1, 2$) for all $n \in N$.

Proofs

Proof of Theorem 7. We may assume, that B and C are positive. (Otherwise we replace n by $n + s$ in (1) with a number s big enough such that $B' = B + sA > 0$ and $D' = D + sA > 0$.) We substitute n by CBn and ADn in (1), resp. Therefore

(8) $$f_1(ACn + 1) + \epsilon f_1(C^2Bn + D) + f_2(n) \to c_1$$

and

(9) $$f_1(A^2Dn + B) + \epsilon f_1(ACn + 1) + f_2(n) \to c_2.$$

The difference of (9) and (8) shows that

$$f_1(C^2Bn + D) - \epsilon f_1(A^2Dn + B) \to c_3.$$

Finally we apply Theorem 4 and Theorem 5, resp.

Proof of Theorem 8. We replace n by Bn in (2). So we have

(10) $$f(An + 1) + f(An - 1) \to c.$$

Now we substitute n by $An, (A + 1)n, A(A + 1)n, A(A + 1)n + 1$ and $A(A + 1)n - 1$ in (10). Therefore we obtain the following assertions:

(11) $$f(A^2n + 1) + f(A^2n - 1) + \alpha f(n) \to c_1,$$

(12) $$f(A(A + 1)n + 1) + f(A(A + 1)n - 1) + \alpha f(n) \to c_2,$$

(13) $$f(A^2(A + 1)n + 1) + f(A^2(A + 1)n - 1) + \alpha f(n) \to c_3,$$

(14) $f(A^2n+1)+f(A^2(A+1)n+A-1)+\alpha f(A(A+1)n+1)\to c_4,$

(15) $f(A^2(A+1)n-A+1)+f(A^2n-1)+\alpha f(A(A+1)n-1)\to c_5.$

By the linear combination of the equations (14)+(15)-(11)-α(12) we have

$$f(A^2(A+1)n-A+1)+f(A^2(A+1)n+A-1)-(\alpha^2+\alpha)f(n)\to c_7.$$

Then we replace n by $(A-1)n$ in this formula, which yields that

(16) $f(A^2(A+1)n-1)+f(A^2(A+1)n+1)-(\alpha^2+\alpha)f(n)\to c_8.$

The difference of (16) and (13) shows that $(\alpha^2+2\alpha)f_n\to c_8$, i.e. $f=0$ if $\alpha\notin\{0,-2\}$.

Proof of Theorem 9.

Case 1. Replacing n by $n+\epsilon$ in (3) we have

(17) $f_1(n+2k+2\epsilon)-f_1(n+2k+\epsilon)+f_2(n+2\epsilon)-f_2(n+\epsilon)=o(log\, n).$

The sum of (3) and (17) yields that

(18) $f_1(n+2k+2\epsilon)-f_1(n+2k)+f_2(n+2\epsilon)-f_2(n)=o(log\, n).$

Replacing n by $2n$ in (18) we get that

(19) $f_1(n+k+\epsilon)-f_1(n+k)+f_2(n+\epsilon)-f_2(n)=o(log\, n).$

The difference of (19) and (3) shows that

$$f_1(n+2k+\epsilon)-f_1(n+k+\epsilon)-f_1(n+k)+f_1(n+2k)=o(log\, n).$$

By Theorem 3 we have that $f_1(n)=c_1\ log\, n$ or $f_1(n)=0$. We substitute this result in (3) to obtain $f_2(n)=c_2\ log\, n$.

Case 2. We replace n by $n+\epsilon$ in (4). Therefore

(20) $f_1(n+2k+2\epsilon)-f_1(n+2k+\epsilon)+f_2(n+\epsilon)-f_2(n)=c.$

The sum of (4) and (20) yields that

(21) $f_1(n+2k+2\epsilon)-f_1(n+2k)+f_2(n+\epsilon)-f_2(n-\epsilon)=2c.$

We replace n by $n-k$ in (4) and by $2n$ in (21). So we have

(22) $f_1(n+k+\epsilon)-f_1(n+k)+f_2(n-k)-f_2(n-k-\epsilon)=c'.$

(23) $f_1(n+k+\epsilon)-f_1(n+k)+f_2(2n+\epsilon)-f_2(2n-\epsilon)=c''.$

The difference of (23) and (22) shows that

$$f_2(2n+\epsilon)-f_2(n-k)=f_2(2n-\epsilon)-f_2(n-k-\epsilon)+c'''(\epsilon=1\text{or }-1),$$

i.e. $f_2(2n+2k+\epsilon)-f_2(n)$ is monotonic. Finally we apply Theorem 6.

Case 3. We replace n by $n+1$ in (5), so we have

(24) $f_1(n+3)-f_1(n)+f_2(n)-f_2(n+1)=c.$

We substitute n by $2n+1$ in the sum of (5) and (24), i.e.

$$f_1(n+2) - f_1(n-1) + f_1(n+3) - f_1(n) + f_2(n-1) - f_2(n+1) = 2c,$$

which follows

$$f_1(2n+3) - f_1(n) + f_1(n+2) - f_1(2n+1) + f_2(n) - f_2(n+1) = 2c. \tag{25}$$

The difference of (25) and (24) shows that

$$f_1(2n+3) - f_1(n+3) = f_1(2n+1) - f_1(n+2) + c,$$

i.e. $f_1(2n+3) - f_1(n)$ is monotonic. Finally we apply Theorem 6.

The proof of the remaining two cases is very similar.

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Author's address: Eötvös University, Dept. of Algebra and Number Theory H-1088, Budapest, Múzeum krt.6-8 HUNGARY

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