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Output Functions of Fuzzy Automata

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Abstract. This paper describes various output functions of (Moore) type of fuzzy automata. A representation theorem for output function of fuzzy automaton is derived. In addition, we prove that these modified output functions of special fuzzy automata are the same as an output function of a classical fuzzy automaton.

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1 The (Moore) type of fuzzy automaton

We consider the following (Moore) type of fuzzy automaton.

Definition 1.1. A fuzzy automaton is a system

$$\mathcal{A} = (S, \Lambda, p, \{F(\lambda) : \lambda \in \Lambda\}, G)$$

where

$S = \{s_1, s_2, \ldots, s_n\}$ is a finite set of states,
Λ is a finite set of inputs,
$p \underset{\sim}{\subseteq} S$ is a fuzzy set called a fuzzy initial state,
$G \underset{\sim}{\subseteq} S$ is a fuzzy set called a fuzzy final state,
$F(\lambda) \underset{\sim}{\subseteq} S \times S$ is a fuzzy transition matrix of order n,
i. e. a fuzzy relation in S.

The elements of $F(\lambda) = ||F_{s,t}(\lambda)||$ are the values of a fuzzy transition function; i. e., a membership function of a fuzzy set in $S \times \Lambda \times S$

$$F : S \times \Lambda \times S \to [0, 1].$$

That is to say, for s, $t \in S$ and $\lambda \in \Lambda$, $F(s, \lambda, t)$ is the grade of transition from state s to state t when the input is λ.

Every fuzzy set A of S is called a fuzzy state of $\mathcal{A}$. If an input signal $\lambda \in \Lambda$ is accepted by $\mathcal{A}$, the present fuzzy state A of $\mathcal{A}$ will be changed to the state $B = A \circ F(\lambda)$, where "$\circ$" is a composition rule of fuzzy relations (e. g. a minimax product).

The fuzzy transition function F can be extended to fuzzy transition function

$$F^* : S \times \Lambda^* \times S \to [0, 1]$$

such that for every $\lambda = \lambda_1 \lambda_2 \ldots \lambda_n \in \Lambda^*$ the following diagram commutes.

$$\begin{array}{ccc} S \times \Lambda \times S & \xrightarrow{F} & [0,\, 1] \\ u \big\downarrow & & \big\| \\ S \times \Lambda^* \times S & \xrightarrow{F^*} & [0,\, 1] \end{array}$$

For $\lambda = \lambda_1 \lambda_2 \dots \lambda_n \in \Lambda^*$ the fuzzy transition matrix is defined as a composition of fuzzy relations, $F^*(\lambda) = F(\lambda_1) \circ F(\lambda_2) \circ \dots F(\lambda_n)$.

A principal identification of fuzzy automaton is provided by its output function

$$f_A : \Lambda^* \to [0, 1]$$

such that for $\lambda = \lambda_1 \lambda_2 \dots \lambda_n \in \Lambda^*$

$$f_A(\lambda) = p \circ F(\lambda_1) \circ \dots \circ F(\lambda_n) \circ G = \bigvee_{s \in S} (\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s)) \wedge G(s)).$$

Clearly, $f_A(\lambda)$ is designated as the grade of transition of $\mathcal{A}$, when $\mathcal{A}$ starts with the initial state p to enter into a state in G after scanning the input sequence λ. Then an input sequence $\lambda \in \Lambda^*$ is said to be accepted by $\mathcal{A}$ with a grade $f_A(\lambda)$.

Now we recall that a (classical) Moore type automaton is a system

$$\mathcal{B} = (S, \Lambda, p, d, G),$$

where S, Λ have the same meaning as in Definition 1.1, $p \in S$ is the initial state, $d : S \times \Lambda \to S$ is a transition function and G is set of final states.
The transition function $d : S \times \Lambda \to S$ can be extended analogously to

$$d^* : S \times \Lambda^* \to S$$

such that for $\lambda = \lambda_1 \lambda_2 \dots \lambda_n \in \Lambda^*$ the following diagram commutes.

$$\begin{array}{ccc} S \times \Lambda & \xrightarrow{d} & S \\ u \big\downarrow & & \big\| \\ S \times \Lambda^* & \xrightarrow{d^*} & S \end{array}$$

where u is a embedding and $d^*(p, \lambda) := d(d^*(p, \lambda_1 \lambda_2 \dots \lambda_{n-1}), \lambda_n)$

Now, given $\lambda = \lambda_1 \lambda_2 \dots \lambda_n \in \Lambda^*$,the automaton $\mathcal{B}$ can process this element λ as follows: $\mathcal{B}$ starts in the initial state p and then inputs the characters of λ one by one, each time changing the state; it is according to the value of the transition function for the current state and the character of λ being input.

Any Moore type automaton $\mathcal{B}$ provides an output function

$$f_B : \Lambda^* \to \{0, 1\}$$

such that the following diagram commutes

$$\begin{array}{ccc} \Lambda^* & \xrightarrow{f_B} & \{0,\, 1\} \\ u \big\downarrow & & \big\uparrow \chi_G \\ S \times \Lambda^* & \xrightarrow{d^*} & S \end{array}$$

where $u(\lambda) = (q, \lambda)$ for every $\lambda \in \Lambda^*$, d^* is the extended transition function and χ_G is a characteristic function.

It should be observed that every Moore type automaton behaves as a fuzzy automaton. The following proposition illustrates it.

Proposition 1.2. *Let $\mathcal{B} = (S, \Lambda, p, d, H)$ be a Moore type automaton. Then there exists a fuzzy automaton $F(\mathcal{B}) = (S, \Lambda, p, \{F(\lambda) : \lambda \in \Lambda\}, G)$ so that*

$$f_{F(B)}(\lambda) = f_B(\lambda) \text{ for all } \lambda \in \Lambda^*.$$

PROOF: See [2]. □

2 Output functions of special fuzzy automata

We shall now begin considering special fuzzy automata. Using the definition 1.1, we introduce two special fuzzy automata and review some results along this line.

Let $\mathcal{B} = (S, \Lambda, q, d, H)$ be a (Moore) type of automaton.
Instead of a set H of S we will take a fuzzy set $G \underset{\sim}{\subseteq} S$ and instead of a initial state $q \in S$ we will take one point set $\{p\}$ (denoted again by p) which will be considered as a fuzzy set $p \underset{\sim}{\subseteq} S$. The resulting system $\mathcal{C} = (S, \Lambda, p, d, G \underset{\sim}{\subseteq} S)$ can be considered as a fuzzy automaton where the fuzzy matrix $\{F(\lambda) : \lambda \in \Lambda\}$ is defined as follows

$$F(\lambda)(s,t) = \begin{cases} 1, & \text{if } d(s,\lambda) = t \\ 0, & \text{if } d(s,\lambda) \neq t. \end{cases} \tag{1}$$

This special fuzzy automaton will be then denoted by $\mathcal{C} = (S, \Lambda, p, d, G \underset{\sim}{\subseteq} S)$.

This fuzzy automaton provides a classical output function as a (Moore) type of fuzzy automaton (see Section 1.)

$$f_C : \Lambda^* \to [0, 1],$$
$$f_C(\lambda) = p \circ F(\lambda_1) \circ \ldots \circ F(\lambda_n) \circ G.$$

On the other hand, for this automaton we may define another output function

$$f'_C : \Lambda^* \to [0, 1]$$

such that the following diagram commutes

$$\begin{array}{ccc} \Lambda^* & \xrightarrow{f'_B} & [0,\ 1] \\ u\big\downarrow & & \big\uparrow \chi_G \\ S \times \Lambda^* & \xrightarrow{d^*} & S \end{array}$$

where $u(\lambda) = (p, \lambda)$ for every $\lambda \in \Lambda^*$ and d^* is the extended transition function.

Then the following proposition shows that these two output functions are the same.

Proposition 2.1. *Let $\mathcal{C} = (S, \Lambda, p, d, G \underset{\sim}{\subset} S)$ be fuzzy automaton with above mentioned output functions f_C, f'_C. Then*

$$f_C(\lambda) = f'_C(\lambda) \text{ for all } \lambda \in \Lambda^*.$$

Proof: Let $\lambda \in \Lambda$ and let $\{F(\lambda) : \lambda \in \Lambda\}$ be a fuzzy matrix associated with this fuzzy automaton (defined above). We observe at first that the rule (1) holds even for $\lambda \in \Lambda^*$ and not for elements of Λ only. This may be proved easily by using the induction principle on the length $||\lambda||$ of $\lambda \in \Lambda$.
Let $f'_C(\lambda) = G(d^*(p, \lambda)) = G(s_0) = \alpha$, $0 < \alpha < 1$. Then $f_C(\lambda) = \bigvee_{s \in S} (\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s)) \wedge G(s)) \geq \bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s_0)) \wedge G(s_0) = G(s_0) = \alpha$.
Now we prove by contradiction that $\alpha \geq f_C(\lambda)$. We consider $\alpha < f_C(\lambda) = \bigvee_{s \in S} (\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s)) \wedge G(s)$. Then exists $s_0 \in S$ so that $\alpha < \bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s_0)) \wedge G(s_0)$. Hence, $G(s_0) > \alpha$ and it is contradiction. □

Now, from classical (Moore) type of automaton $\mathcal{B}$ we can construct another special fuzzy automaton as follows. Instead of an initial state $q \in S$ we will take a fuzzy set $p \underset{\sim}{\subset} S$ and instead of a set H we will take a subset $G \subseteq S$ considered as a fuzzy set $G \underset{\sim}{\subset} S$. The system $\mathcal{D} = (S, \Lambda, p \underset{\sim}{\subset} S, d, G)$ can be considered to be a fuzzy automaton with fuzzy matrix $\{F(\lambda) : \lambda \in \Lambda\}$ defined above. This fuzzy automaton will be denoted by $\mathcal{D} = (S, \Lambda, p \underset{\sim}{\subset} S, d, G)$.

Analogously, this fuzzy automaton provides a classical output function

$$f_D : \Lambda^* \to [0, 1],$$
$$f_D(\lambda) = p \circ F(\lambda_1) \circ \ldots \circ F(\lambda_n) \circ G.$$

Now we define another output function

$$f'_D : \Lambda^* \to [0, 1]$$

such that the following diagram commutes

$$\begin{tikzcd} \Lambda^* \arrow[r, "f'_D"] \arrow[d, "u"'] & {[0, 1]} \\ \prod_{S \in S} (S \times [0, 1]) \times \Lambda^* \arrow[r, "d_1^*"] & \prod_{s \in S} (S \times [0, 1]) \arrow[u, "h"'] \end{tikzcd}$$

where $u(\lambda) = ((s, p(s))_{s \in S}, \lambda)$ for every $\lambda \in \Lambda^*$ and $h((s, \lambda_s)_{s \in S}) := \bigvee_{s \in S} (G(s) \wedge \lambda_s)$ for every $(s, \lambda_s)_{s \in S} \in \prod_{s \in S} (S \times [0, 1])$.

Proposition 2.2. *Let $\mathcal{D} = (S, \Lambda, p \underset{\sim}{\subseteq} S, d, G)$ be a fuzzy automaton with above mentioned output functions f_D, f'_D. Then*

$$f_D(\lambda) = f'_D(\lambda) \textit{ for all } \lambda \in \Lambda^*.$$

PROOF: According to 2.1, we have a fuzzy matrix $\{F(\lambda) : \lambda \in \Lambda\}$.
Let $f_D(\lambda) = \bigvee_{s \in S} (\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z,s)) \wedge G(s)) = \alpha$, $0 < \alpha < 1$. Then exists $s_0 \in S$ so that $\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s_0)) \wedge G(s_0) = \alpha$ and for every $s \in S$ $\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z,s)) \wedge G(s) \leq \bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s_0)) \wedge G(s_0)$. According to assumption $0 < \alpha < 1$. Hence, $G(s_0) = 1$ and $\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s_0)) = \alpha$. Then exists $z_0 \in S$ so that $p(z_0) \wedge F(\lambda)(z_0, s_0) = \alpha$ and for every $z \in S$ $p(z) \wedge F(\lambda)(z, s_0) \leq p(z_0) \wedge F(\lambda)(z_0, s_0)$.

Let $p(z_0) \wedge F(\lambda)(z_0, s_0) = \alpha$. Hence, $p(z_0) = \alpha$ and $F(\lambda)(z_0, s_0) = 1$. Then $f'_D(\lambda) = \bigvee_{z \in S} (G(d^*(z, \lambda)) \wedge p(z)) \geq G(d^*(z_0, \lambda)) \wedge p(z_0) = G(s_0) \wedge p(z_0) = \alpha$.

Now we prove by contradiction that $\alpha \geq f'_D(\lambda)$. We consider $\alpha < f'_D(\lambda) = \bigvee_{z \in S} (G(d^*(z, \lambda)) \wedge p(z))$. Then exists $z_1 \in S$ so that $\alpha < G(d(z_1, \lambda)) \wedge p(z_1)$. Hence, $p(z_1) > \alpha$, $G(d(z_1, \lambda)) > \alpha$. Then $G(d(z_1, \lambda)) = G(s_1) = 1$ and $F(\lambda)(z_1, s_1) = 1$.

Finally, $\bigvee_{s \in S} (\bigvee_{z \in S} (p(z) \wedge F(\lambda)(z,s)) \wedge G(s)) \geq \bigvee_{z \in S} (p(z) \wedge F(\lambda)(z, s_1)) \wedge G(s_1) \geq p(z_1) \wedge F(\lambda)(z_1, s_1) > \alpha$ and this is contradiction. □

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