Acta Mathematica et Informatica Universitatis Ostraviensis

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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 4 (1996), No. 1, 75--82

Persistent URL: `http://dml.cz/dmlcz/120506`

The Cantor series of polyadic numbers

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Abstract. In this paper we prove the possibility of evaluation of polyadic numbers in the Cantor series and later using results of the theory of probability we investigate the properties of these evaluations.

1991 Mathematics Subject Classification: 11B05

1 Introduction

Let $\mathbb{N}$ be the set of positive integers. Denote by Ω the ring of polyadic numbers constructed by E. V. Novoselov in 1960 as the completion of $\mathbb{N}$ with respect to certain metric d given by divisibility. (cf.[1]). For a general survey of polyadic numbers, we refer to [3]. The extension of the metric d on Ω will be denoted by the same symbol. Then it holds the metric space (Ω, d) is compact (cf.[1]).

In the set Ω we can extend in a natural way the operations of addition and multiplication $+, \cdot$, such that $(\Omega, +, \cdot)$ is a ring and such that the operations $+, \cdot$ are continuous with respect to the metric d (cf. [1]). Moreover one has:

Theorem A. *([1]) For every $\alpha \in \Omega$ and $m \in \mathbb{N}$ there exists uniquely determined elements $\beta \in \Omega$ and r integer such that $0 \leq r < m$ and*

$$\alpha = m \cdot \beta + r.$$

The number r from Theorem A will be called the rest of α modulo m and will be denoted by $\alpha \bmod m$. Due to Theorem A the relation of divisibility by $m \in \mathbb{N}$ can be extended in a natural way to Ω. The extention of metric d on Ω is then given by

$$d(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\alpha - \beta)}{2^n}$$

where $\varphi_n(\gamma) = 0$ if $n|\gamma$ and $\varphi_n(\gamma) = 1$ if $n \nmid \gamma$ for $n \in \mathbb{N}$ and $\gamma \in \Omega$. The convergence of the sequence $\{\alpha_n\}$,$\alpha_n \in \Omega$ to $\alpha \in \Omega$ can be characterised as follows [1]:

$$\alpha_n \to \alpha \leftrightarrow \forall N \in \mathbb{N} \exists n_0 \forall n > n_0; \alpha_n \equiv \alpha \pmod{N!}$$

and the operations $+, \cdot$ are continuous according to the metric d.

Let us denote $(\gamma) = \{\gamma \cdot \alpha, \alpha \in \Omega\}$, the principal ideal generated by γ. In [2] it is constructed the Haar probability borel measure P such that

$$P(\alpha + (m)) = \frac{1}{m}, m \in \mathbb{N}, \alpha \in \Omega.$$

In this paper we shall investigate the possibility and the properties of evaluation of the polyadic numbers in the Cantor series. For our considerations the folloving notion will be fundamental:

Definition. Let $\{C_n\}$ be an increasing sequence of positive integers such that $C_n \to 0$ with respect to polyadic norm. This sequence will be called well if $C_n | C_{n+1}$ for $n = 0, 1, \dots$ and $C_0 = 1$.

We shall need the following results from the theory of probability.

Lemma A. *(1. Borel-Cantelli lemma)([4]) Let (X, P) be a probability space and $\{D_n\}$ a sequence of measurable sets. If*

$$\sum_{n=1}^{\infty} P(D_n) < \infty \text{ and } D = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} D_j$$

then $P(D) = 0$.

If (X, P) is a probability space and η is a real valued measurable function on X, called also random variable, then the value $E(\eta) = \int \eta dP$ is caled the mean value of η and the value

$$D^2(\eta) = E((\eta - E(\eta))^2)$$

will be called the dispersion of η.

Lemma B. *([4]) Let (X, P) be a probability space. Let $\{\eta_n\}$ be a sequence of independent random variables such that $E(\eta_n) = 0$ for $n = 1, 2, \dots$. Let $\{b_n\}$ be a nondecreasing sequence of real numbers such that $b_n \to \infty$. If*

$$\sum_{n=1}^{\infty} \frac{D^2(\eta_n)}{b_n^2} < \infty$$

then for almost all $x \in X$ we have

$$\lim_{n \to \infty} \frac{1}{b_n} \sum_{k=1}^{n} \eta_k(x) = 0.$$

Let $p(\alpha)$ be a property of $\alpha \in \Omega$. We shall write only $P(p(\alpha))$ instead of the symbol $P(\{\alpha \in \Omega;\ p(\alpha)\})$ as it is common in the theory of probability. We shall say that almost all elements of Ω have the property p if $P(p(\alpha)) = 1$.

2 The Cantor series

There exist many examples of well sequences. One of such a sequence is the sequence of factorials $\{n!\}$. It is proved [1] that each polyadic number α can be expressed in form

$$\alpha = \sum_{n=0}^{\infty} a_n n!$$

where $0 \le a_n \le n$. The following result will be a generalisation:

Theorem 1. *Let $\{C_n\}$ be a well sequence. Then for each $\alpha \in \Omega$ there exists an uniquely determined sequence $\{a_n\} = \{a_n(\alpha)\}$ of nonnegative integers such that*

$$\alpha = \sum_{n=0}^{\infty} a_n C_n \tag{1}$$

and

$$0 \le a_n < \frac{C_{n+1}}{C_n}, n = 0, 1, \ldots \tag{2}$$

PROOF: By induction we prove that there exists a sequence of nonnegative integers $\{a_n\}$ satisfying (2) such that for $n \in \mathbb{N}$

$$\alpha = a_0 C_0 + \ldots + a_n C_n + \alpha_{n+1} C_{n+1} \tag{3}$$

for some $\alpha_{n+1} \in \Omega$. If $n = 1$ then Theorem A gives

$$\alpha = a_0 + \alpha_1 C_1 = \alpha = a_0 C_0 + \alpha_1 C_1$$

where $0 \le a_0 \le C_1 = \frac{C_1}{C_0}$. Let us suppose that

$$\alpha = a_0 C_0 + \ldots + a_{n-1} C_{n-1} + \alpha_n C_n.$$

Theorem A gives

$$\alpha_n = a_n + \alpha_{n+1} \frac{C_{n+1}}{C_n}$$

where $0 \le a_n < \frac{C_{n+1}}{C_n}$, and so

$$\alpha = a_0 C_0 + \ldots + a_n C_n + \alpha_{n+1} C_{n+1}.$$

But $C_n \to 0$ and so (1) follows. Let

$$\alpha = \sum_{n=0}^{\infty} a'_n C_n, 0 \le a'_n < \frac{C_{n+1}}{C_n}, n = 0, 1, \ldots.$$

Considering that $C_{n+1} | \sum_{k=n+1}^{\infty} a'_k C_k$ the equation (3) gives for $n = 0, 1, \ldots$

$$a_0 C_0 + \ldots + a_n C_n = \alpha \bmod C_{n+1} = a'_0 C_0 + \ldots + a'_n C_n.$$

And so $a_0 = a'_0$ and by induction we obtain $a_n = a'_n$ for $n = 0, 1, \ldots$. The proof is complete. $\square$

From now on, we shall assume that $\{C_n\}$ is a given well sequence.

The representation (1) will be called the Cantor series of α with respect to $\{C_n\}$.

It can be interesting the question why the series (1) gives an integer. If $a \in \mathbb{N}$ then $a < C_n$ for some $n \in \mathbb{N}$. Then $a = a \bmod C_n$ and (3) gives that $a = a_0C_0 + \ldots + a_{n-1}C_{n-1}$. Thus (1) is an evaluation of positive integer if and only if $a_k = 0, k > n$ for some $n \in \mathbb{N}$. Let us consider a negative integer $-a$. Then there exists C_n such that $C_n - a > 0$. And so

$$C_n - a = \sum_{j=0}^{n-1} a_j C_j, 0 \le a_j < \frac{C_{j+1}}{C_j}, j = 0, 1, \ldots n-1, \tag{4}$$

but

$$\sum_{j=n}^{\infty} (\frac{C_{j+1}}{C_j} - 1)C_j = \sum_{j=n}^{\infty}(C_{j+1} - C_j) = -C_n,$$

and so (4) gives

$$-a = \sum_{j=0}^{n-1} a_j C_j + \sum_{j=n}^{\infty} (\frac{C_{j+1}}{C_j} - 1)C_j.$$

Thus due to the uniqueness of evaluation we have proved:

Theorem 2. *The polyadic number α given by (1) is an integer if and only if it holds one of two conditions:*

(i) $\exists n_0 \forall n > n_0; a_n = 0$

(ii) $\exists n_0 \forall n > n_0; a_n = \frac{C_{n+1}}{C_n} - 1$

3 The properties of coefficients of Cantor series

Theorem 1 gives the possibility to consider the coeficients $a_n = a_n(\alpha)$ from (1) as functions of α.

In following we shall investigate the properties of this sequence of functions. We shall need:

Lemma 1. *Let $a_0^0, \ldots, a_n^0$ be a finite sequence such that $0 \le a_j^0 < \frac{C_{j+1}}{C_j} - 1$, $j = 0, \ldots, n$. Put $b = a_0^0 C_0 + \ldots + a_n^0 C_n$. Then*

$$\forall \alpha \in \Omega; \alpha \in b + (C_{n+1}) \Longleftrightarrow a_j(\alpha) = a_j^0, j = 0, \ldots, n$$

PROOF: Clearly

$$0 \le b \le \sum_{j=0}^{n} (\frac{C_{j+1}}{C_j} - 1)C_j = C_{n+1} - 1.$$

Thus $\alpha \in b + (C_{n+1}) \Leftrightarrow b = \alpha \bmod C_{n+1}$. Considering now (3) we obtain the assertion. The proof is complete. □

Lemma 1 implies that the functions $a_n, n = 0, 1, 2, \ldots$ are measurable, and so these functions can be considered as random variables defined on the probability space (Ω, P). The class $b + (C_{n+1})$ from Lemma 1 will be called the class associated with the sequence $a_0^0, \ldots, a_n^0$.

Let $n \in \mathbb{N}$ and $0 \le a < \frac{C_{n+1}}{C_n}$. Let $b_1, \ldots, b_r$ be all numbers of the form $d_0C_0 + \ldots + d_{n-1}C_{n-1} + aC_n$, where $0 \le d_j < \frac{C_{j+1}}{C_j}, j = 0, \ldots, n-1$. Then

$$\{\alpha \in \Omega; a_n(\alpha) = a\} = \bigcup_{i=1}^{r} b_i + (C_{n+1}).$$

It is easy to see that $r = \frac{C_1}{C_0}\frac{C_2}{C_1} \cdots \frac{C_n}{C_{n-1}} = C_n$ and so

$$P(a_n(\alpha) = a) = \frac{C_n}{C_{n+1}}. \tag{5}$$

Theorem 3. *If $\sum_{n=1}^{\infty} \frac{C_n}{C_{n+1}} < \infty$ then for allmost all $\alpha \in \Omega$ any nonnegative integer occures in the sequence $\{a_n(\alpha)\}$ at most finitely times.*

PROOF: Let denote for $k \in \mathbb{N} \cup \{0\}$

$$D_n(k) = \{\alpha; a_n(\alpha) = k\},$$

Clearly $P(D_n(k)) = \frac{C_n}{C_{n+1}}$ or 0. Thus $\sum_{n=1}^{\infty} P(D_n(k)) < \infty$. Put

$$D(k) = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} D_j(k).$$

Trivialy $\alpha \in D(k) \Leftrightarrow a_j(\alpha) = k$ for infinitely many j. Lemma A gives $P(D(k)) = 0$ and so $P(\cup_{k=0}^{\infty} D(k)) = 0$. The proof is complete. □

The condition from Theorem 3 is satisfied for instance for the well sequence $\{(n!)^2\}$.

Lemma 2. *If $m \neq n \in \mathbb{N}$ then the functions a_m, a_n are independent.*

PROOF: Let $m < n$. We consider a_m^0, a_n^0 such that $0 \leq a_m < \frac{C_{m+1}}{C_m}, 0 \leq a_n < \frac{C_{n+1}}{C_n}$. Then

$$\{\alpha \in \Omega; a_m(\alpha) = a_m^0, a_n(\alpha) = a_n^0\} = \bigcup_{i=1}^{q} b_i + (C_{n+1}),$$

where $b_1, \ldots, b_q$ are all numbers in the form

$$d_0C_0 + \ldots + d_{m-1}C_{m-1} + a_m^0 C_m + d_{m+1}C_{m+1} + \ldots + d_{n-1}C_{n-1} + a_n^0 C_n$$

and thus

$$q = \frac{C_1}{C_0} \cdots \frac{C_m}{C_{m-1}} \frac{C_{m+2}}{C_{m+1}} \cdots \frac{C_n}{C_{n-1}} = C_m \frac{C_n}{C_{m+1}}.$$

Therefore by (5) we obtain

$$P(a_m(\alpha) = a_m^0, a_n(\alpha) = a_n^0) = \frac{C_m}{C_{m+1}} \frac{C_n}{C_{n+1}} = P(a_m(\alpha) = a_m^0) P(a_n(\alpha) = a_n^0).$$

And the proof is complete. □

For $\alpha \in \Omega, n \in \mathbb{N}$ and r nonnegative integer let us denote by $N_n(\alpha, r)$ the number of occurance of r in the sequence $a_0(\alpha), \ldots, a_n(\alpha)$. The following assertion will be an analogy of the Low of normal numbers.

Theorem 4. *Let $\sum_{n=1}^{\infty} \frac{C_n}{C_{n+1}} = \infty$ and let r be a nonnegative integer such that $r < \frac{C_{n+1}}{C_n}, n > n_0$ for some n_0. Then for almost all $\alpha \in \Omega$ it holds*

$$\lim_{n \to \infty} \frac{N_n(\alpha, r)}{\sum_{j=1}^{n} \frac{C_j}{C_{j+1}}} = 1.$$

PROOF: Let us define the system of random functions $\{\xi_n\}$ as follows

$$\xi_n(\alpha) = \begin{cases} 1, & \text{if } a_n(\alpha) = r, \\ 0 & \text{for } a_n(\alpha) \neq r, \end{cases}$$

for $n = 1, 2, \ldots$. Lemma 2 gives that these functions are independent. Clearly

$$N_n(\alpha, r) = \sum_{j=1}^{n} \xi_j(\alpha) \tag{6}$$

for $n = 1, 2, \ldots$. Moreover for $n > n_0$ we have by (5)

$$E(\xi_n) = P(\xi_n = 1) = \frac{C_n}{C_{n+1}}.$$

Put for $n = 1, 2, \ldots$

$$\eta_n = \xi_n - \frac{C_n}{C_{n+1}}. \tag{7}$$

Then for $n > n_0$ we have $E(\eta_n) = 0$. And

$$D^2(\eta_n) = E(\eta_n^2) = \frac{C_n}{C_{n+1}}\left(1 - \frac{C_n}{C_{n+1}}\right).$$

Define now

$$b_n = \sum_{j=1}^{n} \frac{C_j}{C_{j+1}}, n = 1, 2, \ldots \tag{8}$$

then $\{b_n\}$ is an increasing sequence and $b_n \to \infty$. Moreover

$$\sum_{n=n_0}^{\infty} \frac{D^2(\eta_n)}{b_n^2} = \sum_{n=n_0}^{\infty} \frac{\frac{C_n}{C_{n+1}}\left(1 - \frac{C_n}{C_{n+1}}\right)}{b_n^2} \le \sum_{n=n_0}^{\infty} \frac{\frac{C_n}{C_{n+1}}}{b_n b_{n-1}} =$$

$$= \sum_{n=n_0}^{\infty} \frac{b_n - b_{n-1}}{b_n b_{n-1}} = \sum_{n=n_0}^{\infty} \frac{1}{b_{n-1}} - \frac{1}{b_n} < \infty.$$

Lemma B now implies that for almost all $\alpha \in \Omega$ it holds

$$\lim_{n \to \infty} \frac{\sum_{k=n_0}^{n} \eta_k(\alpha)}{b_n} = 0.$$

Therefore (6), (7) and (8) imply the the assertion. The proof is complete. □

Corollary. *If we consider the factorial expansion of* $\alpha \in \Omega$

$$\alpha = \sum_{n=1}^{\infty} a_n(\alpha) n!$$

then for almost all $\alpha \in \Omega$ *we have*

$$N_n(\alpha, r) \sim \log n$$

for each r *nonnegative integer.*

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