Acta Mathematica et Informatica Universitatis Ostraviensis

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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 5 (1997), No. 1, 53--59

Persistent URL: http://dml.cz/dmlcz/120513

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Inversions on complete lattices and Girard quantales

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Abstract: We shall introduce inversions on complete lattices and describe some connections with symmetric relations, polarities and quantales. The inversions help us to characterize some generalization of Girard quantales. Finally, we have an opportunity to use an inversion instead of linear negation in Girard quantales and describe an application in linear logic.

Mathematics Subject Classification: 06A23, 06F99

A fundamental definition of polarity on algebraical structures has been described by G.Birkhoff (see [1]). Let us recall that if ρ is a binary relation on a non-empty set M then *a polar* of a set $X \subseteq M$ is $X' = \{m \in M : m\rho x$ for every $x \in X\}$.By induction we can define $X^n = (X^{n-1})'$, for every natural number n.The set of all polars on M belonging to ρ we shall denote by $\rho(M)$.

It is easy to see that the following assertions are equivalent:

(i) ρ is symmetric;

(ii) $X \to X''$ is a closure operator on M;

(iii) $X \subseteq X''$, for every $X \subseteq M$.

From the previous result we can recognize the motivation for the following definition: *A polarity* on a non-empty set M is a symmetric binary relation on M.

We can investigate a polarity from the general point of view (see [4]) and characterize the structure of polars as a complete lattice with an inversion.

We shall describe also quantales with inversions. Let us recall that *a quantale* (see [3], Def. 2.1.1) is a complete lattice $(Q, \vee, \wedge)$ with an associative binary operation . which distribute with arbitrary joins (i.e., $a. \vee b_i = \vee(a.b_i)$ and $(\vee b_i).a = \vee(b_i.a)$ hold, for $a, b_i \in Q$). The top and bottom elements are denoted *0,1*. A frame is a quantale such that $. = \wedge$. We can specify our investigations and give an application of quantales with inversions in linear logic. *Linear logic* is an important part of theoretical computer science and is based on the notion of Girard guantales (see [2] and [3]). J.Y.Girard constructed linear logic with help of linear negation that enable us to define basic connectives in linear logic. A Girard quantale is a quantale with an inversion. We can construct a generalization of Girard linear logic which take advantage of quantales with inversions.

1. Inversions and polarities

Let us introduce an inversion and describe some connections with symmetric relations, polarities and quantales.

1.1 Definition We say that *an inversion* $'$ is given on a complete lattice $(S, \vee, \wedge)$ if a map $' : S \to S$ exists such that $a \leq b \Leftrightarrow a' \geq b'$ and $a'' = a$, for every $a, b \in S$.

Remark. An inversion is an involutive dual order automorphism with properties $(\vee a_i)' = \wedge a_i'$ and $(\wedge a_i)' = (\wedge a_i'')' = (\vee a_i')'' = \vee a_i'$, for $a_i \in S$.

1.2 Proposition *1. If ρ is a symmetric binary relation on a non-empty set M then the set $\rho(M)$ of all polars is a complete lattice with an inversion.*

2. If S is a complete lattice with an inversion then a symmetric binary relation ρ exists on S such that S and $\rho(S)$ are isomorphic complete lattices.

Proof. 1. If we define $\wedge A_i = \cap A_i$ and $\vee A_i = (\cup A_i)''$ for $A_i \in \rho(M)$ then $(\rho(M), \vee, \wedge)$ is a complete lattice with an involution given by polars which is antitone.

2. Let us put $a\rho b \Leftrightarrow b \leq a'$, for $a, b \in S$. Then ρ is a symmetric binary relation on S and $A^\nabla = \{m \in S : m \leq a', a \in A\} = \bigcap\limits_{a\in A} \downarrow a' = \downarrow \bigwedge\limits_{a\in A} a' = \downarrow (\bigwedge\limits_{a\in A} a')'' = \downarrow (\bigvee\limits_{a\in A} a'')' = \downarrow (\vee A)' = \{\vee A\}^\nabla$ holds, where $A \in \rho(M)$ and A^∇ denotes a polar A in ρ, while the denotation a' belongs to the given structure of polars on S.

Let us prove that the map $f : S \to \rho(S)$, defined in the following way $f(a) = \{a\}^{\nabla\nabla}$, is an isomorphisms of complete lattices. From the previous results we have $\{a\}^\nabla = \downarrow a'$ and $f(a) = (\{a\}^\nabla)^\nabla = (\downarrow a')^\nabla = \downarrow a'' = \downarrow a$ and therefore f is an injection. Further, $A = (A^\nabla)^\nabla = [\downarrow (\vee A)']^\nabla = \downarrow (\vee A)'' = \downarrow \vee A = f(\vee A)$ holds for $A \in \rho(S)$ and therefore f is a surjection. Finally, we have $f(\wedge a_i) = \downarrow \wedge a_i = \cap \downarrow a_i = \wedge f(a_i)$ and $f(\vee a_i) = \downarrow \vee a_i = \vee \downarrow a_i = \vee f(a_i)$.

1.3 Theorem *1. Let $(M, .)$ be a semigroup and ρ be a symmetric binary relation on M. Then $\rho(M)$ is a quantale with an inversion according to operations $X * Y = (X.Y)'', \vee X_i = (\cup X_i)'', \wedge X_i = \cap X_i$ for $X, Y, X_i \in \rho(M)$ if and only if the following condition (Q)*

$$X''.Y \cup X.Y'' \subseteq (X.Y)''$$

holds, for every $X, Y \subseteq M$.

2. Let Q be a quantale with an inversion. Then a symmetric relation ρ exists on Q such that Q and $\rho(Q)$ are isomorphic quantales.

Proof. 1. $\Rightarrow$: It follows from 1.2 Proposition and [4], 3.1 and 3.2.

$\Leftarrow$: $\rho(M)$ is a complete lattice with an inversion (see 1.2,1). A map $\rho : expM \to \rho(M)$, such that $\rho(X) = X''$ for $X \subseteq M$, is a closure operator with the following property $X''.Y'' \subseteq (X.Y'')'' \subseteq (X.Y)'''' = (X.Y)''$. Then ρ is a quantic nucleus (see [3], Definition 3.1.1) and $\rho(M) = (expM)_\rho$ is a quantic quotient belonging to the quantale homomorphism $\rho : M \to \rho(M)$ such that $\rho(X) = X''$ for $X \subseteq M$ (see [3], Theorem 3.1.1). Finally, $\rho(M)$ is a quantale.

2. A symmetric binary relation ρ exists on Q such that Q and $\rho(Q)$ are isomorphic complete lattices (see 1.2,2). With regard to the proof of 1.2,2 we have to prove that the map $f : Q \to \rho(Q)$ such that $f(a) = \{a\}^{\nabla\nabla}$ is a semigroup homomorphism: The fact $\{a\}^{\nabla\nabla} = \downarrow a$ implies $f(a) * f(b) = (\{a\}^{\nabla\nabla}.\{b\}^{\nabla\nabla})^{\nabla\nabla} = \{a.b\}^{\nabla\nabla} = f(a.b)$, for $a, b \in Q$.

2. Girard quantales and inversions

In the second part we shall describe some algebraic properties of Girard quantales from point of view of inversions. Let us recall that *a Girard quantale* is a quantale Q which has so called *a cyclic dualizing element d*. If we denote $\vee(t \in Q : a.t \leq b) = a \to_r b, \vee(t \in Q : t.a \leq b) = a \to_l b$ then "cyclic" means that $a \to_r d = a \to_l d$ for every $a \in Q$ and if we denote $a \to_r d = a \to_l d = a^\perp$ then "dualizing" means that $a^{\perp\perp} = a$ for every $a \in Q$. The operator $\perp$ is called *a linear negation* in the Girard linear logic (see [2] and [3], Chapter 6).

2.1 Proposition *A Girard quantale Q is a quantale with an inversion $^\perp$. The cyclic dualizing element d is uniquely determined by the formula $d = \vee(a.a^\perp \vee a^\perp.a : a \in Q)$.*

Proof. $^\perp$ is an inversion on Q.

Now, let us compute the element d. We have $a^\perp = \vee(t \in Q : t.a \leq d) = \vee(t \in Q : a.t \leq d)$ and thus $a.a^\perp = \vee(a.t : a.t \leq d) \leq d$, $a^\perp.a \leq d$ hold. If $z \in Q$ such that $z \geq a.a^\perp \vee a^\perp.a$ for every $a \in Q$ then $z \geq d.d^\perp \vee d^\perp.d = d$ holds because $d^\perp$ is a unit in Q (see [3], p. 140, Corollary). Together $d = \vee(a.a^\perp \vee a^\perp.a : a \in Q)$ holds.

Remark. If we have the other inversion $' : Q \to Q$ such that d' is a unit in Q and $a.a' \vee a'.a \leq d$ holds then $a' \leq a^\perp$, for $a \in Q$.

Example. If $(G, +)$ is a lattice ordered group and $a\rho b \Leftrightarrow |a| \wedge |b| = 0$, where $|a| = a \vee -a$, then we have the classical polarity on l-groups. The binary relation ρ is symmetric on the positive cone G^+. $(G^+, +)$ is a semigroup and let us prove that $(A'' + B) \cup (A + B'') \subseteq (A + B)''$ holds for every $A, B \subseteq G^+$:

If $x \in G^+$ is an element such that $x\rho(a+b)$ for all $a \in A, b \in B$ then $x \wedge (a+b) = 0$ and thus $x \wedge a = x \wedge b = 0$. If $c \in B''$ then $c \wedge d = 0$ for $d \in G^+$ such that $b \wedge d = 0$ for every $b \in B$. We have $x \wedge c = 0$ and $x \wedge a = 0$. That facts imply $x \wedge (a + c) = 0$ because $0 = x \wedge c = x \wedge [(x \wedge a) + c] = x \wedge (x + c) \wedge (a + c) = x \wedge (a + c)$.

We obtain $(A+B)' \subseteq (A+B'')'$, i.e., $(A+B)'' \supseteq (A+B'')'' \supseteq A+B''$. Similarly we can prove that $(A+B)'' \supseteq A''+B$. Proposition 1.3 implies that $\rho(G^+)$ is a quantale with an inversion regarding to the operations $X*Y = (X+Y)''$, $\vee X_i = (\cup X_i)''$, $\wedge X_i = \cap X_i$ for $X, Y, X_i \in \rho(G^+)$. That quantale is not a Girard quantale because polars from $\rho(G^+)$ are not defined consistently with the definition of Girard quantales.

2.2 Proposition *A Girard quantale Q is a Boolean algebra if and only if Q is an idempotent quantale with a cyclic dualizing element $d = 0$.*

Proof. $\Rightarrow$: We have $a.a^\perp = a^\perp.a = a \wedge a^\perp = 0$, for every $a \in Q$ and thus $d = 0$.

$\Leftarrow$: For every $x, y \in Q$ there holds $x.y \leq x.1 = x, x.y \leq 1.y = y$ because $d^\perp = 1$ is a unit. Further, $x \wedge y = (x \wedge y).(x \wedge y) \leq x.y$. Finally, $x.y = x \wedge y$ holds, i.e., Q is a frame and $a, a^\perp$ are complementary elements.

2.3 Corollary *Let L be a frame. Then the following assertions are equivalent:*

1. *L is a Girard quantale.*
2. *L is a Boolean algebra.*
3. *L is a quantale with an inversion and $a' = \vee(t \in L : t \wedge a = 0)$ for $a \in L$.*

Proof. $1 \Rightarrow 2$ follows from 2.2.

$2 \Rightarrow 3$: The map that every element from L maps in its complement is an inversion.

$3 \Rightarrow 1$: 0 is a cyclic dualizing element and $a^\perp = a'$ for $a \in L$.

2.4 Theorem *Let Q be a unital quantale with a unit e. Then the following assertions are equivalent:*

1. *Q is a Girard quantale.*
2. *Q is a quantale with an inversion and $a' = a \rightarrow_l e' = a \rightarrow_r e'$, for every $a \in Q$.*
3. *Q is a quantale with an inversion and for every $a, b, t \in Q$ it holds $t.a \leq b' \Leftrightarrow b.t \leq a'$.*

Proof. $1 \Rightarrow 3$: If Q is a Girard quantale then the linear negation $\perp$ on Q is an inversion and $a \rightarrow_l b^\perp = b \rightarrow_r a^\perp$, for $a, b \in Q$ (see [3], Proposition 6.1.2,(6)). We have $t.a \leq b^\perp \Leftrightarrow t \leq a \rightarrow_l b^\perp = b \rightarrow_r a^\perp \Leftrightarrow b.t \leq a^\perp$.

$3 \Rightarrow 2$: The fact $t.a \leq e' \Leftrightarrow t \leq a'$ implies $a \rightarrow_l e' \leq a'$ and the fact $a.t \leq e' \Leftrightarrow t \leq a'$ implies $a \rightarrow_r e' \leq a'$. Further, $a'.a \leq e'$ and $a.a' \leq e'$ hold and thus $a' \leq a \rightarrow_l e'$ and $a' \leq a \rightarrow_r e'$.

$2 \Rightarrow 1$: Facts $a' = a \rightarrow_l e' = a \rightarrow_r e'$ and $a'' = a$ implies that e' is a cyclic dualizing element and Q is a Girard quantale with the linear negation $'$.

Now, we can investigate unital quantales with inversions similar to Girard quantales. J.Y.Girard in [2] constructs foundation of linear logic on Girard quantales with help of linear negation, multiplicative and additive connectives.

Let us generalize that conception and use an inversion instead of linear negation. We have an opportunity to give a foundation of generalized linear logic on unital quantales with inversions. We shall describe multiplicative and additive connectives on arbitrary semigroup with a symmetric binary relation that has the property (Q) from Theorem 1.3.

If $(M, .)$ is a semigroup and ρ is a symmetric binary relation on M then we can suppose the condition (Q) from Theorem 1.3 and construct a quantale $\rho(M)$ of polars on M regarding to ρ with operations $X * Y = (X.Y)'', \vee X_i = (\cup X_i)'', \wedge X_i = \cap X_i$, for $X, Y, X_i \in \rho(M)$.

Now, we can investigate also additive connectives on $\rho(M)$ in the form $X \wedge Y = X \cup Y$ and $X \vee Y = (X \cap Y)''$. Additive connectives are exactly lattice operations in a quantale with an inversion (see [2], def 1.11).

2.5 Definition Let Q be a quantale with an inversion $'$. Then we define *multiplicative connectives on Q:*

a) Paralelization: $a \sqcup b = (a'.b')'$.

b) Linear implication: $a \rightarrow b = (a.b')'$.

2.6 Proposition *Multiplicative connectives have the following properties:*

1.a)(De Morgan principles) $(a \sqcup b)' = a'.b', (a.b)' = a' \sqcup b'$,

b) $a.b = (a \rightarrow b')', a \sqcup b = a' \rightarrow b$.

2.a) $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$,

b) $(a.b) \rightarrow c = a \rightarrow (b \rightarrow c)$,

c) $a \rightarrow (b \sqcup c) = (a \rightarrow b) \sqcup c$.

3. If Q is a commutative quantale then $a \sqcup b = b \sqcup a$ and $a \rightarrow b = b' \rightarrow a'$ hold.

4. If Q is a unital quantale with a unit e then $a \sqcup e' = e' \sqcup a = a, e \rightarrow a = a$, $a \rightarrow e' = a'$ hold.

Proof. 1. The formulas follows from definitions immediately.

2.a) $a \sqcup (b \sqcup c) = (a'.(b \sqcup c)')' = (a'.(b'.c'))' = ((a'.b').c')' = ((a \sqcup b)'.c')' = (a \sqcup b) \sqcup c$,

b) $(a.b) \rightarrow c = ((a.b).c')' = (a.(b.c'))' = ((a.(b.c'))'')' = a \rightarrow (b.c')' = a \rightarrow (b \rightarrow c)$,

c) $a \rightarrow (b \sqcup c) = (a.(b \sqcup c)')' = (a.(b'.c')'')' = (a.(b'.c'))' = ((a.b').c')' = ((a.b')''.c')' = ((a \rightarrow b)'.c')' = (a \rightarrow b) \sqcup c$.

3. and 4. follows from definitions.

2.7 Proposition *If Q is a quantale with an inversion and $a, b, c \epsilon Q$ then the following formulas hold:*

a) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$,

b) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$,

c) $(a \sqcup b) \wedge (a \sqcup c) = a \sqcup (b \vee c)$,

d) $(a \sqcup b) \vee (a \sqcup c) \leq a \sqcup (b \vee c)$,
e) $(a \to b) \vee (c \to b) \leq (a \wedge c) \to b$,
f) $(a \to b) \vee (a \to c) \leq a \to (b \vee c)$.

Proof. a) $a \to (b \wedge c) = (a.(b \wedge c)')' = (a.(b' \vee c'))' = (a.b' \vee a.c')' = (a.b')' \wedge (a.c')' = (a \to b) \wedge (a \to c)$,

b) $(a \vee b) \to c = ((a \vee b).c')' = (a.c' \vee b.c')' = (a.c')' \wedge (b.c')' = (a \to c) \wedge (b \to c)$,

c) $(a \sqcup b) \wedge (a \sqcup c) = (a'.b')' \wedge (a'.c')' = ((a'.b') \vee (a'.c'))' = (a'.(b' \vee c'))' = (a'.(b \wedge c)')' = a \sqcup (b \wedge c)$,

d) $(a \sqcup b) \vee (a \sqcup c) = (a'.b')' \vee (a'.c')' = (a'.b' \wedge a'.c')' \leq (a'.(b' \wedge c'))' = (a'.(b \vee c)')' = a \sqcup (b \vee c)$,

e) $(a \to b) \vee (c \to b) = (a.b')' \vee (c.b')' = (a.b' \wedge c.b')' \leq ((a \wedge c).b')' = (a \wedge c) \to b$,

f) $(a \to b) \vee (a \to c) = (a.b')' \vee (a.c')' = (a.b' \wedge a.c')' \leq (a.(b' \wedge c'))' = (a.(b \vee c)')' = a \to (b \vee c)$.

Multiplicative and additive connectives have similar properties like on Girard quantales.

2.8 Corollary *Let* $(Q, ., \vee, \wedge)$ *be a quantale with an inversion* $'$. *Then* $(Q, \wedge, \sqcup)$ *is a lattice if and only if* $x.y = x \wedge y$, *for all* $x, y \in Q$.

Proof. $\Leftarrow$: We have $x \sqcup x = (x' \wedge x')' = x'' = x$, $x \sqcup y = (x' \wedge y')' = (y' \wedge x')' = y \sqcup x$, $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$, (see 2.6,2.a)), $x \wedge (x \sqcup y) = x'' \wedge (x' \wedge y')' = (x' \vee (x' \wedge y'))' = x'' = x$ and $x \sqcup (x \wedge y) = (x' \wedge (x \wedge y)')' = (x' \wedge (x' \vee y'))' = x'' = x$ for $x, y, z \in Q$.

$\Rightarrow$: We have $x.x = (x''.x'')'' = (x' \sqcup x')' = x'' = x$, $x.y = (x''.y'')'' = (x' \sqcup y')' = (y' \sqcup x')' = (y''.x'')'' = y.x$, $x = x'' = (x' \sqcup 0)' = (x''.0')'' = x.1$, for $x, y \in Q$.

Further, $x \wedge y = (x \wedge y).(x \wedge y) \leq x.y$ and $x.y \leq x.1 \wedge 1.y = x \wedge y$, i.e., $x \wedge y = x.y$, for $x, y \in Q$.

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