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Measure and measurable functions of S^n

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Abstract. Let $\mathbb{R}$ be the set of real numbers ordered by the usual ordering, $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\Xi = \{-, 0, +\}$ with $- < 0 < +$. The set $S = \hat{\mathbb{R}} \times \Xi \setminus \{(-\infty, -), (+\infty, +)\}$ ordered lexicographically and endowed by some partial operations and the order topology, is said to be the quasi-real line and its elements the quasi-real numbers. We clear the disconnected character of S, we give a measure on S^n and generalize an extension theorem on real valued functions ranging over a more general than S^n, partially ordered set. The last theorem shows that such a function, under easy conditions, is extended into a continuous function.

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Introduction

Consider $\mathbb{R}$ the set of real numbers endowed by the usual ordering, $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\Xi = \{-, 0, +\}$ ordered by $- < 0 < +$.

The set $S = \hat{\mathbb{R}} \times \Xi \setminus \{(-\infty, -), (+\infty, +)\}$ ordered lexicographically, endowed by the order topology and on which a partial addition and multiplication are adequately defined is said to be the *quasi-real line* and its elements the *quasi-real numbers*.

The set S^n is defined for any natural number n, is ordered by the componentwise ordering and from the "topological aspect" it is endowed by the product topology. In [4], §1, have been given the partial operations (for these partial operations see also [7]) and in [6], §3, the ordering of S^n has been presented as a completion of some orderings of $\mathbb{R}^n$. It's a natural task to inquire the topological feature of these new spaces and it is our main purpose in this paper to give easy as well as necessary information about the subject.

For the completeness sake of the paper we summarize in paragraph 1 some properties of the space S. For the demonstration of these properties we used the personal notes of Professor L. Dokas. I thank him again for this.

In paragraph 2 we give a measure on S^n. This measure has not solved the problems we hoped it would do, but the given results are natural and as we expected them to be.

Last, in paragraph 3 we prove an extension theorem of a real valued function, defined on an ordered space more general than S^n. The theorem works as well on S^n or on S^1 and shows how such a function under easy conditions may be extended into a continuous real valued function.

For the terminology, notation and definitions we follow these ones of [3], [4] and [7]

1 Basic topology on S

Consider the set S of quasi-real numbers endowed by the order topology, the topology of open intervals. Each element of S, a quasi-real number, may be represented by a couple (ρ, ξ) or $\rho^{\xi} = r$, where $\rho \in \mathbb{R}$ (called the *real part of r)* and $\xi \in \Xi$ (called the *kind of r*). We note by S^- (resp. S^0, S^+) the quasi-real number of kind $-$(resp. kind 0, kind $+$).

As usual, given a sequence $(\rho_n, \xi_n)_{n\in\mathbb{N}}$ with $\rho_n < \rho, \rho \in \mathbb{R}$ (resp. $\rho_n > \rho, \rho \in \mathbb{R}$) and $\lim \rho_n = \rho$, there holds: $\lim \rho_n^{\xi_n} = \rho^-$ $(\lim \rho_n^{\xi_n} = \rho^+)$.

Denote by S^{-+} the set $S^- \cup S^+$. The following hold (c.f. [4], p.84):

Proposition 1. *The real projection function of S is continuous.*

In fact, the function $(\rho, \xi) \mapsto \rho$, where $\xi \in \Xi$, is continuous.

Proposition 2. *The space S is disconnected. Particularly, the subset S^{-+} is disconnected.*

PROOF: In fact, if $\varepsilon_n > 0$ and $\xi_n \in \Xi$, we have $\lim_{\varepsilon_n \to 0}(\rho - \varepsilon_n)^{\xi_n} = \rho^-$, $\lim_{\varepsilon_n \to 0}(\rho + \varepsilon_n)^{\xi_n} = \rho^+$. Besides every interval $(\rho_1^{\xi_1}, \rho_2^{\xi_2})$ is open and closed if $\xi_1 \neq +$ and $\xi_2 \neq -$.

We also have $\{\rho^-\} = \bigcap_{\varepsilon>0}](\rho - \varepsilon)^-, \rho[$, $\{\rho\} =]\rho^-, \rho^+[$ and $\{\rho^+\} = \bigcap_{\varepsilon>0}]\rho, (\rho+\varepsilon)^+[$. □

Theorem 3. *Every closed interval of S^{-+} is compact.*

PROOF: Let O be an open covering of $[r_1, r_2] \subseteq S^{-+}$. Decompose each $U \in O$ in maximal intervals and so we can regard that the covering is a covering of open intervals. Let $U^{\mathbb{R}}$ be the real projection of U and $O^{\mathbb{R}} = \{U^{\mathbb{R}} : U \in O\}$; if $r_1 = \rho_1^{\xi_1}$ and $r_2 = \rho_2^{\xi_1}$, then $O^{\mathbb{R}}$ is a covering of $[\rho_1, \rho_2]_{\mathbb{R}} = [\rho_1, \rho_2] \cap \mathbb{R}$.

If $\rho \in [\rho_1, \rho_2]_{\mathbb{R}}$ is not an interior point of any $U^{\mathbb{R}} \in O^{\mathbb{R}}$ and $\rho \neq -\infty$ or $\rho \neq +\infty$ (if $\rho = -\infty$ or $\rho = +\infty$ the results are obvious), it means that there exist in O two intervals U_ρ and U_ρ^*, such that ρ^+ and ρ^- belong to the $U_\rho \cup U_\rho^* = V_\rho$ and the real projections of these two intervals are in opposite sides of ρ. More precisely, the projections of these two intervals are of the form $]\alpha, \rho[$, $]\rho, \beta[$. But in this case ρ is an interior point of the interval $V_\rho^{\mathbb{R}} = U_\rho^{\mathbb{R}} \cup U_\rho^{*\mathbb{R}}$. Let P be the family of the intervals belonging to O and of all the sets of the V_ρ we have just defined. It is also a covering of $[r_1, r_2]$ by intervals. Let $P^{\mathbb{R}}$ be the set of the real projections of P's elements, all of which we may consider open by substituting them, if it is need, with their interior.

Let $\overset{o}{V}$ be the set of the points $r = \rho^{\xi}$, such that $\rho \in V^{\mathbb{R}}$, where $V^{\mathbb{R}}$ is open. It is also an interval of S^{-+} which is open because of the continuity of the map $r = (\rho, \xi) \mapsto \rho$ (Prop. 1). Since for every $r \in [r_1, r_2]$ the ρ belongs to the interior of some V, it results that the family $\overset{o}{P} = \{\overset{o}{V} : V^{\mathbb{R}} \in P^{\mathbb{R}}\}$ is an open covering of $[r_1, r_2]$, for which the family $P^{\mathbb{R}}$ is an open covering of $[\rho_1, \rho_2]_{\mathbb{R}}$.

Since the set $[\rho_1, \rho_2]_{\mathbb{R}}$ is compact (even if ρ_1 or ρ_2 is infinite), there exists a finite subfamily $\overset{o}{W}$ of $\overset{o}{P}$, such that the $W^{\mathbb{R}}$ is a covering of $[\rho_1, \rho_2]_{\mathbb{R}}$ and consequently the $\overset{o}{W}$ will constitute a covering of $[r_1, r_2]$. □

Theorem 4. *The space S^{-+} is Hausdorff separable.*

PROOF: It is evident that the space S^{-+} is Hausdorff and if Q is the set of rational numbers, cl $(Q^{-+}) = S$, where cl denotes the closure of the set (the meaning of Q^{-+} is obvious). □

Theorem 5.

1. *Let F^s be a perfect (i. e. closed without isolated points) subset of the topological space S^{-+}. Then the set $F^{\mathbb{R}} = \{x \in \mathbb{R} : x^\xi \in F^s\}$ is perfect, too.*

2. *Inversely, if the subset $F^{\mathbb{R}} \subseteq \mathbb{R}$ is perfect, there is one and only one corresponding set F^s having as real projection the set $F^{\mathbb{R}}$, which is also perfect.*

PROOF: 1) Suppose that $\rho \in F^{\mathbb{R}}$ is isolated. Then there is an $\varepsilon > 0$ such that the interval $]\rho-\varepsilon, \rho+\varepsilon[$ of $\mathbb{R}$ does not contain any point of $F^{\mathbb{R}}$ but ρ. But then the only elements of F^s which are contained in the open interval $](\rho-\varepsilon)^+, (\rho+\varepsilon)^-[\subseteq S^{-+}$ are of the form ρ^ξ, and at least one of the ρ^-, ρ^+ belongs to F^s. So, if $\rho^- \in F^s$, it is the only element of the set $](\rho-\varepsilon)^+, \rho^+[\cap F^s$ and if $\rho^+ \in F^s$, it is the only element of $]\rho^-, (\rho+\varepsilon)^-[\cap F^s$. Thus, if $\rho \in F^{\mathbb{R}}$ is an isolated point, then every $\rho^\xi \in F^s$ is isolated.

2) Let $F^{\mathbb{R}}$ be a perfect subset of $\mathbb{R}$ and ask for a perfect subset F^s of S^{-+}, whose the real projection is $F^{\mathbb{R}}$. Firstly we prove that if such an F^s exists, it is uniquely defined.

In fact, if $\rho^\xi \in F^s$, then $\rho \in F^{\mathbb{R}}$. We prove that $\rho^- \in F^s$ (resp. $\rho^+ \in F^s$) if and only if ρ is a limit from the left (resp. from the right) of $F^{\mathbb{R}} \setminus \{\rho\}$.

Suppose that ρ is such a limit. Then, for each $\varepsilon > 0$, $]\rho-\varepsilon, \rho[\cap F^{\mathbb{R}} \neq \emptyset$ (resp. $]\rho, \rho+\varepsilon[\cap F^{\mathbb{R}} \neq \emptyset$), which implies $](\rho-\varepsilon)^+, \rho^-[\cap F^s \neq \emptyset$ (resp. $]\rho^+, (\rho+\varepsilon)^-[\cap F^s \neq \emptyset$) and since $](\rho-\varepsilon)^+, \rho^-[\cap F^s =](\rho-\varepsilon)^+, \rho^+[\cap(F^s \setminus \{\rho^-\})$

$$(\text{resp.}]\rho^+, (\rho+\varepsilon)^-[\cap F^s =]\rho^-, (\rho+\varepsilon)^-[\cap(F^s \setminus \{\rho^+\})),$$

we observe that ρ^- (resp. ρ^+) belongs to the closure of F^s, and since F^s is closed, $\rho^- \in F^s$. If ρ is not such a limit, then there is an $\varepsilon > 0$ such that $](\rho-\varepsilon)^+, \rho^-[F^s = \emptyset$ (resp. $]\rho^+, (\rho+\varepsilon)^-[\cap F^s = \emptyset$), hence $](\rho-\varepsilon)^+, \rho^+[\cap(F^s \setminus \{\rho^-\}) = \emptyset$ (resp. $]\rho^-, (\rho+\varepsilon)^-[\cap(F \setminus \{\rho^+\}) = \emptyset$).

Thus, if ρ^- (resp. ρ^+) belongs to F^s, it would be an isolated point of S^{-+}, absurd.

Then ρ^- (resp. ρ^+) does not belong to F^s.

Consider the set F^s defined as above (in fact, we consider the left (resp. right) limits ρ^- (resp. ρ^+) of elements of $F^{\mathbb{R}} \setminus \{\rho\}$, $\rho \in F^{\mathbb{R}}$).

Firstly, we will prove that its complement is open. Let $\rho^\xi \notin F^s$. Then, either $\rho \notin F^{\mathbb{R}}$ or $\rho \in F^{\mathbb{R}}$, but without being a limit of elements of $F^{\mathbb{R}}$ from the left into $\mathbb{R}$, if $\xi = -$ and from the right, if $\xi = +$.

If $\rho \notin F^{\mathbb{R}}$, since $F^{\mathbb{R}}$ is closed, there exists an interval $(\rho - \varepsilon,\, \rho + \varepsilon)$ disjoint of $F^{\mathbb{R}}$, hence the neighborhood $](\rho - \varepsilon)^+,\, (\rho + \varepsilon)^-[$ of ρ^ξ does not intersect F^s.

If $\rho \in F^{\mathbb{R}}$ and $\xi = -$ (resp. $\xi = +$), then there is an $\varepsilon > 0$ such that $]\rho - \varepsilon,\, \rho[\cap F^{\mathbb{R}} = \emptyset$ (resp. $]\rho,\, \rho + \varepsilon[\cap F^{\mathbb{R}} = \emptyset$), hence $](\rho - \varepsilon)^+,\, \rho^\xi[\cap F^s = \emptyset$ (resp. $]\rho^\xi,\, (\rho + \varepsilon)^-[\cap F^s = \emptyset$). Thus ρ^ξ is an interior point of $S^{-+} \setminus F^s$ and the set F^s is closed.

It rests to show that F^s has not any isolated point in S^{-+}.

Let $\rho^\xi \in F^s$ be an isolated point of F^s in S^{-+}. Then $\rho \in F^{\mathbb{R}}$ and we have $](\rho - \varepsilon)^+,\, \rho^-[\cap F^s = \emptyset$ or $]\rho^+,\, (\rho + \varepsilon)^-[\cap F^s = \emptyset$, with respect to the fact $\xi = -$ or $\xi = +$. But then in $\mathbb{R}$

$$]\rho - \varepsilon,\, \rho[\cap F^{\mathbb{R}} = \emptyset \text{ or }]\rho,\, \rho + \varepsilon[\cap F^{\mathbb{R}} = \emptyset,$$

respectively and thus ρ is not in $\hat{\mathbb{R}}$ the limit from the left (resp. from the right) of elements of $F^{\mathbb{R}}$ when $\xi = -$ (resp. $\xi = +$). Thus, from the construction of F^s, there must be $\rho^\xi \notin F^s$, which is absurd. Hence F^s is perfect. $\square$

Real functions of a quasi-real variable

Theorem 6. *(C.f. [4], p.85) Every real function f^* of a real variable is the real projection of a continuous real function f of $[\rho_1^{\xi_1},\, \rho_2^{\xi_2}] \subseteq S$ if and only if, for every real number $\rho \in [\rho_1,\, \rho_2]$, the function f^* admits a right and a left limit $(f^*(\rho + 0),\, f^*(\rho - 0))$.*

Proof: Consider a continuous function $f : [\rho_1^{\xi_1},\, \rho_2^{\xi_2}] \subseteq S \to \mathbb{R}$ and f^* its real projection, that is, for every $x \in [\rho_1,\, \rho_2] \cap \mathbb{R}$ it is $f^*(x) = f(x)$. Let $x_1,\, x_2,\, \ldots,\, x_i,\, \ldots$ be an increasing sequence (resp. decreasing sequence) of real numbers which converges to $\rho \in \mathbb{R}$. Then, since f is continuous at the point ρ, we have on the quasi-real line $\lim x_n = \rho^-$ (resp. ρ^+) and $\lim f^*(x_n) = f(\rho^-)$ (resp. $f(\rho^+)$). Hence, the $\lim f^*(x_n)$ does not depend on the choice of the sequence (x_n), that is, there exist $f^*(\rho - 0)$ and $f^*(\rho + 0)$ for every ρ. Similarly, there exist the values $f^*(\rho_1 + 0)$ and $f^*(\rho_2 - 0)$.

Conversely, let f^* be a real function defined on the real interval $[\rho_1,\, \rho_2]$ with side – limits $f^*(\rho + 0)$, $f^*(\rho - 0)$ at a point ρ.

Put

$$f(\rho^-) = f^*(\rho - 0),\, f(\rho) = f^*(\rho),\, f(\rho^+) = f^*(\rho + 0).$$

We give arbitrary values to $f(\rho_1^+)$ and $f(\rho_2^-)$.

It is enough to show that f is continuous on the interval $[\rho_1^+,\, \rho_2^-]$.

Let I be an open interval of $\mathbb{R}$ and $\rho \in I$, and let be $\rho^\xi \in S$ such that $f(\rho^\xi) = \rho$. Since I is open, there is an $\varepsilon > 0$, such that $]\rho - \varepsilon, \rho + \varepsilon[\subseteq I$. If $\xi = 0$, put $V_r =]\rho^-, \rho^+[= \{\rho\}$, hence $f(V_r) = \{\rho\} \subseteq]\rho - \varepsilon, \rho + \varepsilon[$. Let be $\xi = -$; then there exists an $n > 0$, such that for every real $x \in V_r =]\rho - n, \rho[\subseteq S$ there holds:

$$|f^*(x) - f^*(\rho - 0)| = |f^*(x) - f(\rho^-)| \leq \frac{\varepsilon}{2} \text{ and}$$

we have $f^*(\text{cl } (V_r \cap \mathbb{R})) \subseteq [\rho - \frac{\varepsilon}{2}, \rho + \frac{\varepsilon}{2}]$. But then $f(V_r) \subseteq \text{cl } [f^*(\text{cl } (V_r \cap \mathbb{R}))] \subseteq [\rho - \frac{\varepsilon}{2}, \rho + \frac{\varepsilon}{2}] \subseteq]\rho - \varepsilon, \rho + \varepsilon[$.

The proof for $\xi = +$ is analogous. □

2 Measure on S^n

2.1 The measure of a hypercube

Consider again the set S^n as it has been defined above. The topology is always this one of the product topology of the open intervals. Throughout this paragraph we make use of partially defined operations mentioned in the introduction and mainly of the addition. We recall that the addition of two quasi-real numbers is possible if and only if they are not of opposite kind, and it means it is not permitted the one to be of kind $+$ and the other of kind $-$. If the sum $\rho_1^{\xi_1} + \rho_2^{\xi_2}$ exists, then it equals $(\rho_1 + \rho_2)^\xi$, where ξ is the common kind or it is the kind which is not zero. (For the rest definitions and the properties of the partially defined operations one can see to the references mentioned in the introduction).

Definition 1.

1. If I is an interval of S, the intersections $I \cap S^0$, $I \cap S^+$, $I \cap S^-$ are called the *proper intervals* of kind $0, +, -$, respectively and they are symbolized by I^0, I^+, I^-.

2. Every cartesian product $\prod_{i=1}^n I_i$ of any kind of proper intervals $I_i \subseteq S$ is called a *hypercube.*

3. We call *elementary set of hypercubes* every finite union of pairwise disjoint hypercubes.

The intervals I_i may be open, closed or semiclosed.

Proposition 2. *If the subsets A, B of S^n are elementary sets of hypercubes, then the subsets $A \cap B$, $A \cup B$, $S^n \backslash A$ are also elementary sets of hypercubes.*

Proof: Let be $A = \bigcup_{i=1}^m P_i$ and $B = \bigcup_{i=1}^m Q_i$, where P_i and Q_i are pairwise disjoint.

The proof that $A \cap B$ and $A \cup B$ are elementary hypercubes is as for $\mathbb{R}^n$.

We prove that the $S^n \backslash A$ is an elementary set of hypercubes. It is

$$S^n \backslash A = \bigcap_{i=1}^{m} (S^n \backslash P_i),$$

where $P_i = I_1 \times I_2 \times \ldots \times I_n$. It is sufficient to show that $S^n \backslash P_i$ is an elementary hypercube. Consider S as $S^0 \cup S^+ \cup S^-$, thus the set S^n can be considered as a set of finite number of elementary hypercubes of the type $\Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_n$, where each Λ_i is an interval of kind 0, + or −. If, for example, the I_i is of kind + and $I_i = \{x = \rho^+ : \alpha_i \leq \rho \leq \beta_i, \alpha_i, \rho, \beta_i \text{ in } \mathbb{R}\}$, then the set S_i^+ (where S_i is a copy of S in the i-position of the cartesian product S^n) could be divided into the intervals I_i, $\{x = \rho^+ : \rho < a_i\}$ and $\{x = \rho^+ : \rho > bi\}$.

So the set S^n can be considered as the union of finite number of hypercubes, one of which will be the P_i and consequently the set $S^n \setminus P_i$ will be an elementary set of hypercubes, as the finite union of pairwise disjoint hypercubes. □

The proofs of the next statements follow in a great length these ones of an elementary measure theory on $\mathbb{R}^n$, so these proofs are given briefly.

Proposition 3. *If P_1, P_2, …, P_m are hypercubes of S^n, there exist hypercubes Q_1, Q_2, …, Q_r of S^n pairwise disjoint such that*

$$(i) \quad \bigcup_{i=1}^{m} P_i = \bigcup_{j=1}^{n} Q_j, \qquad (ii) \quad P_i \cap Q_j \neq \emptyset \Rightarrow Q_j \subseteq P_i.$$

PROOF: Because the unions of elementary hypercubes can be written as unions of pairwise disjoint hypercubes, for the proof of the proposition it is enough to prove:

(α) There exist elementary sets of hypercubes Γ_1, Γ_2, …, Γ_q, such that $\bigcup_{i=1}^{m} P_i = \bigcup_{j=1}^{q} \Gamma_j$, $P_i \cap \Gamma_j \neq \emptyset \Rightarrow \Gamma_j \subseteq P_i$.

The proposition (α) is evident for $m = 1$. Suppose now there exist elementary sets of hypercubes Δ_1, Δ_2, …, Δ_r pairwise disjoint such that:

$$\bigcup_{i=1}^{m-1} P_i = \bigcup_{j=1}^{r} \Delta_j, \quad P_i \cap \Delta_j \neq \emptyset \Rightarrow \Delta_j \subseteq P_i.$$

Put $\Gamma_{2j-1} = \Delta_j \cap P_m$, $\Gamma_{2j} = \Delta_j \cap (S^n \setminus P_m)$, $j \in \{1, \ldots, r\}$ and Γ_{2r+1} the set of the elements of P_m which do not belong to the union $\bigcup_{j=1}^{r} \Delta_j$.

Thus, $\bigcup_{i=1}^{m} P_i = (\bigcup_{i=1}^{m-1} P_i) \cup P_m = (\bigcup_{j=1}^{r} \Delta_j) \cup P_m = (\bigcup_{j=1}^{r} \Delta_j) \cup \Gamma_{2r+1} = \bigcup_{j=1}^{2r+1} \Gamma_i$.

It remains to show that

$$P_i \cap \Gamma_j \neq \emptyset \Rightarrow \Gamma_j \subseteq P_i. \tag{1}$$

Firstly the subset Γ_{2r+1} is disjoint of the sets P_1, P_2, …, P_{m-1} and is contained in P_m.

We show the relation (1) for Γ_1 (the proof is similar for Γ_3, Γ_5, ..., Γ_{2r-1}) and for Γ_2 (as well, the proof is similar for Γ_4, Γ_6, ..., Γ_{2r}).

It is $\Gamma_1 \subseteq \Delta_1$ and as $\Delta_1 \cap P_i \neq \emptyset$ implies $\Delta_1 \subseteq P_i$, for $i \in \{1, 2, \ldots, m-1\}$, the same relation holds for Γ_1.

On the other hand, by supposition, $\Gamma_2 \cap P_i \neq \emptyset$ for $i \in \{1, \ldots, m-1\}$ implies $\Gamma_2 \subseteq P_i$, while $\Gamma_2 \cap P_m = \emptyset$. □

Definition 4. For every bounded hypercube $P = I_1 \times I_2 \times \ldots \times I_n$ of S^n we define as *n-dimensional measure* (or simply *measure*) of P the number $m(P) = (\lg I_1) \cdot (\lg I_2) \cdot \ldots \cdot (\lg I_n)$, where $\lg I_i = \text{length } I_i = \beta - \alpha$ and I_i is an interval with edges the α^{ξ_i}, $\beta^{\xi_i'}$, for α, β in R and ξ_i, ξ_i' in Ξ.

Remarks 5.

1. The measure is independent of the order of the components of P.
2. If one of the components is a singleton, then the measure is equal to 0.
3. If P, Q are bounded hypercubes with $P \subseteq Q$, then $m(P) \leq m(Q)$.
4. If P is a bounded hypercube and $\varepsilon > 0$, then there exists an open bounded hypercube P', such that $P \subseteq P'$ and $m(P') \leq m(P) + \varepsilon$.

Proposition 6. *If $P = \bigcup_{i=1}^{n} P_i$, with P_i bounded and pairwise disjoint hypercubes, then $m(P) = m(P_1) + \ldots + m(P_n)$.*

2.2 Measurable functions of S^n

1. Any sum of quasi – real numbers may appear as a sum of a quasi – real number of kind – and a quasi – real number of kind +, for, every quasi – real number of kind 0 can be summed up either to the first or to the latter term. So, every sum of quasi – real numbers would take the form of a couple (α^-, β^+), where α, β are real numbers and α^-, β^+ the sums which we talked about before. In the case where one or both of the kinds + and – do not exist, then at least one of the elements of the couple will be of kind 0 or, instead of the couple, will exist a real number.

Now, on the set $S^- \times S^+$ we define an obvious equivalence relation $\sim$ as following:

$$(\alpha^-, \beta^+) \sim (\alpha_1^-, \beta_1^+) \Leftrightarrow \alpha + \beta = \alpha_1 + \beta_1.$$

It is evident that this equivalence would also be extended to the case where, either the one of the two elements of the couple is of kind 0, or the couple is reduced to a real number. So, the couple (α^0, β^+) corresponds to the class of $\alpha + \beta$, while the α^0 corresponds to the class of α . On the other hand, since the existence of one or even two elements of kind 0 in a couple does not play any role in the whole theory, from now on we will write (α^-, β^+) for any case.

In the present paragraph we symbolize by $C_{(\alpha^-,\beta^+)}$ the class of any element (α^-, β^+).

In the set $S^- \times S^+/_\sim$ define an addition $\oplus$ (simply $+$) as following:

$$C_{(\alpha^-,\beta^+)} + C_{(\alpha_1^-,\beta_1^+)} = C_{((\alpha+\alpha_1)^-,\,(\beta+\beta_1)^+)}.$$

Evidently the operation $+$ is well defined.

2. We recall that if X is a non-void set, a subset C of $P(X)$ is said to be a *race* if it fulfils:

1. $\emptyset \in C$
2. $A,\ B \in C \Rightarrow A - B \in C$, that is $A \cap B^c \in C$
3. $A,\ B \in C \Rightarrow A \cup B \in C$.

If, instead of (3), there holds:

$$(3') \quad (\forall n) A_n \in C \Rightarrow \bigcup_{n=1}^{\infty} A_n \in C,$$

then the set C is called *σ-race*.

If, in addition with (1), (2), (3') there holds:

$$(4) \quad A \in C \to X \setminus A \in C,$$

then the set C is called a *tribe*.

We define as a *measurable space* every couple $(X,\ B)$, where B is a tribe on the set X.

If f is a function defined on a measurable space $(X,\ B)$ and ranging in an other measurable space $(X_1,\ B_1)$, then f is called *measurable function* if and only if

$$(\forall A \in B_1)[f^{-1}(A) \in B].$$

Consider the set of the proper intervals of S of any kind and any form. The set of all the countable unions of intervals of this kind constitutes a tribe which we symbolize by $\sigma_S(C)$.

Analogously we form the tribe $\sigma(C)$ in S^n.

Definition 3. A quasi-real function f of S^n is said to be a *step function*, if there exists a finite number of bounded hypercubes $P_1,\ P_2,\ \ldots,\ P_m$, pairwise disjoint such that:

$$f(x) = \begin{cases} c_i, & \text{iff } x \in P_i \\ 0, & \text{iff } x \in S^n \setminus \bigcup_{i=1}^m P_i \end{cases}$$

where $i \in \{1,\ \ldots,\ m\}$, c_i is a quasi-real constant, not infinite.

If X_{P_i} is the characteristic function of P_i, then f is written as $f = c_1 X_{P_1} + \ldots + c_m X_{P_m}$.

Remark 4. If f_1, f_2 are step functions and $f_1 + f_2$, $f_1 \cdot f_2$ can be defined, then they are step functions, too.

Notation 5. We symbolize:

1. by $S(\geq 0)$ and $S(< 0)$ the sets $\{(\rho, \xi) \in S : (\rho, \xi) \geq (0, 0)\}$ and $\{(\rho, \xi) \in S : (\rho, \xi) < (0, 0)\}$ respectively,
2. by $E(S^n, \sigma(C), S(\geq 0))$ the set of all finite measurable quasi-real step functions of S^n,
3. by $M(S^n, \sigma(C), S(\geq 0))$ the set of all measurable quasi-real functions of S^n into $S(\geq 0)$.

Proposition 6. *Let be (X, C^*) a measurable space and f a quasi-real function of X. The function f is measurable iff the sets*

$$X_0 = \{x \in X : f(x) = (\rho, 0) < (\alpha, 0)\} \in C^*,$$

for every $(\alpha, 0)$ in S^0, and

$$X_{(-,+)} = \{x \in X : f(x) = (\rho, \xi') < (\alpha, \xi)\} \in C^*$$

for every $(\alpha, \xi) \in S^{-+}$, are measurable.

PROOF: Evident. □

Remark 7. If f, g belong in $E(S^n, \sigma(C), S(\geq 0))$ or in $M(S^n, \sigma(C), S(\geq 0))$, then the sum $f + g$ and the product $f \cdot g$ are defined if and only if the kinds of $f(x)$ and $g(x)$ are not opposite.

Definition 8. Let $f = c_1X_1 + \ldots + c_\rho X_\rho$ be a quasi-real step function of S^n, where $X_1, \ldots, X_\rho$ are characteristic functions of bounded pairwise disjoint hypercubes $P_1, \ldots, P_\rho$, respectively. If the numbers $c_1, \ldots, c_\rho$ can be summed up, then we call *integral of f on S^n* the quasi-real number $c_1m(P_1) + c_2m(P_2) + \ldots + c_\rho \cdot m(P_\rho)$ and symbolize by $\int f$.

Proposition 9. *Let f_1, f_2, f be quasi-real step functions on S^n and $\lambda \in S$. Then:*

1. *If the $\int(f_1 + f_2)$ can be defined, then $\int(f_1 + f_2) = \int f_1 + \int f_2$.*
2. *If the $\int \lambda f$ can be defined, then $\int \lambda f = \lambda \int f$.*

Evidently, every step function is measurable.

10. We define now a function Φ_1 of the space $E(S^n, \sigma(C), S(\geq 0))$ into $S^- \times S^+/_\sim$ as following:

If $f \in E(S^n, \sigma(C), S(\geq 0))$ is of the form $f = \sum_{i=1}^{m}(\alpha_i, \xi_i) \cdot X_{A_i}$, where X_{A_i} is the characteristic function of A_i, then

$$\Phi_1(f) = C_{(\alpha^-, \beta^+)},$$

where $\alpha^- = \sum_{i=1}^{\varepsilon} \alpha_{\varepsilon_i}^- \cdot m(A_{\varepsilon_i})$ (resp. $\beta^+ = \sum_{\rho=\varepsilon+1}^{m} \alpha_{\varepsilon_\rho}^+ \cdot m(A_{\varepsilon_\rho})$), where A_{ε_i} (resp. A_{ε_ρ}) are hypercubes of kind$-$ or 0 (resp. $+$ or 0).

Proposition 11. *Let be f, g in $E(S^n, \sigma(C), S(\geq 0))$. There holds:*

1. $\Phi_1(\lambda f) = \lambda \Phi_1(f)$, *for every* $\lambda \in S^0$.
2. *If $f+g$ can be defined, then*

$$\Phi_1(f+g) = \Phi_1(f) + \Phi_1(g).$$

3. *If $f \leq g$, then $\Phi_1(f) \leq \Phi_1(g)$.*

PROOF: 1) In fact, let be $f = \sum_{i=1}^{m}(\alpha, \xi)_i X_{A_i}$ and $\Phi_1(f) = C_{(\alpha_1^-, \alpha_2^+)}$, where $\alpha_1^- = \sum_{i=1}^{n}(\alpha^-)_{\varepsilon_i} m(A_{\varepsilon_i})$ and $\alpha_2^+ = \sum_{\rho=\varepsilon+1}^{m}(\alpha^+)_{\varepsilon_\rho} m(A_{\varepsilon_\rho})$.

$$\lambda \Phi_1(f) = \lambda C_{(\alpha_1^-, \alpha_2^+)} = C_{(\lambda\alpha_1^-, \lambda\alpha_2^+)} \text{ and}$$
$$\lambda f = \sum_{i=1}^{m}(\lambda\alpha, \xi)_i X_{A_i}, \text{ hence}$$
$$\Phi_1(\lambda f) = C_{(\lambda\alpha_1^-, \lambda\alpha_2^+)} \Rightarrow \Phi_1(\lambda f) = \lambda \Phi_1(f).$$

The proof of (2) and (3) is analogous. □

Remarks 12. (1) We could transfer the notions of the almost everywhere equal functions or of the functions of the same measure, or of the functions which are almost everywhere continuous, for two functions which are equal or they are of the same measure in domains which differ in sets of measure 0, or for functions which are continuous in a set and discontinuous in another of measure 0. Besides if two functions differ in a set of measure 0and the first is measurable, then the other will be measurable too.
(2) We remark that each class $C_{(\alpha^-, \beta^+)}$ corresponds to a real number, the number $\alpha + \beta$ and hence the values of β_1 also correspond to a real number.

13. Now we attempt to define an integral for arbitrary measurable functions.

Consider a function $f \in M(S^n, \sigma(C), S(\geq 0))$ and for every x we symbolize by $f^0(x)$ the real projection of $f(x)$. (That is, if $f(x) = \alpha^\xi \in S$, $f^0(x) = \alpha$). In this way, in such a function f we assign a real function f^0, which we will call in the present section, the *real part of f*.

Proposition. *The real part of a measurable function is a measurable function, too.*

PROOF: Let be f and f^0 as above and E an open subset of S. The set E is the union of elementary hypercubes , say $(I_i)_{i\in I}$. The inverse image of I_i via f, is a measurable set in S^n, while the $(f^0)^{-1}(I_i)$ differs from $f^{-1}(I_i)$ possibly at the end points of I_i, that is, it remains measurable. □

Proposition 14. *For every function f^0 (real part of a measurable S-function f on S^n), there exists a sequence $(f_i^0)_i$ of real step functions with the same domain, converging to f^0.*

PROOF: As it is known by the proposition of 13, there exists a sequence $(\bar{f}_i^0)_{i\in\mathbb{N}}$ of real step functions defined on $\mathbb{R}^n$, converging to the restriction of f^0 on $\mathbb{R}^n$. We extend each $\bar{f}_i^0$ on S^n, by putting for every $r \in \mathbb{R}^n$, $f_i^0(r) = \bar{f}_i^0(r)$, and $f_i^0(r^+) = f_i^0(r^-) = \bar{f}_i^0(r)$.

So, we define the $(f_i^0)_{i\in\mathbb{N}}$ corresponded to $(\bar{f}_i^0)_{i\in\mathbb{N}}$, which also converges to f^0. □

Definition 15. Consider a function $f \in M(S^n, \sigma(C), S(\geq 0))$, which has kind ξ (where $\xi \in \{+, -\}$) or 0 everywhere but a set of measure zero; let be f^0 its real part and $(f_n^0)_{n\in\mathbb{N}}$ the sequence of the step functions which converges to f^0 (according to Prop. 14). For this sequence it is meaningful the function β_1 (Defin. 10).

Define a function $\Phi : M(S^n, \sigma(C), S(\geq 0)) \to S^- \times S^+/\sim$ with value

$$\Phi(f) = (\sup \Phi_1(f_n^0))^\xi,$$

where ξ is the kind of f.

Symbolize $\Phi(f) = \int f(x)\mathrm{d}m(x)$.

2.3 Properties of measurable functions

Proposition 1. *Every continuous function of S into S is measurable.*

PROOF: Since the tribes defined on the domain and the range of the function coincide and the function is continuous, the proof is evident. □

Proposition 2. *Let $(f_i)_{i\in\mathbb{N}}$ be an increasing sequence of measurable step functions of S^n and $f \in M(S^n, \sigma(C), S(\geq 0))$ such that $f \leq \sup_i f_i$.*

Then $\beta_1(f) \leq \sup_i \beta_1(f_i)$.

PROOF: We prove it in two steps:

1st step Let $f = (\alpha, \xi)X_A$, where $(\alpha, \xi) > (0, 0)$ and $A \in \sigma(C)$. (If $(\alpha, \xi) = (0, 0)$, then the results are evident).

We also suppose $(\rho_1, 0) < (\alpha, \xi)$ and $(\rho_2, \xi') < (\alpha, \xi)$, $\xi' \neq 0$. Now, since the functions f_i are measurable, we have (Prop. 6)

$$[f_i \geq (\rho_1, 0)]_0 = \{x \in S^n : f_i(x) = (\rho, 0) \geq (\rho_1, 0)\} \in \sigma(C)\text{and}$$

$$[f_i \geq (\rho_2, \xi')]_{(-,+)} = \{x \in S^n : f_i(x) = (\rho, \xi^*) \geq (\rho_2, \xi')\} \in \sigma(C).$$

Next, we construct an increasing sequence of elements of $\sigma(C)$ as follows (for notation see Prop. 6):

$$A_i = A \cap \{[f_i \geq (\rho_1,\, 0)]_0 \cup [f_i \geq (\rho_2,\, \xi')]_{(-,+)}\}.$$

It is $\bigcup_i A_i = A$ and $m(A) = \sup_i m(A_i)$.

But

$$\begin{aligned} \Phi_1(f_i) \geq (\rho_1,\, 0) m(A \cap [f_i \geq (\rho_1,\, 0)]_0) \\ +(\rho_2,\, \xi')m(A \cap [f_i \geq (\rho_2,\, \xi')]_{(-,+)}). \end{aligned}$$

In fact, putting $X_{A_i} = I(A_i)$, and $(\rho,\, \xi) = \rho^\xi$, it is:

$$\begin{aligned} f_i \geq \rho_1^0 I(A \cap [f_i \geq \rho_1^0]_0) + \rho_2^{\xi'} I(A \cap [f_i \geq \rho_2^{\xi'}]_{(-,+)}) \\ \text{and} \quad \Phi_1(\rho_1^0 I(A \cap [f_i \geq \rho_1^0]_0) + \rho_2^{\xi'} I(A \cap [f_i \geq \rho_2^{\xi'}]_{(-,+)})) = \\ = \rho_1^0 m(A \cap [f_i \geq \rho_1^0]_0) + \rho_2^{\xi'} m(A \cap [f_i \geq \rho_2^{\xi'}]_{(-,+)}). \end{aligned}$$

Since $(\rho,\, 0) \leq (\alpha,\, \xi)$ and $(\rho_2,\, \xi') \leq (\alpha,\, \xi)$, we have:

$$\sup_i \Phi_1(f_i) \geq (\alpha,\, \xi) m(A) = \Phi_1(f).$$

2nd step Let now $f = \sum_k (\alpha,\, \xi)_k I(A_k)$, where the sets A_k are pairwise disjoint and belong to $\sigma(C)$.

According to the 1st step, and since $f_i \cdot I(A_k)$ and $f \cdot I(A_k)$ are, respectively, the restrictions of f_i and f in A_k, there holds:

$$\sup_i \Phi_1(f_i \cdot I(A_k)) \geq \Phi_1(f \cdot I(A_k)).$$

On the other hand,

$$\begin{aligned} f_i \geq \textstyle\sum_k f_i \cdot I(A_k), \text{ hence} \\ \Phi_1(f_i) \geq \textstyle\sum_k \Phi_1(f_i \cdot I(A_k)), \text{ that is} \\ \sup_i \Phi_1(f_i) \geq \sup_i \textstyle\sum_k \Phi_1(f_i \cdot I(A_k)) = \\ = \textstyle\sum_k \sup_i \Phi_1(f_i \cdot I(A_k)), \end{aligned}$$

because the sequence $\Phi_1(f_i \cdot I(A_k))_{i \in \mathbb{N}}$ is increasing.

Thus, $\sup_i \Phi_1(f_i) \geq \sum_k \Phi_i(f \cdot I(A_k)) = \Phi_1(f)$. □

Proposition 3. *The integral of a positive measurable function is zero if and only if this function is almost everywhere equal to zero.*

Proof: Let be $f \in M(S^n,\, \sigma(C),\, S(\geq 0))$ and $f \neq 0$.

We have:

$$\begin{aligned} [f \neq 0] = \textstyle\bigcup_k [f \geq (\frac{1}{k},\, 0)]_0 \cup \bigcup_k [f \geq (\frac{1}{k},\, +)]_{(-,+)} = \\ = \textstyle\bigcup_k ([f \geq (\frac{1}{k},\, 0)]_0 \cup [f \geq (\frac{1}{k},\, +)]_{(-,+)}). \end{aligned}$$

The sequence of the sets: $([f \geq (\frac{1}{k}, 0)]_0 \cup [f \geq (\frac{1}{k}, +)]_{(-,+)})_{k \in \mathbb{N}}$ is increasing, hence $m([f \neq 0]) = \sup_k m([f \geq (\frac{1}{k}, 0)]_0 \cup [f \geq (\frac{1}{k}, +)]_{(-,+)})$.

Besides, we have:

$$f \geq (\frac{1}{k}, 0) \cdot I([f \geq (\frac{1}{k}, 0)]_0) + (\frac{1}{k}, +) \cdot I([f \geq (\frac{1}{k}, +)]_{(-,+)}),$$

hence

$$\int f dm \geq (\frac{1}{k}, 0) m([f \geq (\frac{1}{k}, 0)]_0) + (\frac{1}{k}, +) m([f \geq (\frac{1}{k}, +)]_{(-,+)})$$

$$0 \geq (\frac{1}{k}, 0) m([f \geq (\frac{1}{k}, 0)]_0) + (\frac{1}{k}, +) m([f \geq (\frac{1}{k}, +)]_{(-,+)}),$$

which implies $m([f \geq (\frac{1}{k}, 0)]_0) = 0$ and $m([f \geq (\frac{1}{k}, +)]_{(-,+)}) = 0$, that is, $m([f \neq 0]) = 0$.

Conversely, if $m([f \neq 0]) = 0$, then $\int f dm = 0$.

In fact, $f \leq \sup_k k \cdot I([f \neq 0])$, hence $\int f dm < \sup_k k \cdot m([f \neq 0]) = 0$, that is $\int f dm = 0$. □

3 A generalization on the extension of a real function

In this last section we give a generalization of Theorem 6, §1.

Notation and definitions 1. Let $(E, \leq)$ be a structure of partial ordering without jumps and with the topology of the open intervals on it; $(\tilde{E}, \leq)$ is the well known MacNeille's completion of the given structure. (For the completion one may see in [1] p. 126). For each cut (A, B) we symbolize by $(A_i)_{i \in I}$, $(B_j)_{j \in J}$ the decompositions into up-directed or down-directed, maximal by containment, subsets of A and B respectively. For each $e \in E$ we symbolize by $(A_i^e)_{i \in I}$, $(B_j^e)_{j \in J}$ the decompositions into up- and down-directed maximal by containment subsets of the sets $] \longleftarrow e[$, $]e \rightarrow [$, respectively.

We say that the real function F has *limit in the cut* (A, B) *at the direction* A_i the number $l \in \mathbb{R}$, if and only if

$$(\forall \varepsilon > 0)(\exists x_0 \in A_i)(\forall x \in A_i)[x_0 \leq x \rightarrow |F(x) - l| \leq \varepsilon]$$

$$(\lim_{A_i} F(x) = l \text{ by symbols}).$$

Analogous is the definition for the limit at the direction B_j.

We say that the real function F *has limit in the point* $e \in E$ *at the direction* A_i^e *when* x *tends to* e *and* $x \neq e$, the number $l \in \mathbb{R}$, if and only if

$$(\forall \varepsilon > 0)(\exists x_0 \in A_i^e)(\forall x \in A_i^e)[x_0 \leq x \rightarrow |F(x) - l| \leq \varepsilon]$$

$$\text{(symb. } \lim_{A_i^e,\, x\neq e} F(x) = l).$$

Analogous is the definition for the limit at the direction B_j, when x tends to e and $x \neq e$.

Theorem 2. *We consider a real function F defined on a partially ordered set E, without jumps and carrying the topology of the open intervals. If F is monotone and has a limit in each Mac Neille's cut (A_i, B_j) at all the directions A_i, $i \in I$, B_j, $j \in \Im$ and it has a limit in each $e \in E$ at all the directions A_i^e, $i \in I$, B_j^e, $j \in \Im$, when x tends to e and $x \neq e$, then there exists a completion E_{ku} of E, into which F can be continuously extended to a real function $\tilde{F}$.*

PROOF: Following the above symbolisms, consider the Mac Neille's completion $\tilde{E}$ of E. Next, on the set $\tilde{E}$ we consider for each $e \in \tilde{E}$ the decompositions of $]\longleftarrow, e[$ and $]e, \rightarrow[$ into maximal directed sets, which have no ends. Put $\Xi(\tilde{E}) = \Xi_-(\tilde{E}) \cup \Xi_+(\tilde{E})$, where $\Xi_-(\tilde{E})$ (resp. $\Xi_+(\tilde{E})$) is the set of the right directed subsets of $]\longleftarrow e[$ resp. left directed subsets of $]e \rightarrow [$).

The set $E_{ku} = \tilde{E} \cup \Xi(E)$ is an extension of E, which is ordered by an extension of the given $\leq$; to do that, it is enough, for each $x \in \Xi_-(e)$, $x = A_i$ or $x = A_i^e$ (referring to a cut $(A_i, B_j)_{(i,j)}$ or $(A_i^e, B_j^e)_{(i,j)}$), one to put $x \leq (A_i, B_j)$ or $x \leq e$. (Analogous results are for $x \in \Xi_+(e)$, $x = B_j$ or $x = B_j^e$). □

Definition of $\tilde{F}$. If $x \in E$, we put $\tilde{F}(x) = F(x)$.

Let $x \in \tilde{E} \setminus E$. Then x is a cut $(A = (A_i), B = (B_j))$ which is a gap and there exists the limit of F at all the directions. Because of the monotony of F we have that, if $\lim_{A_i} F(x) = l_i$ and $\lim_{B_j} F(x) = m_j$, then $l_i \leq m_j$.

Put $\tilde{F}(x) = e$, with $l_i \leq e \leq m_j$ for every $i \in I$, $j \in \Im$.

Let now $x = A_i' \in E_{ku}$ be a right directed subset of some $]\longleftarrow, e[$, without end. The trace A_i of A_i' on E will not have an end, too. If $e \in E$, then, by supposition, there exists the $\lim_{A_i} F(x)$. If $e \notin E$, then A_i belongs to a Mac Neille's cut, which is a gap and, by supposition, there exists the $\lim_{A_i} F(x)$, too. In both of the cases, if l is this limit, we put $\tilde{F}(x) = l$.

Analogously we define the $\tilde{F}(x)$, when $x = B_j'$ is a left directed subset of $]e \rightarrow [$ without end.

The function $\tilde{F}$ is an extension of F.
We will prove its continuity.

If $e \in E$, the continuity is obvious.

If $e \in \tilde{E}$, $\tilde{F}(e) = l$ and $e = ((A_i)_i, (B_j)_j) \in \tilde{E}$, then $\lim_{A_i} \tilde{F}(x) \leq l \leq \lim_{B_j} \tilde{F}(x)$ and hence there will exist $\alpha_i \in A_i$, $b_j \in B_j$ such that

$$\tilde{F}(]\alpha_i, \beta_j[) \subseteq]\tilde{F}(\alpha_i), \tilde{F}(\beta_j)[.$$

Now, let $e \in E_{ku} \setminus \tilde{E}$, say $e = A_i'$, a right directed subset of $\tilde{E}$. The set A_i' will belong to the decomposition $((A_i)_i, (B_j)_j)$ of an $x \in \tilde{E}$, and then there will hold $\tilde{F}(e) \leq \tilde{F}(x) = l$. It is obvious that the trace A_i on E of A_i', will have an element $\alpha_i \in A_i$, such that $\tilde{F}(\alpha_i) = F(\alpha_i) \leq \tilde{F}(e)$ and hence $\tilde{F}(]\alpha_i, x[) \subseteq]\tilde{F}(\alpha_i), \tilde{F}(x)[$.

The monotony of $\tilde{F}$ is evident.

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