Transformation of Spaces of Vector Functions of Scalar Argument

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Abstract. The present paper studies the generalized Kummer's transformation of regular spaces of real vector functions of scalar argument and connection to the transformation of the spaces of the solutions of the systems of linear differential equations of the first order.

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Introduction

In paper [4] the properties of the transformation of solutions of the system of n linear differential first-order equations in the real domain are studied

$$(a_n) \qquad d\underline{y}/dt = M(t)\underline{y} \qquad d\underline{Y}/dT = N(T)\underline{Y} \qquad (A_n)$$

where

$$\begin{aligned} \underline{y}(t) &= (y_1(t),\, y_2(t),\, \ldots,\, y_n(t))^T, \\ \underline{Y}(T) &= (Y_1(T),\, Y_2(T),\, \ldots,\, Y_n(T))^T, \end{aligned}$$

$M(t)$, $N(T)$ are square matrices of the n-th order defined on the intervals j, J. The transformation studied is of the following form:

$$\underline{u}(t) = K(t)\underline{U}[Z(t)], \tag{0.1}$$

where $\underline{u}$ resp. $\underline{U}$ is the solution of (a_n) resp. (A_n) defined on j resp. J; K is a square matrix of the n-th order defined on j, and Z is some function mapping the interval $i \subset j$ into the interval $I \subset J$.

The present paper studies the transformation (0.1) of regular spaces of real vector functions of the scalar argument. The spaces of solutions of (a_n), (A_n) are the special cases of these spaces. The considered transformation is, in the main, the generalized Kummer's transformation the properties of which are studied in the papers [1] and [3].

1 Basic considerations

Convention 1.1. In this paper, i, j, I, J will represent open intervals which may be bounded as well as unbounded. Only the column vectors will be considered here, and $\underline{o}$ will represent the zero vector.

Definition 1.1. Let $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_l$ be real vector functions of scalar argument (here after refered to as vector functions or vectors) defined on the interval $j \subset E_1$. If l real numbers $c_1, c_2, \dots, c_l$ exist, such that

$$\sum_{k=1}^{l} c_k^2 \neq 0 \text{ and } \sum_{k=1}^{l} c_k \underline{u}_k = \underline{o}$$

on the whole interval j, then we say that the functions $\underline{u}_k$ $(k = 1, 2, \dots, l)$ are linearly independent on j. If l numbers $c_1, c_2, \dots, c_l$ do not exist, such that $\sum_{k=1}^{l} c_k^2 \neq 0$ and the interval $i \subset j$, on which $\sum_{k=1}^{l} c_k \underline{u}_k = \underline{o}$, then we say that the vector functions $\underline{u}_k$ $(k = 1, 2, \dots, l)$ are linearly independent on j.

Definition 1.2. Let $j \subset E_1$. Let the vector functions $\underline{u}_k$ $(k = 1, 2, \dots, n)$ be continuous and linearly independent on the interval j. The set $\mathbf{R}$ of all the vector functions of the form $\underline{y} = \sum_{k=1}^{n} c_k \underline{u}_k$, where c_k $(k = 1, 2, \dots, n)$ are arbitrary real numbers will be called the linear space of continuous vector functions of the dimension n (in brief, the vector space $\mathbf{R}$).

Theorem 1.1. *Let $\mathbf{R}$ be a vector space defined in the definition 1.2. Let $\underline{y}_k \in \mathbf{R}$ $(k = 1, 2, \dots, n)$. Then $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$ are either linearly independent or linearly dependent.*

PROOF: According to the definition 1.2, there exist numbers c_{kl} $(k, l = 1, 2, \dots, n)$ such that

$$\underline{y}_k = c_{k1}\underline{u}_1 + c_{k2}\underline{u}_2 + \cdots + c_{kn}\underline{u}_n \qquad (k = 1, 2, \dots, n). \tag{1.1}$$

If it is assumed that, for example, $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$ are independent, then there exist numbers a_k $(k = 1, 2, \dots, n)$ and the interval $i \subset j$ such that

$$\sum_{k=1}^{n} a_k^2 \neq 0 \text{ and } \sum_{k=1}^{n} a_k \underline{y}_k = \underline{o}$$

on the interval i. If we substitute $\underline{y}_k$ into the last equality according to (1.1), we get

$$\sum_{i=1}^{n} \underline{u}_l \left(\sum_{k=1}^{n} a_k c_{kl} \right) = \underline{o}.$$

As the vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ are linearly independent, the following must be true:

$$\sum_{k=1}^{n} a_k c_{kl} = 0 \quad (l = 1, 2, \dots, n), \text{ where } \sum_{k=1}^{n} a_k^2 \neq 0.$$

Should then the system (1.2) with the unknowns a_k have an untrivial solution, its determinant must be equal to zero, i. e.

$$\begin{vmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ c_{12} & c_{22} & \dots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{vmatrix} = 0$$

One column of the determinant is the linear combination of the other columns. If, for example, the m-th column is the linear combination of the other columns, then according to (1.1), $\underline{y}_m$ is the linear combination of the vectors $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_{m-1}, \underline{y}_{m+1}, \dots, \underline{y}_n$, that is $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$ are linearly dependent on j. □

Theorem 1.2. *Let $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$ be linearly independent vectors of the space* $\mathbf{R}$. *Each vector $\underline{y} \in \mathbf{R}$ may be expressed in the form*

$$\underline{y} = \sum_{k=1}^{n} a_k \underline{y}_k ,$$

where a_k $(k = 1, 2, \dots, n)$ are suitable real numbers.

PROOF: Since $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$ are linearly independent, we can write

$$\underline{y}_k = c_{k1}\underline{u}_1 + c_{k2}\underline{u}_2 + \dots + c_{kn}\underline{u}_n \quad (k = 1, 2, \dots, n), \tag{1.3}$$

where

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} \neq 0$$

Let

$$\underline{y} = c_1\underline{u}_1 + c_2\underline{u}_2 + \dots + c_n\underline{u}_n . \tag{1.4}$$

Having solved the system (1.3) with respect to $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ and substituting into (1.4), we get the required expression. □

Definition 1.3. Let each ordered n-tuple $(\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n)$ of linearly independent vectors of the space $\mathbf{R}$ be called the basis of the space $\mathbf{R}$.

Definition 1.4. The vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ defined in definition 1.3 are called the basis vectors.

Definition 1.5. The numbers a_k $(k = 1, 2, \dots, n)$ from the theorem 1.2 are called the coordinates of the vector $\underline{y}$ in the basis $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$.

Definition 1.6. The matrix arranged from the basis vectors of $\mathbf{R}$ so that its k-th column is formed by the components of the k-th basis vector is called the basis matrix of the space $\mathbf{R}$.

Theorem 1.3. *Let $b_1(t)$ be a basis matrix of the space $\mathbf{R}$. The matrix $b_2(t)$ is the basis matrix of space $\mathbf{R}$ if and only if a regular constant matrix C of the n-th order exists, such that*

$$b_2(t) = b_1(t) \cdot C \tag{1.5}$$

PROOF: Let $b_1(t)$ be the basis matrix formed from the vectors $\underline{x}_k(t)$ $(k = 1, 2, \ldots, n)$ and $b_2(t)$ be the basis matrix formed from the vectors $\underline{y}_k(t)$ $(k = 1, 2, \ldots, n)$. There exist numbers c_{kl} $(k, l = 1, 2, \ldots, n)$, such that

$$\begin{aligned}
\underline{y}_1(t) &= c_{11}\underline{x}_1(t) + c_{12}\underline{x}_2(t) + \cdots + c_{1n}\underline{x}_n(t) \\
\underline{y}_2(t) &= c_{21}\underline{x}_1(t) + c_{22}\underline{x}_2(t) + \cdots + c_{2n}\underline{x}_n(t) \\
&\;\;\vdots \\
\underline{y}_k(t) &= c_{k1}\underline{x}_1(t) + c_{k2}\underline{x}_2(t) + \cdots + c_{kn}\underline{x}_n(t) \\
&\;\;\vdots \\
\underline{y}_n(t) &= c_{n1}\underline{x}_1(t) + c_{n2}\underline{x}_2(t) + \cdots + c_{nn}\underline{x}_n(t)
\end{aligned} \tag{1.6}$$

The system (1.6) may be written in the following form:

$$\begin{pmatrix} y_{k1}(t) \\ y_{k2}(t) \\ \vdots \\ y_{kn}(t) \end{pmatrix} = c_{k1} \begin{pmatrix} x_{11}(t) \\ x_{12}(t) \\ \vdots \\ x_{1n}(t) \end{pmatrix} + c_{k2} \begin{pmatrix} x_{21}(t) \\ x_{22}(t) \\ \vdots \\ x_{2n}(t) \end{pmatrix} + \cdots + c_{kn} \begin{pmatrix} x_{n1}(t) \\ x_{n2}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

$(k = 1, 2, \ldots, n)$ as well as $\underline{y}_k = b_1(t) \cdot (c_{k1}, c_{k2}, \ldots, c_{kn})^T$, $(k = 1, 2, \ldots, n)$. This is in fact the same relation as that of (1.5), but recorded in another way, such that C is a constant matrix, the k-th column of which is formed by the elements $c_{k1}, c_{k2}, \ldots, c_{kn}$. Obviously, the matrix C is regular, in the opposite case the vectors $\underline{y}_k(t)$ $(k = 1, 2, \ldots, n)$ would be linearly dependent.

Let $b_1(t)$ be the basis matrix, C a regular constant matrix of the n-th order and (1.5) holds true. If we designate $\underline{x}_1(t), \underline{x}_2(t), \ldots, \underline{x}_n(t)$ the columns of the matrix $b_1(t)$ and $\underline{y}_1(t), \underline{y}_2(t), \ldots, \underline{y}_n(t)$ the columns of the matrix $b_2(t)$, then (1.5) may be written in the form (1.6). Obviously, $\underline{y}_k(t)$ $(k = 1, 2, \ldots, n)$ are independent vectors of the space $\mathbf{R}$. In the opposite case, the matrix C would have to be singular. □

Corollary 1.1. *Let $\mathbf{R}$ be the vector space defined on the interval j, $b(t)$ a basis matrix. If the matrix $b(t)$ is regular (singular) at the point $t = t_0$, $t_0 \in j$, then all the basis matrices are regular (singular) at the point $t = t_0$.*

Definition 1.7. The point $t_0 \in j$ is called the regular (singular) point of the domain of definition of the space **R** if all the basis matrices of the space **R** are regular (singular) at the point $t = t_0$.

Definition 1.8. Space **R**, the domain of definition of which contains regular points only, is called regular. A space, the domain of definition of which contains at least one singular point, is called singular.

Convention 1.2. In the following discussion only the regular spaces will be considered.

Theorem 1.4. *Let P, Q be two matrices of the same type and for each $\underline{x} \in \mathbf{R}$ holds $P\underline{x} = Q\underline{x}$. Then $P = Q$ holds.*

PROOF: Let $\underline{x}_k$ $(k = 1, 2, \ldots, n)$ be basis vectors of the space **R** . According to this assumption the following then holds,

$$P\underline{x}_k = Q\underline{x}_k \qquad (k = 1, 2, \ldots, n)$$

and therefore also

$$Pb(t) = Qb(t)$$

holds, where $b(t)$ is basis matrix formed from the vectors $\underline{x}_k$ $(k = 1, 2, \ldots, n)$. Multiplying the last equality from the right hand side by the inversion matrix to $b(t)$ we obtain the statement already proved. □

2 Transformation of regular spaces

Definition 2.1. Let two vector spaces **R** and **S** of the dimension n be given and defined on the intervals j, J. Let there exist the function $Z(t)$ and the matrix $K(t)$ possessing the following properties:

(i) $K(t)$ is the square matrix of the n-th order defined on some interval $i \subset j$,

(ii) $Z(t)$ is defined, continuous and strictly monotonic on the interval i,

(iii) the domain of functional values of the function $Z(t)$ is some interval $I \subset J$,

(iv) for every $\underline{U} \in \mathbf{S}$ there exists $\underline{u} \in \mathbf{R}$, such that

$$\underline{u}(t) = K(t)\underline{U}[Z(t)] \tag{2.1}$$

for every $t \in i$,

(v) for every $\underline{u} \in \mathbf{R}$ there exists $\underline{U} \in \mathbf{S}$, such that (2.1) holds on the interval i.

Then we say that the space $\mathbf{S}$ may be transformed on the interval I onto the space $\mathbf{R}$ on the interval i. The relation (2.1) is called the transformation $\mathbf{T}(K,\, Z,\, I,\, i)$ of the vector $\underline{U}$ onto the vector $\underline{u}$. We write $\underline{u} = \mathbf{T}(\underline{U})$.

Theorem 2.1. *Let $\mathbf{T}(K,\, Z,\, I,\, i)$ be the transformation from the definition 2.1. Let $\underline{u}_1,\ \underline{u}_2 \in \mathbf{R}$ and $\underline{u}_k \in \mathbf{T}(\underline{U}_k)$ $(k = 1,\, 2)$. Then $\mathbf{T}(c_1\underline{U}_1 + c_2\underline{U}_2) = c_1\underline{u}_1 + c_2\underline{u}_2$.*

PROOF:

$$\begin{aligned}
\mathbf{T}(c_1\underline{U}_1 + c_2\underline{U}_2) &= K(t)c_1\underline{U}_1\,[Z(t)] + c_2\underline{U}_2\,[Z(t)] = \\
&= c_1K(t)\underline{U}_1\,[Z(t)] + c_2K(t)\underline{U}_2\,[Z(t)] = \\
&= c_1\underline{u}_1(t) + c_2\underline{u}_2(t)
\end{aligned}$$

□

Theorem 2.2. *Let $\mathbf{R}$, $\mathbf{S}$ be the spaces of the dimension n, $\mathbf{T}(K,\, Z,\, I,\, i)$ the transformation from the definition 2.1. If $\underline{U}_k$ $(k = 1,\, 2,\, \ldots,\, n)$ is a basis of $\mathbf{S}$, then $\underline{u}_k = \mathbf{T}(\underline{U}_k)$ $(k = 1,\, 2,\, \ldots,\, n)$ is a basis of $\mathbf{R}$.*

PROOF: Assume that $\underline{u}_k$ $(k = 1,\, 2,\, \ldots,\, n)$ is not a basis of $\mathbf{R}$. We choose $\underline{u} \in \mathbf{R}$ so that it cannot be expressed as a linear combination of the vectors $\underline{u}_k$ $(k = 1,\, 2,\, \ldots,\, n)$. According to property (v) of the definition 2.1, there exists $\underline{U} \in \mathbf{S}$ such that $\underline{u} = \mathbf{T}(\underline{U})$. If we consider that $\underline{U}$ can be expressed as a linear combination of the vectors of the basis of the space $\mathbf{S}$, the following may be written:

$$\begin{aligned}
\underline{u}(t) &= K(t)\underline{U}\,[Z(t)] = K(t)\sum_{k=1}^{n} c_k\underline{U}_k\,[Z(t)] = \\
&= \sum_{k=1}^{n} c_kK(t)\underline{U}_k\,[Z(t)] = \sum_{k=1}^{n} c_k\underline{u}_k(t).
\end{aligned}$$

Hence $\underline{u}$ is the linear combination of $\underline{u}_k$ $(k = 1,\, 2,\, \ldots,\, n)$. This is a contradiction, and the proof is thus completed. □

Theorem 2.3. *Let $\mathbf{R}$, $\mathbf{S}$ be the spaces of the dimension n, $\mathbf{T}(K,\, Z,\, I,\, i)$ the transformation from the definition 2.1. If the vectors $\underline{U}_k \in \mathbf{S}$ $(k = 1,\, 2,\, \ldots,\, l)$ are linearly independent, then $\underline{u}_k = \mathbf{T}(\underline{U}_k) \in \mathbf{R}$ $(k = 1,\, 2,\, \ldots,\, l)$ are linearly independent as well.*

PROOF: Let $\underline{u}_1,\, \underline{u}_2,\, \ldots,\, \underline{u}_l$ be linearly dependent. Let us complete the set of the vectors $\underline{U}_1,\, \underline{U}_2,\, \ldots,\, \underline{U}_l$ by the vectors $\underline{U}_{l+1},\, \ldots,\, \underline{U}_n$, such that $\underline{U}_k$ $(k = 1, 2, \ldots, n)$ would be a basis. Let $\underline{u}_k = \mathbf{T}(\underline{U}_k)$ $(k = 1, 2, \ldots, n)$. According to the theorem 2.2 $\underline{u}_k$ $(k = 1, 2, \ldots, n)$ is a basis of $\mathbf{R}$, which contradicts the statement that $\underline{u}_1$, $\underline{u}_2,\, \ldots,\, \underline{u}_l$ are linearly dependent. □

Theorem 2.4. *The space $\mathbf{S}$ of the dimension n on the interval I may be transformed onto the space $\mathbf{R}$ of the dimension n on the interval i if there exist the* *matrix $K(t)$, the function $Z(t)$ with the properties (i), (ii), (iii) of the definition 2.1, and the linearly independent vectors $\underline{U}_1, \underline{U}_2, \ldots, \underline{U}_n \in \mathbf{S}$ and the linearly independent vectors $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n \in \mathbf{R}$, such that for every $t \in i$ the following holds:*

$$\underline{u}_k(t) = K(t)\underline{U}_k[Z(t)] \qquad (k = 1, 2, \ldots, n).$$

PROOF: Since the matrix $K(t)$ and the function $Z(t)$ already possess the properties (i), (ii), (iii) of the definition 2.1, properties (iv) and (v) remain to be proved as follows:

I. Let $\underline{U} \in \mathbf{S}$. Since $\underline{U}_1, \underline{U}_2, \ldots, \underline{U}_n$ is basis of $\mathbf{S}$, there exist numbers c_k ($k = 1, 2, \ldots, n$) such that

$$\underline{U} = \sum_{k=1}^{n} c_k \underline{U}_k.$$

The following holds:

$$K(t)\underline{U}[Z(t)] = K(t)\textstyle\sum_{k=1}^n c_k\underline{U}_k[Z(t)] =$$
$$= \textstyle\sum_{k=1}^n c_k K(t)\underline{U}_k[Z(t)] = \sum_{k=1}^n c_k \underline{u}_k(t).$$

Obviously $\underline{u} = \sum_{k=1}^n c_k\underline{u}_k \in \mathbf{R}$. The property (v) of the definition 2.1 is proved analogously.

II. Validity of the inverse theorem is evident. □

Definition 2.2. Two transformations $\mathbf{T}_1$, $\mathbf{T}_2$ from the definition 2.1 are called identical if for every $\underline{U} \in \mathbf{S}$ the following holds: $\mathbf{T}_1(\underline{U}) = \mathbf{T}_2(\underline{U})$.

Theorem 2.5. *Let $\mathbf{T}_1(K_1, Z, I, i)$, $\mathbf{T}_2(K_2, Z, I, i)$ be two identical transformations of the space $\mathbf{S}$ on the interval I onto the space $\mathbf{R}$ on the interval i. Then $K_1(t) = K_2(t)$ for every $t \in i$.*

PROOF: For every $\underline{U} \in \mathbf{S}$ $\mathbf{T}_1(\underline{U}) = \mathbf{T}_2(\underline{U})$ ($= \underline{u} \in \mathbf{R}$) holds, i. e. $K_1(t)\underline{U}[Z(t)] = K_2(t)\underline{U}[Z(t)]$ for every $t \in i$. According to the theorem 1.4, $K_1(t) = K_2(t)$ holds as was proved. □

Theorem 2.6. *Let $\mathbf{T}(K, Z, I, i)$ be a transformation of the space $\mathbf{S}$ onto the space $\mathbf{R}$. Then the matrix $K(t)$ is continuous on the interval i and its rank is $h = n$.*

PROOF: According to the theorem 2.4 there exist basis vectors $\underline{u}_k \in \mathbf{R}$ ($k = 1, 2, \ldots, n$) $\underline{U}_k \in \mathbf{S}$ ($k = 1, 2, \ldots, n$) such that

$$\underline{u}_k(t) = K(t)\underline{U}_k[Z(t)] \qquad (k = 1, 2, \ldots, n).$$

If we designate the basis matrix created from the vectors $\underline{u}_k$ ($\underline{U}_k$) ($k = 1, 2, \ldots, n$) as b (B), the following may be written:

$$b(t) = K(t)B[Z(t)].$$

Since the matrices b, B are continuous and possess the rank n (the spaces $\mathbf{R}$, $\mathbf{S}$ are regular) the matrix $K(t)$ $(= b(t)B^{-1}[Z(t)])$ must be continuous and of the rank n as well. □

Theorem 2.7. *Let* $\mathbf{S}$, $\mathbf{R}$ *be spaces defined on* J, j *and* $Z(t)$ *is a continuous and strictly monotonic function mapping* j *onto* J. *Then there exists the transformation* $\mathbf{T}(K, Z, J, j)$ *mapping the space* $\mathbf{S}$ *on the interval* J *onto the space* $\mathbf{R}$ *on the interval* j *where* $K(t) = b(t)B^{-1}[Z(t)]$ *and* $B(T)$, $b(t)$ *are the basis matrices of the spaces* $\mathbf{S}$, $\mathbf{R}$.

PROOF: Let $B(T)$, $b(t)$ be basis matrices of the spaces $\mathbf{S}$, $\mathbf{R}$. Let

$$K(t) = b(t)B^{-1}[Z(t)]. \tag{2.2}$$

Multiplying (2.2) from the right hand side by the matrix $B[Z(t)]$, we have

$$b(t) = K(t)B[Z(t)]. \tag{2.3}$$

Designating the k-th column of the matrix b (B) as $\underline{u}_k$ $(\underline{U}_k)$, then according to (2.3) we have

$$\underline{u}_k(t) = K(t)\underline{U}_k[Z(t)].$$

According to the theorem 2.4 the transformation $\mathbf{T}(K, Z, J, j)$ maps the $\mathbf{S}$ on J onto $\mathbf{R}$ on j. □

Theorem 2.8. *If* $\mathbf{T}(K, Z, I, i)$ *maps* $\mathbf{S}$ *on* I *onto* $\mathbf{R}$ *on* i, *then* $\mathbf{T}(cK, Z, I, i)$ *where* c *is a constant, maps* $\mathbf{S}$ *on* I *onto* $\mathbf{R}$ *on* i *as well.*

PROOF: Let $\mathbf{T}(K, Z, I, i)$ maps $\mathbf{S}$ on I onto $\mathbf{R}$ on i. Then according to the theorem 2.4, $b(t) = K(t)B[Z(t)]$ where b, B are basis matrices of the spaces $\mathbf{R}$, $\mathbf{S}$. Multiplying the last equality by the constant c we obtain:

$$b(t).(cE) = cK(t)B[Z(t)], \tag{2.4}$$

where E is the unit matrix. On the left hand side of (2.4) there is — according to the theorem 1.3 — a basis matrix of the space $\mathbf{R}$. This completes the proof. □

3 Inverse transformation

Definition 3.1. Let $\underline{U} \in \mathbf{S}$ be mapped onto $\underline{u} \in \mathbf{R}$ by means of the transformation $\mathbf{T}(\overline{K}, \overline{Z}, I, i)$. Every transformation $\tilde{\mathbf{T}}(K, Z, i, I)$ which maps $\underline{u} \in \mathbf{R}$ onto $\underline{U} \in \mathbf{S}$ is called the inverse transformation to $\mathbf{T}$ with respect to the vector $\underline{U}$.

Definition 3.2. Let $\underline{U}_k \in \mathbf{S}$ $(k = 1, 2, \ldots, l)$ be transformed onto $\underline{u}_k \in \mathbf{R}$ $(k = 1, 2, \ldots, l)$ using the transformation $\mathbf{T}(K, Z, I, i)$. The transformation $\tilde{\mathbf{T}}(\overline{K}, \overline{Z}, i, I)$ which maps $\underline{u}_k \in \mathbf{R}$ $(k = 1, 2, \ldots, l)$ onto $\underline{U}_k \in \mathbf{S}$ $(k = 1, 2, \ldots, l)$ is called the inverse transformation to $\mathbf{T}$ with respect to the vectors $\underline{U}_1, \underline{U}_2, \ldots, \underline{U}_l$.

Theorem 3.1. *The transformation $\tilde{\mathbf{T}}(\overline{K}, \overline{Z}, i, I)$ which is inverse to $\mathbf{T}(K, Z, I, i)$ with respect to the vectors $\underline{U}_1, \underline{U}_2, \ldots, \underline{U}_l$ is inverse with respect to each non-trivial linear combination of these vectors.*

PROOF: Let c_k $(k = 1, 2, \ldots, l)$ be the constants. Let us designate:

$$\begin{aligned} \underline{u}_k(t) &= K(t)\underline{U}_k(t)[Z(t)], \qquad (k = 1, 2, \ldots, l) \\ \underline{U}(T) &= \sum_{k=1}^{l} c_k \underline{U}_k(T) \\ \underline{u}(t) &= \sum_{k=1}^{l} c_k \underline{u}_k(t) \end{aligned}$$

It holds:

$$\underline{U}_k(T) = \overline{K}(T)\underline{u}_k[\overline{Z}(T)] \qquad (k = 1, 2, \ldots, l) \tag{3.1}$$

According to (3.1) as well as to the designation indicated, the following may be written:

$$\begin{aligned} \underline{U}(T) &= \sum_{k=1}^{l} c_k \underline{U}_k(T) = \sum_{k=1}^{l} c_k \overline{K}(T)\underline{u}_k[\overline{Z}(T)] = \\ &= \overline{K}(T)\sum_{k=1}^{l} c_k \underline{u}_k[\overline{Z}(T)] = \overline{K}(T)\underline{u}[\overline{Z}(T)]. \end{aligned}$$

□

Definition 3.3. Let $\mathbf{T}(K, Z, I, i)$ be the transformation of the space $\mathbf{S}$ on I onto the space $\mathbf{R}$ on i. The inverse transformation to all vectors of the space $\mathbf{S}$ is called the inverse transformation to $\mathbf{T}$ and is designated as $\mathbf{T}^{-1}(\overline{K}, \overline{Z}, i, I)$.

Theorem 3.2. *The transformation $\tilde{\mathbf{T}}(\overline{K}, \overline{Z}, i, I)$ is inverse to $\mathbf{T}(K, Z, I, i)$ iff $\tilde{\mathbf{T}}$ is inverse with respect to a basis of the space.*

PROOF: The statement follows directly from theorem 3.1. □

Note 3.1. *It is possible to put $\overline{Z}(T) = Z^{-1}(T)$ throughout part 3 of this paper, where Z^{-1} is the inverse function to Z.*

Theorem 3.3. *Let $\mathbf{T}(K, Z, I, i)$ be the transformation from the definition 2.1, $\mathbf{T}^{-1}(\overline{K}, Z^{-1}, i, I)$ be the inverse transformation to it and Z^{-1} be the inverse function to Z. Then the matrix $\overline{K}(T)$ is inverse to the matrix $K[Z^{-1}(T)]$.*

PROOF: Let b, B be basis matrices of the spaces $\mathbf{R}$, $\mathbf{S}$, such that $b(t) = K(t)B[Z(t)]$. Then $B(T) = \overline{K}(T)b[Z^{-1}(T)] = \overline{K}(T)K[Z^{-1}(T)]B(T)$, hence $\overline{K}(T)K[Z^{-1}(T)] = E$. □

4 Transformation of solutions of systems of linear differential equations

Convention 4.1. The following systems of linear differential equations of the first order in the real domain are considered here

$$(a_n) \qquad \underline{y} = M(t)\underline{y} \qquad \dot{\underline{Y}} = N(t)\underline{Y} \qquad (A_n)$$

where $\underline{y}(t) = (y_1(t),\, y_2(t),\, \dots,\, y_n(t))^T$, $\underline{Y}(t) = (Y_1(t),\, Y_2(t),\, \dots,\, Y_n(t))^T$ and the prime designates the derivation of t, and the point derivation of T. The matrices $M(t)$, $N(T)$ are assumed to be continuous on the intervals j, J. Let the numbers $t_0 \in j$, $T_0 \in J$ be given. The solutions $\underline{u}(t)$, $\underline{U}(T)$ of the systems (a_n), (A_n) are uniquely defined on j, J by the initial conditions

$$(a^*) \qquad \underline{u}(t_0) = \underline{u}_0 \qquad \underline{U}(T_0) = \underline{U}_0 \qquad (A^*)$$

where $\underline{u}_0 = (u_{10},\, u_{20},\, \dots,\, u_{n0})^T$, $\underline{U}_0 = (U_{10},\, U_{20},\, \dots,\, U_{n0})^T$ and u_{k0}, U_{k0} $(k = 1, 2, \dots, n)$ are arbitrary numbers.

Note 4.1. *The spaces of the solutions of the systems (a_n), (A_n) are regular (see e. g. [2]) therefore everything what was said about the regular vector spaces holds for the spaces of the solutions of the systems (a_n), (A_n) as well.*

Theorem 4.1. *Let*

a) $\mathbf{S}$ *be the space of the solutions of the system (A_n),* $\mathbf{R}$ *be the space of the solutions of the system (a_n);*

b) $Z(t) \in C^{(1)}(j)$ *mapping j onto J and $Z'(t) \neq 0$ for $t \in j$;*

c) $T(K,\, Z,\, J,\, j)$ *maps* $\mathbf{S}$ *on J onto* $\mathbf{R}$ *on j;*

Then the matrix $K(t)$ is the solution of the matrix differential equation

$$K'(t) = M(t)K(t) - K(t)N[Z(t)]Z'(t) \tag{4.1}$$

PROOF: Let $\Phi_M(t)$, $\Phi_N(t)$ be fundamental matrices of the systems (a_n), (A_n). According to the theorem 2.7 it follows that

$$K(t) = \Phi_M(t)\Phi_N^{-1}[Z(t)].$$

Derivating the last equality and applying the formula

$$d\Phi^{-1}(t)/dt = -\Phi^{-1}(t)\Phi'(t)\Phi^{-1}(t)$$

we obtain the statement being proved.

$$\begin{aligned} K'(t) &= \Phi'_M(t)\Phi_N^{-1}[Z(t)] + \Phi_M(t)(\Phi_N^{-1}[Z(t)])' = \\ &= M(t)\Phi_M(t)\Phi_N^{-1}[Z(t)] - \Phi_M(t)\Phi_N^{-1}[Z(t)]\dot{\Phi}_N[Z(t)]\Phi_N^{-1}[Z(t)]Z'(t) = \\ &= M(t)\Phi_M(t)\Phi_N^{-1}[Z(t)] - \\ &\quad -\Phi_M(t)\Phi_N^{-1}[Z(t)]N[Z(t)]\Phi_N[Z(t)]\Phi_N^{-1}[Z(t)]Z'(t) = \\ &= M(t)K(t) - K(t)N[Z(t)]Z'(t). \end{aligned}$$

□

Note 4.2. *Let $t_0 \in j$, $T_0 \in J$ and $Z(t_0) = T_0$. Obviously*

$$\Phi_M(t_0) = K(t_0)\Phi_N Z(t_0).$$

Theorem 4.2. *Let for every solution $\underline{U}(T)$ of the sys- tem (A_n) defined by the initial conditions (A^*) there exists a solution $\underline{u}(t)$ of the system (a_n), such that*

$$\underline{u}(t) = K(t)\underline{U}[Z(t)]. \tag{4.2}$$

If the initial values of the solution $\underline{u}(t)$ are of the form (a^*) then the matrix $K(t_0) = K_0$ satisfies the relation $\underline{u}_0 = K_0\ \underline{U}_0$.

Proof: Since $\underline{u}(t)$ satisfies the equation (4.2) and the system (a_n), the following must hold:

$$\underline{u}'(t) = \{K'(t) - K(t)N[Z(t)]Z'(t)\}\,\underline{U}[Z(t)]$$

and

$$\underline{u}'(t) = M(t)K(t)\underline{U}[Z(t)].$$

Comparing the righ sides of both equalities and applying the theorem 1.4 we obtain (4.1). □

Note 4.3. *Let Φ_N be a fundamental matrix of the system (A_n). Obviously the matrix*

$$\omega(t) = K(t)\Phi_N[Z(t)]$$

satisfies the system (a_n), however, it need not be its fundamental matrix.

Theorem 4.3. *Let $A(t)$, $B(t)$ be square matrices of the n-th order and continuous on j . Then the matrix differential equation*

$$X' = A(t)X - XB(t) \tag{4.3}$$

has a solution on j , given uniquely by the initial conditions $X(t_0) = X_0$. The solution X of the equation (4.3) may be expressed in the form $X(t) = \omega(t)\Phi_B^{-1}(t)$ where $\omega(t)$ is a square matrix satisfying the system $\underline{y}' = A(t)\underline{y}$, and $\Phi_B(t)$ is a fundamental matrix of the system $\underline{Y}' = B(t)\underline{Y}$.

Proof: I. Let

$$(A) \qquad \underline{y}' = A(t)\underline{y} \qquad \underline{Y}' = B(t)\underline{Y} \qquad \mathbf{(B)}$$

be the systems defined on the interval j, $\omega(t)$ be a square matrix satisfying the system (A), $\Phi_B(t)$ be a fundamental matrix of the system (B), and $X_0 = \omega(t_0)\Phi_B^{-1}(t_0)$. (Evidently the matrices $\omega(t)$, $\Phi_B(t)$ of the required properties do exist). Analogously, as in the proof of the theorem 4.1, it is possible to show that the matrix $X(t) = \omega(t)\Phi_B^{-1}(t)$ is the solution of (4.3).

II. Let there exist matrices $X_1(t)$, $X_2(t)$, these being the solution of (4.3), such that $X_1(t_0) = X_2(t_0) = X_0$, $X_1(t_1) \neq X_2(t_1)$ where t_0, $t_1 \in j$. Let us designate

$$\begin{aligned} \omega_1(t) &= X_1(t)\Phi_B(t) \\ \omega_2(t) &= X_2(t)\Phi_B(t). \end{aligned}$$

According to the note 4.3 the matrices $\omega_1(t)$, $\omega_2(t)$ satisfy the system (A). As $\omega_1(t_0) = \omega_2(t_0)$ holds, and the solutions of the system (A) are given uniquely by the initial conditions, the following must hold: $\omega_1(t_1) = \omega_2(t_1)$. Hence we have $X_1(t_1) = \omega_1(t_1)\Phi_B^{-1}(t_1) = w_2(t_1)\Phi_B^{-1}(t_1) = X_2(t_1)$ and thus we have reached a contradiction with the assumption $X_1(t_1) \neq X_2(t_1)$ and the proof is completed. □

Theorem 4.4. *Let $X(t)$ be the solution of (4.3) on j. If $X(t_0)$ is regular for some $t_0 \in j$, then $X(t)$ is regular on the whole interval j.*

Proof: See [4]. □

Theorem 4.5. *Let*

a) $\mathbf{S}$ be the space of the solutions of the system (A_n), $\mathbf{R}$ be the space of the solutions of the system (a_n);

b) $Z(t) \in C^{(1)}(j)$ maps j onto J and $Z'(t) \neq 0$ for $t \in j$;

c) the matrix $K(t)$ be the solution of (4.1) defined by the initial conditions $K(t_0) = K_0$, and let K_0 be regular.

Then $\mathbf{T}(K, Z, J, j)$ maps $\mathbf{S}$ on J onto $\mathbf{R}$ on j.

Proof: Let Φ_N be a fundamental matrix of the system (A_n). According to the theorem 2.6 $K(t)$ is regular on the whole interval j , and therefore also the matrix $\Phi(t) = K(t)\Phi_N[Z(t)]$ is regular on j . According to note 4.3, $\Phi(t)$ satisfies the system (a_n). With regard to regularity, $\Phi(t)$ is the fundamental matrix of the system (a_n) and this — with respect to the theorem 2.4 - proves our statement. □

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