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A note on stability properties of integrated semigroups

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Abstract. Asymptotic stability of a certain class of integrated semigroups is discussed by means of Lyapunov functionals.

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1 Introduction

We consider the abstract Cauchy problem, (ACP):

$$\dot{u} = \tilde{A}u(t), \quad t > 0; \qquad u(0) = x$$

in a Hilbert space H and discuss the asymptotic stability of the integrated semigroup $\tilde{S}(t)$ associated with (ACP).

Definition 1. Let $\tilde{S}(t)$ be a nondegenerate N-times integrated semigroup on H such that $|\tilde{S}(t)| \leq M \exp(\omega t)$ for $t \geq 0$, some $\omega \in R$ and some $M \geq 1$. If a closed linear operator $\tilde{A}$ has the resolvent $R(\mu; \tilde{A})$ and satisfies

$$R(\mu; \tilde{A}) = \mu^N \int_0^\infty \exp(-\mu t)\tilde{S}(t)x dt \quad \text{for } \mu > \omega \text{ and } x \in H,$$

then $\tilde{A}$ is called the *generator* of $\tilde{S}(t)$.

We need the following propositions.

Proposition 1. *(F. Neubrander) By a solution of (ACP) is meant an H-valued, strongly continuous function $u(\cdot) : [0, +\infty) \to H$ which satisfies the evolution equation (DE) for $t > 0$ and the initial condition (IC). If $\tilde{A}$ is the generator of a nondegenerate, exponentially bounded N-times integrated semigroup $\tilde{S}(t)$ on H, then (ACP) is well-posed in the sense that there exist constants M and ω such that for each $x \in D(\tilde{A}^{N+1})$ there exists a unique solution $u(\cdot)$ such that $|u(t)| \leq M \exp(\omega t)|x|_N$, where $|x|_N$ denotes the graph norm of the space $D(\tilde{A}^N)$ defined by*

$$|x|_N = |x| + |\tilde{A}x| + \ldots + |\tilde{A}^N x|.$$

Proposition 2. *(S. Oharu) (a) The space $D(\tilde{A}^N)$ is a Banach space under the graph norm $|\cdot|_N$. We write Y for the Banach space, namely,*

$$Y = \left[D\left(\tilde{A}^N\right)\right].$$

(b) Since $\tilde{A}$ is the generator of a nondegenerate, exponentially bounded N-times integrated semigroup $\tilde{S}(t)$ on H, $\tilde{A}$ has a nonemty resolvent set $\rho(A)$.
(c) If $D(\tilde{A})$ is dense in H, $D(\tilde{A}^k)$ is dense in $D(\tilde{A}^j)$ for every pair of nonnegative integers j and k with $j \leq k$. In particular, Y is dense in H.

2 On stability properties of integrated semigroups

Given a (C_0) semigroup $T(t)$ on a Banach space X, a continuous functional $V : X \to R$ is called a Lyapunov functional for $T(t)$, if $V(0) = 0$ and

$$\dot{V}(x) \equiv \limsup_{t\downarrow 0} t^{-1}(V(T(t)x) - V(x)) \leq 0 \qquad \text{for } x \in X.$$

Moreover, a continuous functional $V : X \to R$ is said to be *quadratic* if there exists a bounded linear operator B from X into its dual space X^* such that

$$(Bx)(y) = \overline{(By)(x)} \qquad \text{for } x,\, y \in X$$

and $V(x) = (Bx)(x)$ for $x \in X$. If a Lyapunov functional V for $T(t)$ is quadratic, we say that V is a *quadratic Lyapunov functional* for $T(t)$.

It is the main feature of the argument of this paper to use quadratic Lyapunov functionals to discuss asymptotic stability of integrated semigroups.

Definition 2. We say that a (C_0)-semigroup $T(t)$ on a Banach space X is asymptotically stable, if $T(t)x \to 0$ in X as $t \to \infty$ for each $x \in X$, and that an N-times integrated semigroup $\tilde{S}(t)$ on H is asymptotically stable, if $(N!/t^N)\tilde{S}(t)x \to 0$ in H as $t \to \infty$ for each $x \in H$.

In what follows, we put the following conditions on the operator $\tilde{A}$:
(A) The operator $\tilde{A}$ has a dense domain $D(\tilde{A})$ in H and is the generator of a nondegenerate, exponentially bounded N-times integrated semigroup $\tilde{S}(t)$ on H.
(B) There exists a bounded linear operator $\tilde{B}$ from the Banach space Y into H with the three properties below:

$$\langle \tilde{B}y_1,\, y_2\rangle_H = \overline{\langle \tilde{B}y_2,\, y_1\rangle_H} = \langle y_1,\, \tilde{B}y_2\rangle_H \qquad \text{for } y_1,\, y_2 \in Y, \tag{B1}$$

$$\langle \tilde{B}y,\, y\rangle_H \geq \gamma|y|^2 \qquad \text{for } y \in Y \text{ and some } \gamma > 0, \tag{B2}$$

$$Re\langle \tilde{B}y,\ (\tilde{A}-\omega I)y\rangle_H \le 0 \qquad \text{for } y \in Y \text{ and some } \omega \in R. \tag{B3}$$

An inner product $\langle \cdot,\ \cdot\rangle_X$ is defined on the subspace $D(\tilde{A}^N)$ by

$$\langle y_1,\ y_2\rangle_X = \langle \tilde{B}y_1,\ y_2\rangle_H \qquad \text{for } y_1,\ y_2 \in D(\tilde{A}^N).$$

In fact, $\langle y,\ y\rangle_X = \langle \tilde{B}y,\ y\rangle \ge 0$ for $y \in D(\tilde{A}^N)$ and $\langle y,\ y\rangle_X = 0$ implies $\overline{|y| = 0}$ and $y = 0$ by (B2). For $y_1,\ y_2 \in D(\tilde{A}^N,\ \langle y_2,\ y_1\rangle_X = \langle \tilde{B}y_2,\ y_1\rangle_H = \overline{\langle \tilde{B}y_1,\ y_2\rangle_H} = \overline{\langle y_1,\ y_2\rangle_X}$ by (B1). Also, $\langle \alpha_1 y_1 + \alpha_2 y_2,\ z\rangle_X = \alpha_1\langle y_1,\ z\rangle_X + \alpha_2\langle y_2,\ z\rangle_X$ for y_1, $y_2 \in D(\tilde{A}^N)$ and $\alpha_1,\ \alpha_2 \in C$ by the linearity of $\tilde{B}$.

One can then define a norm $|\cdot|_X$ on $D(\tilde{A}^N)$ by $|y|_X = \langle y,\ y\rangle_X^{1/2}$. Let X be a completion of $D(\tilde{A}^N)$ with respect to the norm $|\cdot|_X$. Then the inner product $\langle \cdot,\ \cdot\rangle_X$ is naturally induced on X and X becomes a Hilbert space. From (B2) it follows that X can be regarded as a dense subspace of H and is continuously embedded in H. On the other hand, $\tilde{B}$ is a bounded linear operator from Y into H, and so the Banach space Y is continuously embedded in X. Therefore X is an intermediate Hilbert space in the sense that

$$Y \hookrightarrow X \hookrightarrow H.$$

The part A of $\tilde{A}$ in X is defined by

$$D(A) = \{x \in D(\tilde{A}) \cap X;\ \tilde{A}x \in X\}, \qquad Ax = \tilde{A}x \quad \text{for } x \in D(A)$$

and has the following properties:

Proposition 3. *(a) The operator A is a densely defined, closed linear operator in the Hilbert space X.*
(b) The subspace $D(\tilde{A}^{N+1})$ is a core of the operator A in X.
(c) Let ω be the constant appearing in (B3). Then $A-\omega$ is dissipative in X.
(d) The range condition holds in the sense that $R(\imath - \lambda A) = X$ for $\lambda > 0$.

From Proposition 3 and the Lumer–Phillips theorem it follows that A generates a (C_0)-semigroup $T(t)$ on X such that $|T(t)| \le \exp(\omega t)$ for $t \ge 0$ and the constant $\omega \in R$ appearing in condition (B3). If $\omega > 0$, the semigroup $T(t)$ is said to be quasi-contractive; if $\omega \le 0$, it is said to be a contradiction semigroup. Now the relationship between the (C_0)-semigroup $T(t)$ on X and the original integrated semigroup $\tilde{S}(t)$ on H may be stated as follows:

Proposition 4. *Let $T(t)$ be a (C_0)-semigroup on X generated by A.*
(a) For $x \in X$ and $t \ge 0$, $\tilde{S}(t)$ maps X into itself and

$$\tilde{S}(t)x = \int_0^t dt_{N-1}\int_0^{t_{N-1}} \cdots dt_1 \int_0^{t_1} T(s)x ds \qquad \textit{for } t \ge 0 \textit{ and } x \in X,$$

where the integral is taken in the sense of Bochner and in the Hilbert space X.

(b) For each $t \geq 0$, let $S(t)$ be the restriction of $\tilde{S}(t)$ to the Hilbert space X. Then $S(t)$, $t \geq 0$, form a nodegenerate, N-times integrated semigroup on X and A is its generator. Moreover, $S(t)$ is exponentially bounded in the sense that

$$|S(t)x|_X \leq (N!)^{-1} t^N \exp(\omega t)|x|_X \qquad \text{for } t \geq 0 \text{ and } x \in X.$$

Applying Propositions 3 and 4, we obtain the following result concerning asymptotic stability of $\tilde{S}(t)$.

Proposition 5. *Let ω be the constant appearing in (B3) and assume that $\omega < 0$.*
(a) The C_0-semigroup $T(t)$ on the Hilbert space X is exponantially stable.
(b) Suppose that $\sup\{t^{-N} N! |\tilde{S}(t)x|_H : t \geq 0\} < +\infty$ for $x \in H$. Then $\tilde{S}(t)$ is asymptotically stable on the Hilbert space H.

PROOF: (a) The result of Walker [3] implies that the search for a quadratic Lyapunov functional reduces to a search for a bounded linear operator $B : X \to X^*$ such that $(Bx)(y) = \overline{(By)(x)}$ for all x, $y \in X$ and $Re(Bx)(Ax) \leq 0$ for all $x \in D(A)$, where A is the infitesimal generator. Now, the result follows from Walker's theorem T.4.1. [3].
(b) We consider the integral $(c - A)^N \int_0^t dt_{N-1} \int_0^{t_{N-1}} \cdots dt_1 \int_0^{t_1} T(s) R(c, A)^N x ds$, for some $c > \omega$, $t \geq 0$. According to the results of Thieme for $N = 1$ and Nicaise [6] for generally N, $S(t)$ will be N-times integrated semigroup on X, and $\tilde{S}(t)$ will be N-times integrated semigroup on H. These results follow if we apply the transformations between integrated semigroups and C-semigroups with $C = R(c, A)^N$ as it is done in Tanaka, Miyadera [1] and Miyadera [7]. Under the condition $\sup\{t^{-N} N! |\tilde{S}(t)x|_H : t \geq 0\} < +\infty$ for $x \in H$, from the Proposition 4 follows that $\tilde{S}(t)$ is asymptotically stable on the Hilbert space H. □

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