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Complete solution of parametrized Thue equations

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Abstract: We give a survey on recent results concerning parametrized Thue equations. Moreover, we solve completely the family

$$X(X-Y)(X-aY)(X-(a+1)Y)-Y^4=\pm 1.$$

Key Words: Diophantine equations, Symbolic Computation, Linear Forms in Logarithms

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1. Introduction

Let $F \in \mathbb{Z}[X,Y]$ be a homogeneous, irreducible polynomial of degree $n \geq 3$ and m be an integer. Then the diophantine equation

$$F(x,y)=m \tag{1}$$

is called a *Thue equation* in honour of A. THUE, who proved in 1909 [31]:

Theorem 1.1. (Thue) *(1) has only finitely many solutions* $(x,y) \in \mathbb{Z}^2$. THUE's proof is based on his approximation theorem: Let α be an algebraic number of degree $n \geq 2$ and $\varepsilon > 0$. Then there exists a constant $c(\alpha,\varepsilon)$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha,\varepsilon)}{q^{n/2+1+\varepsilon}}.$$

Since this approximation theorem is not effective,THUE's theorem is not.

Studying linear forms in logarithms of algebraic numbers, A. BAKER could give an effective upper bound for the solutions of such an equation in 1968 [1]:

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Theorem 1.2. (Baker) *Let $\kappa > n+1$ and $(x,y) \in \mathbb{Z}^2$ be a solution of (1). Then*

$$\max\{|x|,|y|\} < Ce^{\log^\kappa m},$$

where $C = C(n,\kappa,F)$ is an effectively computable number.

Since that time, these bounds have been improved; BUGEAUD and GYŐRY [6] recently gave the following bound:

Theorem 1.3. (Bugeaud-Győry) *Let $B \geq \max\{|m|, e\}$, α be a root of $F(X,1)$, $K := \mathbb{Q}(\alpha)$, $R := \mathrm{reg}_K$ the regulator of K and r the unit rank of K. Let $H \geq 3$ be an upper bound for the absolute values of the coefficients of F.*

Then all solutions $(x,y) \in \mathbb{Z}^2$ of (1) satisfy

$$\max\{|x|,|y|\} < \exp\Big(C_1 \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB))\Big)$$

and

$$\max\{|x|,|y|\} < \exp\Big(C_2 \cdot H^{2n-2} \cdot \log^{2n-1} H \cdot \log B\Big),$$

with $C_1 = 3^{r+27}(r+1)^{7r+19}n^{2n+6r+14}$ and $C_2 = 3^{3(n+9)}n^{18(n+1)}$.

BOMBIERI and SCHMIDT [5] proved that the number of solutions of (1) with $m = \pm 1$ is $O(n)$:

Theorem 1.4. (Bombieri-Schmidt) *There is an absolute constant C_0 such that for all $n \geq C_0$ the diophantine equation $F(X,Y) = \pm 1$ has at most $431 \cdot n$ solutions.*

Up to maybe the constant 431, this is best possible, since the equation

$$X^n + (X-Y)(2X-Y)\dots(nX-Y) = \pm 1$$

has at least the $2n+2$ solutions $\pm\{(1,1),\dots,(1,n),(0,\pm 1)\}$.

However, the bounds obtained by BAKER's method are rather large, thus the solutions practically cannot be found by simple enumeration. BAKER and DAVENPORT [2] proposed for a similar problem a method to reduce drastically the bound by using continued fraction reduction. PETHŐ and SCHULENBERG [26] replaced the continued fraction reduction by the LLL-algorithm and gave a general method to solve (1) in the totally real case for $m = 1$ and arbitrary n. TZANAKIS and DE WEGER [32] describe the general case. Finally, BILU and HANROT [4] observed that Thue equations imply not only one, but $r-1$ independent linear forms in logarithms of algebraic numbers in the same very small size. They were able to replace the LLL-algorithm by the much faster continued fraction method and solve Thue equations up to degree 1000.

In 1990, THOMAS investigated for the first time a parametrized family of Thue equations. Since then, the following families have been studied:

1. $X^3 - (a-1)X^2Y - (a+2)XY^2 - Y^3 = 1$.

THOMAS [29] and MIGNOTTE [21] proved that for $a \geq 4$, the only solutions are $(0,-1)$, $(1,0)$ and $(-1,+1)$, while in the cases $0 \leq a \leq 4$ there exist some nontrivial solutions, too, which are given explicitely in [29].

2. $|X^3 - (a-1)X^2Y - (a+2)XY^2 - Y^3| \le 2a+1$.

All solutions of this Thue inequality have been found by MIGNOTTE, PETHŐ, and LEMMERMEYER [23].

3. $X^3 - (a+1)X^2Y + aXY^2 - Y^3 = 1$.

LEE [15] and independently MIGNOTTE and TZANAKIS [24] proved that for $a \ge 3.33 \cdot 10^{23}$, only trivial solutions exist. Very recently, MIGNOTTE [20] could solve this equation completely.

4. $X(X - n^aY)(X - n^bY) \pm Y^3 = 1$.

This family was investigated by THOMAS [30]. He proved that for $0 < a < b$ and $n \ge \left(2 \cdot 10^6 \cdot (a+2b)\right)^{4.85/(b-a)}$ nontrivial solutions cannot exist. He also investigated this family with n^a and n^b replaced by polynomials in n of degrees a and b, respectively.

5. $X^4 - aX^3Y - X^2Y^2 + aXY^3 + Y^4 = \pm 1$.

This quartic family was solved by PETHŐ [25] for large values of a; MIGNOTTE, PETHŐ, and ROTH [22] solved it completely.

6. $X^4 - aX^3Y - 3X^2Y^2 + aXY^3 + Y^4 = \pm 1$ has been solved for $a \ge 9.9 \cdot 10^{27}$ by PETHŐ [25].

7. $X^4 - aX^3Y - 6X^2Y^2 + aXY^3 + Y^4 \in \{\pm 1, \pm 4\}$.

This equation has been solved by LETTL and PETHŐ [16]; CHEN and VOUTIER [7] solved it independently by using a hypergeometric method instead of BAKER's method.

8. $X(X - Y)(X - aY)(X - bY) - Y^4 = \pm 1$.

All solutions of this two-parametric family are known for $10^{2 \cdot 10^{28}} < a+1 < b \le a(1 + (\log a)^{-4})$, cf. PETHŐ and TICHY [27]. The case $b = a+1$ will be considered in this paper.

9. WAKABAYASHI [34] proved that if $|x^4 - a^2x^2y^2 + y^4| \le a^2 - 2$ and $a \ge 8$ then $|y| \le 1$.

10. HALTER-KOCH, LETTL, PETHŐ, and TICHY [11] investigated for distinct integers $a_1 = 0, a_2, \ldots, a_{n-1}$ and an integral parameter $a_n = a$ the equation

$$\prod_{i=1}^{n}(X - a_iY) \pm Y^n = \pm 1.$$

11. $X(X^2 - Y^2)(X^2 - a^2Y^2) - Y^5 = \pm 1$.

For $a > 3.6 \cdot 10^{19}$, all solutions have been found by HEUBERGER [12].

12. $X^6 - 2aX^5Y - (5a+15)X^4Y^2 - 20X^3Y^3 + 5aX^2Y^4 + (2a+6)XY^5 + Y^6 \in \{\pm 1, \pm 27\}$ was investigated by LETTL, PETHŐ, and VOUTIER, they found all solutions for $a \ge 89$ by hypergeometric methods [18]. For $a < 89$ they used BAKER's method [17].

2. General approach and Linear Forms in Logarithms of algebraic numbers

2.1. Thue equations

In this section, we give a short survey on the general approach to solve a single Thue equation (cf. GAÁL [10]). In order to keep notation simple, we only consider the equation

$$F(X,Y) = \pm 1, \tag{2}$$

where

$$f(X) := F(X,1) = \sum_{i=0}^{n} a_i X^i$$

is a monic polynomial with real zeros $\alpha^{(1)}, \dots, \alpha^{(n)}$.

Let $(x,y) \in \mathbb{Z}^2$ be a solution of (2). We define $j \in \{1, \dots, n\}$ by

$$\left| \frac{x}{y} - \alpha^{(j)} \right| = \min_{i \in \{1,\dots,n\}} \left| \frac{x}{y} - \alpha^{(i)} \right|. \tag{3}$$

Then we have $|y| \left|\alpha^{(i)} - \alpha^{(j)}\right| \le \left|x - \alpha^{(i)} y\right| + \left|x - \alpha^{(j)} y\right| \le 2\left|x - \alpha^{(i)} y\right|$ and

$$\left|x - \alpha^{(j)} y\right| = \frac{1}{\prod_{i \neq j} \left|x - \alpha^{(i)} y\right|} \le \frac{2^{n-1}}{|y|^{n-1} \prod_{i\neq j} \left|\alpha^{(i)} - \alpha^{(j)}\right|} \le \frac{c_1}{|y|^{n-1}}, \tag{4}$$

where $c_1, \dots$ denote positive effectively computable constants depending on $K := \mathbb{Q}(\alpha^{(1)})$. For $y > (2c_1)^{1/(n-2)}$, x/y is a convergent of $\alpha^{(j)}$ by LAGRANGE's theorem. This yields

$$y(\alpha^{(j)} - \alpha^{(i)}) - \frac{c_1}{|y|^{n-1}} < x - \alpha^{(i)} y < y(\alpha^{(j)} - \alpha^{(i)}) + \frac{c_1}{|y|^{n-1}}. \tag{5}$$

We have

$$F(X,Y) = Y^n f\left(\frac{X}{Y}\right) = Y^n \prod_{i=1}^{n} \left(\frac{X}{Y} - \alpha^{(i)}\right) = \prod_{i=1}^{n} (X - \alpha^{(i)} Y) = \mathrm{N}_{K/\mathbb{Q}}(X - \alpha^{(1)} Y).$$

Set $\beta^{(i)} := x - \alpha^{(i)} y$ for $i = 1, \dots, n$, then $\beta^{(1)}$ is a unit in $\mathfrak{O} := \mathbb{Z}[\alpha^{(1)}]$. Thus by DIRICHLET's theorem, we obtain

$$\beta^{(1)} = \pm \varepsilon_1^{u_1} \dots \varepsilon_r^{u_r}, \qquad u_1, \dots, u_r \in \mathbb{Z}, \tag{6}$$

where $\varepsilon^{(1)}, \dots, \varepsilon^{(r)}$ are fundamental units of $\mathfrak{O}$. Considering (6) for all conjugates and taking logarithms, we get the following system of linear equations in the u_i:

$$\log \left|\beta^{(i)}\right| = u_1 \log \left|\varepsilon_1^{(i)}\right| + \dots + u_r \log \left|\varepsilon_r^{(i)}\right| \qquad i \neq j. \tag{7}$$

Using (5), we derive the estimate

$$U := \max_{1 \le i \le r} |u_i| \le c_2 \max_{i \neq j} \left|\log \left|\beta^{(i)}\right|\right| \le c_3 \log |y|. \tag{8}$$

For $k \neq l \in \{1, \dots, n\} \setminus \{j\}$, together with (4) and (5) SIEGEL's identity

$$1 - \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \cdot \frac{x - \alpha^{(l)} y}{x - \alpha^{(k)} y} = \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}} \cdot \frac{x - \alpha^{(j)} y}{x - \alpha^{(k)} y} \tag{9}$$

yields

$$\left|1 - \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \frac{\beta^{(l)}}{\beta^{(k)}}\right| = \left|\frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}}\right| \left|\frac{\beta^{(j)}}{\beta^{(k)}}\right| \leq \frac{c_4}{|y|^n}. \tag{10}$$

Using a lower bound for linear forms in logarithms of algebraic numbers (see section 11), (6), (10), (4), (5) and finally (8) we have

$$\begin{aligned} \exp(-c_5 \log U) &< \left| \log \left|\frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}}\right| + u_1 \log \left|\frac{\varepsilon_1^{(l)}}{\varepsilon_1^{(k)}}\right| + \dots + u_r \log \left|\frac{\varepsilon_r^{(l)}}{\varepsilon_r^{(k)}}\right| \right| \\ &\leq 2 \left|1 - \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \cdot \frac{\beta^{(l)}}{\beta^{(k)}}\right| \\ &\leq \frac{2c_4}{|y|^n} = \exp(c_6 - n \log |y|) < \exp(c_6 - c_7 U). \end{aligned} \tag{11}$$

This estimate can only hold for $U \leq c_8$ yielding an upper bound for $|y|$ by (7) and (5).

2.2. Parametrized families of Thue equations

Given a parametrized family of Thue equations, one has to perform the same steps as in the case of one single Thue equation using asymptotic bounds for the quantities involved.

The main extra tool are estimates of the form

$$U > Ca^g \log a$$

for some positive constant C and some $g \in \mathbb{N}$. THOMAS [30] calls this fact 'stable growth'. The lower bounds for U usually contradict the upper bound for U from the linear form estimates.

Stable growth can be seen considering asymptotic expansions of the u_i resulting from (7), but at the time of this writing, we cannot give any condition on the familiy which guaranties stable growth for $n \geq 4$; for the case $n = 3$ see THOMAS [30].

2.3. Linear forms in logarithms

We give a brief survey on some lower bounds for linear forms in logarithms of algebraic numbers which are currently used for solving diophantine equations.

For an algebraic number γ with minimal polynomial $\sum_{i=0}^{d} a_i X^i$ and conjugates $\gamma = \gamma^{(1)}, \dots, \gamma^{(d)}$, the absolute logarithmic Weil height of γ is defined by

$$h(\gamma) := \frac{1}{d} \log \left[a_d \prod_{i=1}^{d} \max\left(1, \left|\gamma^{(i)}\right|\right) \right].$$

In general situations, one can use the following estimate of BAKER and WÜSTHOLZ [3]:

Theorem 2.1. (Baker-Wüstholz) *Let $\gamma_1, \dots, \gamma_n$ be algebraic numbers, not 0 or 1, $K = \mathbb{Q}(\gamma_1, \dots, \gamma_n)$ and d the degree $[K : \mathbb{Q}]$. For $i = 1, \dots, n$ let*

$$h_i \geq \max\left(h(\gamma_i), \frac{|\log(\gamma_i)|}{d}, \frac{1}{d}\right).$$

Let $b_1, \dots, b_n \in \mathbb{Z}$, $\Lambda = b_1 \log \gamma_1 + \dots + b_n \log \gamma_n \neq 0$ and $B \geq \max |b_j|$. Then we have

$$\log|\Lambda| > -C(n,d) h_1 \cdots h_n \log B, \tag{12}$$

where

$$C(n,d) = 18(n+1)! n^{n+1} (32d)^{n+2} \log(2nd).$$

In many concrete families, it is possible to reduce the number of logarithms in the linear form and to use estimates for linear forms in few logarithms. VOUTIER [33] considers three logarithms:

Theorem 2.2. (Voutier) *Let γ_1, γ_2 and γ_3 be positive algebraic numbers and put $D := [\mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) : \mathbb{Q}]$. Let b_1, b_2 and b_3 be integers with $b_3 \neq 0$ and let h_1, h_2, h_3, B and $E > 1$ be real numbers which satisfy*

$$h_i \geq \max\left(\frac{\log E}{D}, h(\gamma_i), \frac{E|\log \gamma_i|}{D}\right) \qquad 1 \leq i \leq 3,$$
$$B \geq \max\left\{2, E^{1/D}, \frac{\log^2 E}{D^2}\left(\frac{|b_1|}{h_3} + \frac{|b_3|}{h_1}\right)\left(\frac{|b_2|}{h_3} + \frac{|b_3|}{h_2}\right)\right\}$$

and $E < 4.6^D$. If $\log\gamma_1$, $\log\gamma_2$, and $\log\gamma_3$ are linearly independent over $\mathbb{Q}$, then

$$\log|b_1 \log \gamma_1 + b_2 \log \gamma_2 + b_3 \log \gamma_3| > -\frac{2.4 \cdot 10^6 \cdot D^5 \log^2 B}{\log^4 E} \cdot h_1 \cdot h_2 \cdot h_3.$$

LAURENT, MIGNOTTE and NESTERENKO [14] settle the case of two logarithms:

Theorem 2.3. (Laurent-Mignotte-Nesterenko) *Let γ_1 and γ_2 be multiplicatively independent and positive algebraic numbers, b_1 and $b_2 \in \mathbb{Z}$ and*

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$, for $i = 1, 2$ let

$$h_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}$$

and

$$b' \geq \frac{b_1}{D\,h_2} + \frac{b_2}{D\,h_1}.$$

If $|\Lambda| \neq 0$*, then we have*

$$\log|\Lambda| \geq -24.34 \cdot D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 h_1 h_2.$$

The following very deep conjecture is due to LANG and WALDSCHMIDT (cf. LANG [13]):

Conjecture 2.4. *Let* K *be an algebraic number field of degree* m*,* $\beta_1, \dots, \beta_k \in K$ *and* $b_1, \dots, b_k \in \mathbb{Z}$*. Let* $B_1, \dots, B_k, B \in \mathbb{R}$ *be real numbers such that*

$$\log B_i \geq h(\beta_i), \quad i = 1, \dots, k \qquad \text{and} \qquad B \geq \max\{|b_1|, \dots, |b_k|, e\}.$$

Then there exists a constant $c(k,m) > 0$ *such that*

$$\log|b_1 \log\beta_1 + \cdots + b_k \log\beta_k| > -c(k,m)(\log B_1 + \cdots + \log B_k)\log B,$$

provided that $b_1 \log\beta_1 + \cdots + b_k \log\beta_k \neq 0$.

Assuming this conjecture, HALTER-KOCH, LETTL, PETHŐ and TICHY [11] could prove:

Theorem 2.5. *Let* $n \geq 3$*,* $a_1 = 0, a_2, \dots, a_{n-1}$ *be distinct integers and* $a_n = a$ *an integral parameter. Let* $\alpha = \alpha(a)$ *be a zero of* $P(x) = \prod_{i=1}^n (x - a_i) - d$ *with* $d = \pm 1$ *and suppose that the index* I *of* $\langle \alpha - a_1, \dots, \alpha - a_{n-1}\rangle$ *in* $\mathfrak{O}^\times$*, the group of units of* $\mathfrak{O} := \mathbb{Z}[\alpha]$*, is bounded by a constant* $J = J(a_1, \dots, a_{n-1}, n)$ *for every* a *from some subset* $\Omega \in \mathbb{Z}$*. Assume further that the Lang-Waldschmidt conjecture is true. Then for all but finitely many values* $a \in \Omega$ *the diophantine equation*

$$\prod_{i=1}^{n} (x - a_i y) - dy^n = \pm 1$$

has only trivial solutions, except when $n = 3$ *and* $|a_2| = 1$*, or when* $n = 4$ *and* $(a_2, a_3) \in \{(1,-1), (\pm 1, \pm 2)\}$*, in which cases it has exactly one more solution for every value of* a*.*

If $\mathbb{Q}(\alpha)$ is primitive over $\mathbb{Q}$ — especially if n is prime — then there exists a bound $J = J(a_1, \dots, a_{n-1}, n)$ for the index I by lower bounds for the regulator of $\mathfrak{O}$ (cf. POHST and ZASSENHAUS [28], chapter 5.6, (6.22)). Applying the theory of Hilbertian fields and results on thin sets, primitivity is proved for almost all choices (in the sense of density) of the parameters, cf. [11].

3. A quartic Family of Thue equations

In [27], PETHŐ and TICHY considered the two parametric family of Thue equations

$$F_{a,b}(X,Y) := X(X-Y)(X-aY)(X-bY) - Y^4 = \pm 1. \tag{13}$$

They proved the following theorem:

Theorem 3.1. *Assume that*

$$10^{2\cdot 10^{28}} < a+1 < b \le a\left(1 + \frac{1}{\log^4 a}\right).$$

Then (13) has only the trivial solutions

$$F_{a,b}(\pm 1, 0) = 1,$$
$$F_{a,b}(0,\pm 1) = F_{a,b}(\pm 1, \pm 1) = F_{a,b}(\pm a, \pm 1) = F_{a,b}(\pm b, \pm 1) = -1.$$

The case $b = a+1$ was not covered by that paper, because its Galois group is different from the general case. In the remainder of this paper, we will investigate this case and we find all solutions for all integers a. We will prove:

Theorem 3.2. *Let a be an integer. Then the diophantine equation*

$$F_a(X,Y) := X(X-Y)(X-aY)(X-(a+1)Y) - Y^4 = \pm 1 \tag{14}$$

only has the trivial solutions

$$F_a(\pm 1, 0) = 1,$$
$$F_a(0,\pm 1) = F_a(\pm 1, \pm 1) = F_a(\pm a, \pm 1) = F_a(\pm (a+1), \pm 1) = -1.$$

Let $a < 0$ be an integer and put $A = -a$. Then $A \ge 0$ and we have

$$\begin{aligned} F_a(X,Y) &= X(X-Y)(X+AY)(X+(A-1)Y) - Y^4 \\ &= Z(Z-Y)(Z-AY)(Z-(A+1)Y) - Y^4 = F_A(Z,Y), \end{aligned}$$

with $Z = X + AY$. Hence it is enough to solve (14) for non-negative values of the parameter.

In [27] it was proved that all solutions $(x,y) \in \mathbb{Z}^2$ of (14) with $|y| \le 1$ are exactly the solutions listed in Theorem 3.2.

3.1. Properties of the quartic number field

We put

$$f_a(X) := F_a(X,1) = X(X-1)(X-a)(X-(a+1)) - 1$$

and we will investigate some properties of the number field $\mathbb{Q}(\alpha)$, where α is a root of f_a.

It is easy to observe that f_a is irreducible for $a \neq 0$, the case $a = 0$ yields precisely the solutions of Theorem 3.2 and will not be considered below.

If $a \geq 3$, all conjugates of α are real, we need sharper approximations for the roots of f_a than those established in [27], Lemma 2.1.

Lemma 3.3. *Let $a \geq 7$ and $\alpha := \alpha^{(1)} < \alpha^{(2)} < \alpha^{(3)} < \alpha^{(4)}$ be the zeros of f_a. Then the following estimates hold:*

$$-\frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^5} - \frac{6}{a^6} < \alpha^{(1)} < -\frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^5} - \frac{4}{a^6}$$
$$1 + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^5} + \frac{4}{a^6} < \alpha^{(2)} < 1 + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^5} + \frac{6}{a^6}$$
$$a - \frac{1}{a^2} - \frac{1}{a^3} - \frac{1}{a^5} - \frac{6}{a^6} < \alpha^{(3)} < a - \frac{1}{a^2} - \frac{1}{a^3} - \frac{1}{a^5} - \frac{4}{a^6}$$
$$a + 1 + \frac{1}{a^2} - \frac{1}{a^3} - \frac{1}{a^5} + \frac{4}{a^6} < \alpha^{(4)} < a + 1 + \frac{1}{a^2} - \frac{1}{a^3} - \frac{1}{a^5} + \frac{6}{a^6}$$

Proof. These inequalities can easily be verified by considering the sign of f_a at the given bounds. □

Sometimes, approximations of higher order will be needed; they can be obtained performing two or three symbolic Newton steps starting at $0, 1, a, a+1$ respectively, calculating an asymptotic expansion by Maple and verifying as in the proof above.

By [27], Theorem 2.1, we know that the Galois group of f_a is isomorphic to the dihedral group D_8. Indeed, we have $\alpha^{(4)} = -\alpha^{(1)} + a + 1$ and $\alpha^{(3)} = -\alpha^{(2)} + a + 1$, since $f(-X+a+1) = f(X)$. Therefore we have $\mathrm{Gal}(E/\mathbb{Q}) = \langle (14), (1243) \rangle$, where E is the splitting field of f_a. Moreover, we have $K := \mathbb{Q}(\alpha^{(1)}) = \mathbb{Q}(\alpha^{(1)}, \alpha^{(4)})$, hence $\mathrm{Gal}(E/K) = \langle (23) \rangle$. Thus there exists exactly one quadratic subfield of K, say $\mathbb{Q}(\varepsilon)$, and this subfield is invariant under $\langle (14), (23) \rangle$. This leads to $\varepsilon = -\alpha^{(1)}\alpha^{(4)}$.

In the sequel, we will work in the order $\mathfrak{O} := \mathbb{Z}[\alpha]$. First we investigate the structure of its unit group $\mathfrak{O}^\times$. The corresponding part of [27] cannot be used, because it depends on the fact that $\mathbb{Q}(\alpha)$ is primitive over $\mathbb{Q}$ in that situation. However, the structure is the same:

Theorem 3.4. *Let $a \neq 0$ be an integer, α be a root of $f_a = X(X-1)(X-a)(X-a-1) - 1$ and $\mathfrak{O} := \mathbb{Z}[\alpha]$. If $|a| \geq 3$, we have*

$$\mathfrak{O}^\times = \langle -1, \alpha, \alpha - 1, \alpha - a \rangle,$$

for the remaining values of a, we have

$$|a| = 1: \qquad \mathfrak{O}^\times = \langle -1, \alpha, \alpha - 1 \rangle$$
$$|a| = 2: \qquad \mathfrak{O}^\times = \left\langle -1, \alpha, \sqrt{\alpha(\alpha - a)} \right\rangle.$$

Proof. We will first discuss the case $a \geq 49$. To prove that $\alpha, \alpha - 1$ and $\alpha - a$ are independent units, consider the regulator

$$R_\alpha := \begin{vmatrix} \log\left|\alpha^{(1)}\right| & \log\left|\alpha^{(1)} - 1\right| & \log\left|\alpha^{(1)} - a\right| \\ \log\left|\alpha^{(2)}\right| & \log\left|\alpha^{(2)} - 1\right| & \log\left|\alpha^{(2)} - a\right| \\ \log\left|\alpha^{(3)}\right| & \log\left|\alpha^{(3)} - 1\right| & \log\left|\alpha^{(3)} - a\right| \end{vmatrix}.$$

By Lemma 3.3, we obtain for $a \geq 25$

$$-4\log^3 a - \frac{3}{a} < R_\alpha < -4\log^3 a + \frac{1}{a}.$$

Thus $R_\alpha \neq 0$ and the units are independent.

We will use the following result of MAHLER [19]:

Proposition 3.5. (Mahler) *Let γ be an algebraic integer of degree $d \geq 2$ with conjugates $\gamma = \gamma^{(1)}, \ldots, \gamma^{(d)}$ and*

$$M(\gamma) := \prod_{k=1}^{d} \max\left\{1, \left|\gamma^{(k)}\right|\right\}.$$

Then

$$\left|\operatorname{discr}_{\mathbb{Z}[\gamma]}\right| \leq d^d \cdot M(\gamma)^{2(d-1)}.$$

Since

$$\operatorname{discr} f_a = a^8 + 6a^6 - 15a^4 - 152a^2 - 240 \geq a^8$$

and $\operatorname{discr} f_a \leq \operatorname{discr}\gamma$, MAHLER's result implies

$$M(\gamma) \geq \frac{a^{4/3}}{4^{2/3}} \tag{15}$$

for all $\gamma \in \mathfrak{O}$.

To prove that the units are independent, we will first consider another system of independent units. We will prove that these units generate $\mathfrak{O}^\times$ using [28], chapter 5, Theorem 7.1.

First, we prove that $\eta_1 := \alpha(\alpha - a) > 0$ can be extended to a system of fundamental units. To prove this, we have to show that there is no $\gamma \in \mathfrak{O}^\times$ and no $n \geq 2$ such that $\eta_1 = \gamma^n$. By (15) and Lemma 3.3 we have

$$\frac{a^{4/3}}{4^{2/3}} \leq M(\gamma) = M(\eta_1)^{1/n} \leq \left(\frac{103}{100}a^2\right)^{1/n} \leq \sqrt{\frac{103}{100}}a, \tag{16}$$

which is a contradiction for $a \geq 16$.

Next, let $\eta_2 := \alpha(\alpha - 1) > 0$. We prove that η_1, η_2 can be extended to a system of fundamental units of $\mathfrak{O}$. We have to prove that $\gamma^n = \eta_1^k \eta_2$ has no solution with $\gamma \in \mathfrak{O}^\times$, $n \geq 2$ and $|k| \leq n/2$. For $n \geq 44$, we can argue as in (16) and we get a contradiction for $a \geq 48$, since $M(\eta_1^k \eta_2) \leq M(\eta_1)^k M(\eta_2)$. For $4 \leq n \leq 43$, we explicitely bound $M(\eta_1^k\eta_2)$ for all possible choices of k by Lemma 3.3 and we find a contradiction for $a \geq 24$. For $n = 2$ and $k = 0$, we find that

$$\pm\sqrt{\eta_2^{(1)}} \pm \sqrt{\eta_2^{(2)}} \pm \sqrt{\eta_2^{(3)}} + \sqrt{\eta_2^{(4)}}$$

is no integer for all choices of the signs, which is impossible since η_2 is an algebraic integer; the case $k = 1$ can be excluded since $\eta_1^{(2)} < 0$. If $n = 3$, we find

$$\begin{array}{lrl} k = 0 & d_1^2 - 4d_2 = -8 + \vartheta_1 \frac{1}{a^{2/3}} & \vartheta_1 \in [7/18, 8/18] \\ k = 1 & -2d_1 - 3ad_1 + 3d_2 = -3a^2 - 4a + 1 - \vartheta_2 \frac{1}{a^{2/3}} & \vartheta_2 \in [-341/54, -11/3] \\ k = -1 & 3ad_1 - 2d_1 + 3d_2 = -3a^2 + 4a + 1 - \vartheta_3 \frac{1}{a^{2/3}} & \vartheta_3 \in [-11/3, -55/27], \end{array}$$

where d_i are the symmetric functions in $\gamma^{(i)}$, i. e. $d_1 := \sum_{i=1}^4 \gamma^{(i)}$ and $d_2 := \sum_{1\leq i<j\leq 4} \gamma^{(i)}\gamma^{(j)}$, hence $d_i \in \mathbb{Z}$, which is a contradiction for $a \geq 16$.

To finish the proof of the case $a \geq 49$, we use an idea of LETTL and PETHŐ [16]. Assume that $-1, \eta_1, \eta_2$ and some $\eta \in \mathfrak{O}^\times$ generate $\mathfrak{O}^\times$. Consider $\mathcal{N} : \mathfrak{O}^\times \to \langle\varepsilon\rangle ; \gamma \mapsto \left|N_{K/\mathbb{Q}(\varepsilon)}(\gamma)\right| = \left|\gamma^{(1)}\gamma^{(4)}\right|$. We see that $\mathcal{N}(\mathfrak{O}^\times) \subseteq \langle\varepsilon\rangle$ by PETHŐ [25], Lemma 3.2. η_1 and η_2 were chosen such that $\mathcal{N}(\eta_1) = \mathcal{N}(\eta_2) = 1$. Put $\alpha = \pm\eta_1^k\eta_2^l\eta^m$, then we see

$$\mathcal{N}(\eta)^m = \mathcal{N}(\eta_1^k\eta_2^l\eta^m) = \mathcal{N}(\pm\alpha) = \varepsilon,$$

hence $m = \pm 1$ and we have

$$\mathfrak{O}^\times = \langle -1, \eta_1, \eta_2, \eta\rangle = \langle -1, \eta_1, \eta_2, \alpha\rangle = \langle -1, \alpha, \alpha - 1, \alpha - a\rangle .$$

Next, we consider $3 \leq a \leq 48$. We used Pari (cf. COHEN [8]) to compute the regulator R of K for every a, we calculated the regulator R_α explicitely and got

$$I := \left[\mathfrak{O}^\times : \langle -1, \alpha, \alpha - 1, \alpha - a\rangle\right] = \frac{R_\alpha}{R_{\mathfrak{O}}} \leq \frac{R_\alpha}{R} = M,$$

where $M = 1$ for all a except

a	4	11	14	29	36
M	3	5	3	7	3

In these cases we explicitely solved

$$\gamma^n = \alpha^{k_1}(\alpha - 1)^{k_2}(\alpha - a)^{k_3}$$

for all $|k_1|, |k_2|, |k_3| < n/2$ and $2 \leq n \leq M$. We did not find any solution $\gamma \in \mathfrak{O}$ with $\gcd(n, k_1, k_2, k_3) = 1$. Hence, $I = 1$ in these cases. The last step took about 6 minutes on a Pentium 200 running Linux.

The case $a \leq -3$ follows from the positive case considering $f_{-a}(\alpha - a) = 0$.

The remaining cases $|a| \in \{1, 2\}$ can be proved using Kant [9]. □

3.2. Approximation properties of the solutions

Let $(x,y) \in \mathbb{Z}^2$ be a solution of (14), $y \geq 2$. As in (3), we define the *type* j of (x,y) such that

$$\left|\alpha^{(j)} - \frac{x}{y}\right| = \min\left\{\left|\alpha^{(k)} - \frac{x}{y}\right|, 1 \leq k \leq 4\right\}.$$

By $F_a(-X+(a+1)Y, Y) = F_a(X,Y)$ and Lemma 3.3 we see that if (x,y) is a solution of (14) of type 3 or 4, then $(-x+(a+1)y, y)$ is a solution of type 2 or 1, respectively. Thus in order to prove Theorem 3.2, we have to show that there exists no solution (x,y) of type 1 or 2 with $y \geq 2$.

Since we have

$$F_a(x,y) = \mathrm{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(x - \alpha y) = 1,$$

Theorem 3.4 yields

$$x - \alpha y = \pm \eta_1^{u_1} \eta_2^{u_2} \eta_3^{u_3}, \tag{17}$$

where $\eta_1 = \alpha$, $\eta_2 = \alpha - 1$ and $\eta_3 = \alpha - a$.

3.3. Upper bounds for a linear form in logarithms

We shall now derive upper bounds for the linear form

$$\Lambda_{pqj} := u_1 \log\left|\frac{\eta_1^{(p)}}{\eta_1^{(q)}}\right| + u_2 \log\left|\frac{\eta_2^{(p)}}{\eta_2^{(q)}}\right| + u_3 \log\left|\frac{\eta_3^{(p)}}{\eta_3^{(q)}}\right| + \log\left|\frac{\alpha^{(j)} - \alpha^{(q)}}{\alpha^{(j)} - \alpha^{(p)}}\right|, \tag{18}$$

where p and q will be chosen according to the type j of the solution (x,y). Furthermore, we will investigate relations between the u_i.

Lemma 3.6. *Let $a \geq 100$ and (x,y) be a solution of (14) with $y \geq 2$ of type j. The following estimates hold, according to the value of j:*

$j=1$: Let $U := u_1$ and $V := u_3 - u_2$. Then we have $\frac{3}{2} a \log a \,|-U+3V| < U$ and $U - 1 > 3a^2 \log a$. Putting $p = 2$ and $q = 3$, we have

$$\log|3\Lambda_{231}| < -\frac{8}{3} U \log a + \log(4.5 a^{14/3}). \tag{19}$$

$j=2$: Let $U := u_2 - u_3$. Then we have $|u_1| < U$, $\frac{7}{5} a \log a\,|U + 3u_1| < U$ and $U - 1 > 3a^2 \log a$. Putting $p = 1$ and $q = 4$ in this case, we get

$$\log|3\Lambda_{142}| < -\frac{8}{3} U \log a + \frac{20}{9}\frac{U}{a} + \log(4.5 a^{14/3}). \tag{20}$$

Proof.

$j = 1$: By Lemma 3.3, we see $\lfloor \alpha^{(1)} \rfloor = -1$ and $\left\lfloor \frac{1}{\alpha+1} \right\rfloor = 1$, thus the continued fraction expansion of $\alpha^{(1)}$ starts with $\left[-1, 1, \alpha_2^{(1)}\right]$ where

$$a^2 + a < \alpha_2^{(1)} < a^2 + 1.1a.$$

Since x/y is a principal convergent of $\alpha^{(1)}$ by (4), we have

$$y \geq a^2. \tag{21}$$

Then (4) yields — as in [27], (4.8) —

$$\alpha^{(1)} - \alpha^{(\nu)} - \frac{8}{a^{10}} < \left|\frac{\beta^{(\nu)}}{y}\right| < \alpha^{(1)} - \alpha^{(\nu)} + \frac{8}{a^{10}}. \tag{22}$$

Taking logarithms of the conjugates of (17), we obtain the following system of linear equations in the u_i:

$$\begin{aligned}
\log\left|\beta^{(4)}\right| &= u_1 \log\left|\eta_1^{(4)}\right| + u_2 \log\left|\eta_2^{(4)}\right| + u_3 \log\left|\eta_3^{(4)}\right| \\
\log\left|\frac{\beta^{(2)}}{\beta^{(4)}}\right| &= u_1 \log\left|\frac{\eta_1^{(2)}}{\eta_1^{(4)}}\right| + u_2 \log\left|\frac{\eta_2^{(2)}}{\eta_2^{(4)}}\right| + u_3 \log\left|\frac{\eta_3^{(2)}}{\eta_3^{(4)}}\right| \\
\log\left|\frac{\beta^{(3)}}{\beta^{(4)}}\right| &= u_1 \log\left|\frac{\eta_1^{(3)}}{\eta_1^{(4)}}\right| + u_2 \log\left|\frac{\eta_2^{(3)}}{\eta_2^{(4)}}\right| + u_3 \log\left|\frac{\eta_3^{(3)}}{\eta_3^{(4)}}\right|
\end{aligned}$$

By (22) and Lemma 3.3, we have good estimates for $\log\left|\beta^{(\nu)}/\beta^{(4)}\right|$ in terms of a. Solving this system by Cramer's rule, we obtain

$$\begin{aligned}
Ru_1 &= \left(6\log^2 a + \vartheta_{11}\frac{\log a}{a}\right)\log\left|\beta^{(4)}\right| + 4\log^3 a + \vartheta_{12}\frac{1}{a} \\
Ru_2 &= \left(-2\log^2 a + \vartheta_{21}\frac{\log a}{a}\right)\log\left|\beta^{(4)}\right| + 2\frac{\log^2 a}{a^2} + \vartheta_{22}\frac{\log^2 a}{a^3} \\
Ru_3 &= \left(-2\frac{\log a}{a} + \vartheta_{31}\frac{\log a}{a^2}\right)\log\left|\beta^{(4)}\right| + 4\frac{\log a}{a^3} + \vartheta_{32}\frac{\log^2 a}{a^4}.
\end{aligned}$$

where R is the determinant of the system matrix,

$$R = 4\log^3 a + 5\frac{\log^2 a}{a^2} - 2\frac{\log a}{a^2} + \vartheta_0\frac{1}{a^2}, \tag{23}$$

and the ϑ lie in the following intervals:

ϑ_0	ϑ_{11}	ϑ_{12}	ϑ_{21}	ϑ_{22}	ϑ_{31}	ϑ_{32}
$[-0.1, 0.01]$	$[-5, -3]$	$[-2, 0]$	$[-2, -1]$	$[2, 3]$	$[2, 3]$	$[-6, 1]$

By (21) and (22) we have $\log\left|\beta^{(4)}\right| > 3\log a$.

We have

$$\begin{aligned}
R(u_1 - 1) &> 5\log^2 a \log\left|\beta^{(4)}\right| - 10\frac{\log^2 a}{a^2} > 0 \\
R(u_1 - 1 + 3u_2 - 5u_3) &> \left(2\frac{\log a}{a^2} - \frac{4}{a^2}\right)\log\left|\beta^{(4)}\right| - 4\frac{\log^2 a}{a^2} > 0 \\
R\big(u_1 - 1 - 3a^2\log a(u_1 - 1 + 3u_2 - 5u_3)\big) &> 11\log a \log\left|\beta^{(4)}\right| > 0,
\end{aligned}$$

hence $u_1 - 1 > 3a^2 \log a(u_1 - 1 + 3u_2 - 5u_3) \geq 3a^2 \log a$.

By (23) we have

$$u_1 \cdot 4\log^3 a \leq Ru_1 \leq 6\log^2 a \log\left|\beta^{(4)}\right| + 4\log^3 a,$$

hence

$$\log\left|\beta^{(4)}\right| \geq \frac{2}{3}\log a(u_1 - 1) \geq 2a^2\log^2 a. \tag{24}$$

We have

$$R(-U+3V) > 3\frac{\log a}{a}\log\left|\beta^{(4)}\right| - 5\log^3 a > 0$$
$$R\left(U - \frac{3}{2}a\log a(-U+3V)\right) > 5\frac{\log^2 a}{a}\log\left|\beta^{(4)}\right| > 0,$$

which implies that $U > \frac{3}{2}a\log a\,|-U+3V|$.

Finally, using (17), Siegel's identity, Lemma 3.3 and (22), we get

$$\Lambda_{231} = \log\left|\frac{\alpha^{(1)}-\alpha^{(3)}}{\alpha^{(1)}-\alpha^{(2)}}\cdot\frac{\beta^{(2)}}{\beta^{(3)}}\right| \leq 1.1a\left|\frac{\beta^{(4)}}{\beta^{(2)}}\right|\left|\frac{\beta^{(4)}}{\beta^{(3)}}\right|^2\frac{1}{\left|\beta^{(4)}\right|^4}.$$

Together with (24), we obtain the requested bound for Λ_{231}.

$j = 2$: The continued fraction expansion of $\alpha^{(2)}$ starts with $\left[1, \alpha_1^{(2)}\right]$ where $a^2 - a < \alpha_1^{(2)} < a^2 - 0.9a$, which yields

$$y \geq 0.9a^2. \tag{25}$$

This leads to

$$\alpha^{(2)} - \alpha^{(\nu)} - \frac{14}{a^{10}} < \left|\frac{\beta^{(\nu)}}{y}\right| < \alpha^{(2)} - \alpha^{(\nu)} + \frac{14}{a^{10}}. \tag{26}$$

Taking logarithms of the conjugates of (17), we obtain the following system of linear equations in the u_i:

$$\log\left|\beta^{(4)}\right| = u_1\log\left|\eta_1^{(4)}\right| + u_2\log\left|\eta_2^{(4)}\right| + u_3\log\left|\eta_3^{(4)}\right|$$
$$\log\left|\frac{\beta^{(1)}}{\beta^{(4)}}\right| = u_1\log\left|\frac{\eta_1^{(1)}}{\eta_1^{(4)}}\right| + u_2\log\left|\frac{\eta_2^{(1)}}{\eta_2^{(4)}}\right| + u_3\log\left|\frac{\eta_3^{(1)}}{\eta_3^{(4)}}\right|$$
$$\log\left|\frac{\beta^{(3)}}{\beta^{(4)}}\right| = u_1\log\left|\frac{\eta_1^{(3)}}{\eta_1^{(4)}}\right| + u_2\log\left|\frac{\eta_2^{(3)}}{\eta_2^{(4)}}\right| + u_3\log\left|\frac{\eta_3^{(3)}}{\eta_3^{(4)}}\right|$$

Solving this system by Cramer's rule, we obtain

$$\begin{aligned}
Ru_1 &= \left(2\log^2 a + \vartheta_{11}\frac{1}{a}\right)\log\left|\beta^{(4)}\right| - 2\frac{\log^2 a}{a^2} + \vartheta_{12}\frac{\log^2 a}{a^3}\\
Ru_2 &= \left(-6\log^2 a + \vartheta_{21}\frac{\log a}{a}\right)\log\left|\beta^{(4)}\right| - 4\log^3 a + \vartheta_{22}\frac{1}{a}\\
Ru_3 &= \left(2\frac{\log a}{a} + \vartheta_{31}\frac{\log a}{a^2}\right)\log\left|\beta^{(4)}\right| - 4\frac{\log a}{a^3} + \vartheta_{32}\frac{\log^2 a}{a^4}.
\end{aligned}$$

where

$$R = -4\log^3 a - 5\frac{\log^2 a}{a^2} + 2\frac{\log a}{a^2} + \vartheta_0\frac{1}{a^2}, \tag{27}$$

and the ϑ lie in the following intervals:

ϑ_0	ϑ_{11}	ϑ_{12}	ϑ_{21}	ϑ_{22}	ϑ_{31}	ϑ_{32}
$[0, 0.1]$	$[-1, 1]$	$[1, 2]$	$[-3, -1]$	$[-1, 2]$	$[2, 3]$	$[-6, 1]$

Consider

$$\begin{aligned}
RU &< -5\log^2 a\log\left|\beta^{(4)}\right| - 3\log^3 a < 0\\
R(U + 3u_1) &< -4\frac{\log a}{a}\log\left|\beta^{(4)}\right| < 0\\
R\big(5U - 7a\log a(U + 3u_1)\big) &< \left(-2\log^2 a + 42\frac{\log^2 a}{a}\right)\log\left|\beta^{(4)}\right|\\
&\quad - 19\log^3 a + 28\log^4 a < 0,
\end{aligned}$$

and we get $7a\log a\,|U + 3u_1| < 5U$, hence $U \geq (7/5)a\log a$.

We have

$$R(U-1) > \left(-6\log^2 a - 5\frac{\log a}{a}\right)\log\left|\beta^{(4)}\right|,$$

hence (27) implies

$$\log\left|\beta^{(4)}\right| > \left(\frac{2}{3}\log a - \frac{5}{9}\frac{1}{a}\right)(U-1) \geq \frac{2}{3}a\log^2 a. \tag{28}$$

We derive

$$\begin{aligned}
R(2u_3 + 3u_1 + U - 1) &< -\frac{\log a}{a^2}\log\left|\beta^{(4)}\right| + 5\frac{\log^2 a}{a^2} < 0\\
R\big(U - 1 - 3a^2\log a(2u_3 + u_1 + U - 1)\big) &< -12\log a\log\left|\beta^{(4)}\right| < 0,
\end{aligned}$$

which implies that $U - 1 > 3a^2\log a\,|2u_3 + 3u_1 + U - 1| \geq 3a^2\log a$.

Finally, using (17), SIEGEL's identity, Lemma 3.3 and (26), we get

$$\Lambda_{142} = \log\left|\frac{\alpha^{(2)} - \alpha^{(4)}}{\alpha^{(2)} - \alpha^{(1)}}\cdot\frac{\beta^{(1)}}{\beta^{(4)}}\right| \leq 1.2a\left|\frac{\beta^{(4)}}{\beta^{(1)}}\right|\left|\frac{\beta^{(4)}}{\beta^{(3)}}\right|\frac{1}{\left|\beta^{(4)}\right|^4}.$$

Together with (28), we obtain the requested bound for Λ_{142}.

□

3.4. Lower bounds for a linear form in logarithms

Lemma 3.7. *Let $a \geq 100$, (x, y) a solution of (14) of type j with $y \geq 2$. Then the following estimates hold according to j:*
$j = 1$:

$$\log|3\Lambda_{231}| > -l_1(a)\left(3 + \frac{2U}{3a\log a}\right) \tag{29}$$

$$\log|3\Lambda_{231}| > -l_1'(a)\log^2\left(\frac{0.18}{a\log^3 a}U^2\right), \tag{30}$$

where

$$l_1(a) := 24\,924.2(\log a - 1.2)^2 \log\left(\frac{303}{100}a^8\right)\log a$$

$$l_1'(a) := 1.7 \cdot 10^{11}\log(1.01a)\log^2 a.$$

$j = 2$:

$$\log|3\Lambda_{142}| > -l_2(a)\left(3 + \frac{U}{\frac{7}{5}a\log a}\right) \tag{31}$$

$$\log|3\Lambda_{142}| > -l_2'(a)\log^2\left(\frac{U^2}{a\log a}\right), \tag{32}$$

where

$$l_2(a) := 199\,393.3(\log a - 1.5)^2 \log\left(\frac{101}{100}a\right)\log a$$

$$l_2'(a) := 8.413 \cdot 10^9 \log\left(\frac{101}{100}a\right)\log^2 a.$$

Proof.
$j = 1$: We rewrite Λ_{231} in the following way:

$$\Lambda_{231} = u_1 \log\left|\frac{\eta_1^{(2)}}{\eta_1^{(3)}}\frac{\eta_2^{(2)}}{\eta_2^{(3)}}\right| + (u_2 - u_1)\log\left|\frac{\eta_2^{(2)}}{\eta_2^{(3)}}\frac{\eta_3^{(2)}}{\eta_3^{(3)}}\right| + (u_3 - u_2 + u_1)\log\left|\frac{\eta_3^{(2)}}{\eta_3^{(3)}}\right| + \log\left|\frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}}\right|$$

Since $\eta_3^{(3)} = (-\alpha^{(2)} + a + 1) - a = -\eta_2^{(2)}$ and $\eta_3^{(2)} = -\eta_2^{(3)}$, the second term vanishes and we are left with

$$3\Lambda_{231} = U\log|\gamma_1| + (-U + 3V)\log|\gamma_2| + 3\log|\gamma_3| \tag{33}$$

$$= U\log|\gamma_1| + \log\left|\gamma_3^3\gamma_2^{-U+3V}\right|, \tag{34}$$

where

$$\gamma_1 := \left(\frac{\eta_1^{(2)}}{\eta_1^{(3)}}\right)^3 \left(\frac{\eta_2^{(2)}}{\eta_2^{(3)}}\right)^3 \left(\frac{\eta_3^{(2)}}{\eta_3^{(3)}}\right)^4, \qquad \gamma_2 := \frac{\eta_3^{(2)}}{\eta_3^{(3)}}, \qquad \gamma_3 := \frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}}.$$

We have

$$h(\gamma_1) \le \frac{1}{4}\log\left(\frac{303}{100}a^8\right), \qquad h(\gamma_2) \le \log a, \qquad h(\gamma_3) \le \log a,$$

$$\log|\gamma_1| \le \frac{3}{a}, \qquad \log|\gamma_2| \le 3\log a, \qquad \log|\gamma_3| \le \log a.$$

Applying Theorem 2.3 with

$$h_1 := \frac{1}{4}\log\left(\frac{303}{100}a^8\right), \qquad h_2 := \log a\left(3 + \frac{2U}{3a\log a}\right), \qquad b' := \frac{1}{4}a,$$

we get (29). Putting

$$h_1 := \frac{1}{4}\log\left(\frac{303}{100}a^8\right), \qquad h_2 := h_3 := \log a, \qquad E := \frac{8}{3}$$

and applying Theorem 2.2, we obtain (30).

$j = 2$: We rewrite Λ_{142} in the following way:

$$\Lambda_{142} = u_1 \log\left|\frac{\eta_1^{(1)}\eta_2^{(1)}}{\eta_1^{(4)}\eta_2^{(4)}}\right| + (u_2 - u_1)\log\left|\frac{\eta_2^{(1)}}{\eta_2^{(4)}}\frac{\eta_3^{(1)}}{\eta_3^{(4)}}\right|$$
$$+ (u_1 - U)\log\left|\frac{\eta_3^{(1)}}{\eta_3^{(4)}}\right| + \log\left|\frac{\alpha^{(2)} - \alpha^{(4)}}{\alpha^{(2)} - \alpha^{(1)}}\right|.$$

Since $\eta_3^{(4)} = (-\alpha^{(1)} + a + 1) - a = -\eta_2^{(1)}$ and $\eta_2^{(4)} = -\eta_3^{(1)}$, the second term vanishes and we are left with

$$3\Lambda_{142} = \log\left[\left|\frac{\alpha^{(2)} - \alpha^{(4)}}{\alpha^{(2)} - \alpha^{(1)}}\right|^3 \left|\frac{\alpha^{(4)} - (a+1)}{\alpha^{(1)} - (a+1)}\right|^{3u_1 + U}\right]$$
$$- U\log\left|\left(\frac{\alpha^{(1)} - a}{\alpha^{(4)} - a}\right)^3 \left(\frac{\alpha^{(4)} - (a+1)}{\alpha^{(1)} - (a+1)}\right)\right|.$$

Proceeding as above, we get the estimates.

□

3.5. 'Large' solutions

We compare the upper and lower bounds for the linear forms given in Lemma 3.6 and in Lemma 3.7: If $j = 1$, we get from (19) and (29)

$$3l_1(a) + \log(4.5a^{14/3}) \geq U\left(\frac{8}{3}\log a - \frac{2l_1(a)}{3a\log a}\right).$$

If $a \geq 11\,313\,890$, the right hand side is positive, so we can insert the lower bound for U from Lemma 3.6, which yields

$$3l_1(a) + \log(4.5a^{14/3}) \geq 3a^2\log a\left(\frac{8}{3}\log a - \frac{2l_1(a)}{3a\log a}\right).$$

This is a contradiction for $a \geq 11\,313\,892$. So there exists no solution (x, y) of type $j = 1$ and $y \geq 2$ for a greater than this bound.

For $j = 2$, we get in exactly the same way a contradiction for $a \geq 6\,700\,703$.

3.6. 'Small' solutions

To find all solutions with $100 \leq a \leq 11\,313\,891$, we proceed as in MIGNOTTE, PETHŐ and ROTH [22].

First we establish an explicit upper bound for U in both cases:

Lemma 3.8. *Let (x, y) be a solution of (14) of type j with $y \geq 2$.*
$j = 1$: Let $100 \leq a \leq 11\,313\,891$. Then we have $U \leq 5 \cdot 10^{16}$.
$j = 2$: Let $100 \leq a \leq 6\,700\,702$. Then we have $U \leq 2.5 \cdot 10^{15}$.
Proof. If $j = 1$, we obtain from (30) and (19)

$$\frac{8}{3}U\log a \leq (l_1'(a) + 1)\log^2\left(\frac{0.18}{a\log^3 a}U^2\right). \tag{35}$$

Since $U > 3a^2\log a$, we can use $\log x \leq \sqrt{[5]}x$ and we get $U \leq 7 \cdot 10^{53}$. Inserting that on the right hand side of (35), we get $U \leq 9 \cdot 10^{17}$ and repeating this process, we get the estimate of the lemma.

The case $j = 2$ can be treated in the same way. □

Lemma 3.9. *Let $\delta_1, \delta_2, M \in \mathbb{R}$, A and B integers and*

$$|A + B\delta_2 + \delta_1| < M. \tag{36}$$

Furthermore, let $Q \in \mathbb{N}$, $\tilde{\delta}_1, \tilde{\delta}_2 \in \mathbb{Q}$ with $\left|\delta_i - \tilde{\delta}_i\right| < Q^{-2}$ for $i = 1, 2$ and p/q a principal convergent of $\tilde{\delta}_2$ with $q \leq Q$. Then we have

$$q\|q\tilde{\delta}\| \leq Q^2M + 1 + 2B, \tag{37}$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

Proof. Multiplying (36) by q, we have

$$\left|q\tilde{\delta}_1 + q(\delta_1 - \tilde{\delta}_1) + qA - B(p - q\tilde{\delta}_2) + Bp - Bq(\tilde{\delta}_2 - \delta_2)\right| \le QM,$$

hence

$$q\|q\tilde{\delta}_1\| \le Q^2M + q^2\left|\delta_1 - \tilde{\delta}_1\right| + B + Bq^2\left|\tilde{\delta}_2 - \delta_2\right|$$

and the assertion follows. □

Let $j = 1$. By (33) and (19) we have (36) with

$$A := U, \qquad B := -U + 3V, \qquad M = 10^{-900}, \qquad \delta_1 := 3\frac{\log|\gamma_3|}{\log|\gamma_1|}, \qquad \delta_2 := \frac{\log|\gamma_2|}{\log|\gamma_1|}.$$

We choose Q depending on the value of a:

$$\begin{array}{rl} 100 \le a < \quad 60\,000 & Q = 10^{30} \\ 60\,000 \le a < \quad 100\,000 & Q = 2\cdot 10^{15} \\ 100\,000 \le a < 11\,313\,891 & Q = 10^{16} \end{array}$$

For each a, we compute rational approximations $\tilde{\delta}_i$ of δ_i and convergents p/q of $\tilde{\delta}_2$ with $q \le Q$. For most values of a, we find such a convergent with

$$q\|q\tilde{\delta}_1\| > 2 + \frac{7\cdot 10^{16}}{a\log a},$$

and this is a contradiction to (37) by Lemma 3.6 and Lemma 3.8. For the remaining values of a, we repeat this argument with $Q = 10^{30}$ and get the corresponding contradiction.

The case $j = 2$ is treated in exactly the same way, we only give the values of Q that we have used:

$$\begin{array}{rl} 100 \le a < \quad 60\,000 & Q = 10^{30} \\ 60\,000 \le a < 6\,700\,703 & Q = 10^{15}. \end{array}$$

Hence there are no nontrivial solutions if $a \ge 100$. For the case $1 \le a \le 99$, we used a program of Hanrot solving Thue equations following the algorithm of Bilu and Hanrot [4], where the fundamental units — which are known by Theorem 3.4 — can be explicitly given. This took 50 seconds on a Pentium 200 and only gave the trivial solutions known from Theorem 3.2. Thus Theorem 3.2 is proved.

The computations were performed on a DEC Alpha workstation and on a Pentium 200 running Linux of TU Graz. We have used MAPLE V in the formal computations, Pari's library mode for the exclusion of the existence of 'small' solutions. We have done this part of the calculations twice, first we only used rational numbers in the continued fraction procedure (this took 21 days for $j = 1$ and 8 days for $j = 2$ on the DEC Alpha), then we used high precision real numbers (18 hours for $j = 1$ and 7 hours for $j = 2$ on the Pentium).

Remark (Added in proof): In the numerical calculations we have used a result of Voutier on linear forms in three logarithms, which is not published yet. Applying the general theorem of Baker-Wüstholz [3] instead of Voutier [33], the numerical computations can be performed, however, computation time increases significantly.

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