Compact Imbeddings in Weighted Sobolev Spaces and Nonlinear Boundary Value Problems

PAVEL DRÁBEK, ALOIS KUFNER

Abstract. Weighted Sobolev spaces are a useful tool in the investigation of degenerated and/or singular elliptic differential operators. Information about the compactness of certain imbeddings in these spaces makes it possible to extend existence results to certain classes of *nonlinear* boundary value problems.

It is the aim of this paper to point out the role of such imbeddings, even for the case of ordinary differential equations, where the situation is more transparent and easier to handle, and to give some criteria for the compactness or certain imbeddings.

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(Dedicated to the memory of Svatopluk Fučík)

1 The space

Let $p > 1$ and $w = w(x)$ be a *weight function* on $(0, 1)$, i. e. a function measurable and positive almost everywhere in $(0, 1)$. Denote by

$$L^p(0, 1; w) = L^p(w) \tag{1.1}$$

the set of all functions $u = u(x)$ on $(0, 1)$, for which the norm

$$||u||_{p,w} = \left(\int_0^1 |u(t)|^p w(t)\mathrm{d}t\right)^{\frac{1}{p}}$$

is finite. Then $L^p(w)$ is called a *weighted Lebesgue space.*

Further, let $k \in \mathbb{N}$ and consider the set of all functions $u \in AC^{k-1}(0, 1)$ (i. e. functions whose derivatives of order $k - 1$ are absolutely continuous on $[0, 1]$) for which the expression

$$||u^{(k)}||_{p,w} = \left(\int_0^1 |u^{(k)}|^p w(t)\mathrm{d}t\right)^{\frac{1}{p}} \tag{1.2}$$

is finite. Note that the expression (1.2) is only a *seminorm* on $AC^{k-1}(0, 1)$. Therefore, let us introduce two subsets M_0, M_1 of the set $\{0, 1, \ldots, k - 1\}$ which satisfy the condition

$$\operatorname{card} M_0 + \operatorname{card} M_1 = k \tag{1.3}$$

and the so called *Pólya condition*

$$\sum_{j=0}^{m}(d_{0j}+d_{1j})\geq m+1, \qquad m=0,1,\ldots,k-1, \tag{1.4}$$

where $d_{ij}=1$ if $j\in M_i$, $d_{ij}=0$ if $j\notin M_i$, $i=0,1$. Now, if we denote by $B^{k,p}(M_0,M_1,w)$ the set of all functions $u\in AC^{k-1}(0,1)$ for which $\|u^{(k)}\|_{p,w}$ is finite and which satisfy the boundary conditions

$$u^{(i)}(0)=0 \quad \text{for} \quad i\in M_0, \qquad u^{(j)}(1)=0 \quad \text{for} \quad j\in M_1, \tag{1.5}$$

then (1.2) is a *norm* on $B^{k,p}(M_0,M_1,w)$ which is equivalent with the usual norm in the wieghted Sobolev space (for details, in particular for the important role of the Pólya condition, see [3]) and the completion of $B^{k,p}(M_0,M_1,w)$ with respect to the norm (1.2) will be denoted by

$$W^{k,p}(M_0,M_1,w) \tag{1.6}$$

and called a *weighted Sobolev space.*

If we assume in addition that the weight function w satisfies

$$w^{1-p'}\in L^1_{\mathrm{loc}}(0,1) \tag{1.7}$$

with $p'=\frac{p}{p-1}$, then $W^{k,p}(M_0,M_1,w)$ is a *Banach space* (see [7]).

In several papers (see e.g. [6], [9] and mainly [5]), the so called *k-th order Hardy inequality*

$$\left(\int_0^1|u(t)|^q v_0(t)\mathrm{d}t\right)^{\frac{1}{q}}\leq c\left(\int_0^1|u^{(k)}(t)|^p v_k(t)\mathrm{d}t\right)^{\frac{1}{p}} \tag{1.8}$$

is investigated and conditions on p,q,v_0,v_k are given which guarantee that (1.8) holds for all $u\in B^{k,p}(M_0,M_1,v_k)$.

Inequality (1.8) can be considered as a continuous imbedding of a weighted Sobolev space into a weighted Lebesque space:

$$W^{k,p}(M_0,M_1,v_k)\hookrightarrow L^q(v_0). \tag{1.9}$$

In Section 3 of this paper, we will derive conditions on the parameters $p,q>1$ and on the weight functions v_0,v_k which guarantee that the imbedding (1.9) is also *compact.* In the following Section 2 we will illustrate the importance of such information in connection with the investigation of certain nonlinear boundary value problems.

2 A nonlinear boundary value problem

Let us consider the nonlinear Dirichlet boundary value problem

$$\begin{aligned}\left(a(t)|u''(t)|^{p-2}u''(t)\right)'+|u(t)|^{q-2}u(t) &= g(t,u(t),u'(t)), \quad t\in(0,1), &(2.1)\\ u(0)=u'(0)=u(1)=u'(1) &= 0, &(2.2)\end{aligned}$$

where $p, q > 1$ are real numbers, $a = a(t)$ is a positive and measurable function in $(0,1)$ and $g = g(t,\xi,\eta)$ is a bounded Carathéodory function in $(0,1)\times \mathbb{R}^2$, i.e., measurable in t for any $(\xi,\eta)\in\mathbb{R}^2$ and continuons in (ξ,η) for a.e. $t\in(0,1)$. In contrast to analogous results on related topics we will consider a function $a(t)$ which may have *singularities* and/or *degeneracies* in $(0,1)$. To this end we will work with the notion of the weak solution to the problem (2.1) and we will look for it in a weighted Sobolev space $X := W^{2,p}(M_0, M_1, a)$ with $M_0 = M_1 = \{0,1\}$.

Let us assume, in this section, that the *compact imbeddings*

$$X \hookrightarrow\hookrightarrow L^q \text{ and } X \hookrightarrow\hookrightarrow W_0^{1,r,q} \tag{2.3}$$

hold with some $r > 1$. Here we denote by L^q the usual Lebesgue space and by $W_0^{1,r,q}$ an anisotropic Sobolev space with the norm

$$||v||_{1,r,q} := ||v||_{L^q} + ||v'||_{L^r}.$$

For the sake of brevity we shall write

$$||u||_X := ||u''||_{p,a}$$

for any $u \in X$. Then $(X, ||.||_X)$ is a *uniformly convex* Banach space.

The function $u\in X$ is called the *weak solution* of (2.1) if the integral identity

$$\int_0^1 a(t)|u''(t)|^{p-2}u''(t)v''(t)\mathrm{d}t + \int_0^1 |u(t)|^{q-2}u(t)v(t)\mathrm{d}t = \tag{2.4}$$

$$= \int_0^1 g(t,u(t),u'(t))v(t)\mathrm{d}t \tag{2.5}$$

holds for any $v\in X$.

It follows from (2.3) that all integrals in (2.4) make sense. Hence the following operators $J, S, G : X \to X^*$ are well defined:

$$\begin{aligned}(J(u),v) &= \textstyle\int_0^1 a(t)|u''(t)|^{p-2}u''(t)v''(t)\mathrm{d}t,\\ (S(u),v) &= \textstyle\int_0^1 |u(t)|^{q-2}u(t)v(t)\mathrm{d}t,\\ (G(u),v) &= \textstyle\int_0^1 g(t,u(t),u'(t))v(t)\mathrm{d}t\end{aligned}$$

for any $u, v\in X$ (here $(.,.)$ denotes the duality between X^* and X). We obtain immediately from (2.4) that the weak solvability of (2.1) is equivalent to the solvability of the *operator equation*

$$J(u) + S(u) - G(u) = 0. \tag{2.6}$$

Let us summarize some *useful properties* of the operators J, S and G.

Lemma 2.1. *The operators J and S are odd and the operator G is bounded.*

Lemma 2.2. *The operators J and G are compact.*

PROOF: The compactness of S follows directly from the compact imbedding $X \hookrightarrow\hookrightarrow L^q$ and the continuity of the Nemytskii operator from L^q into $L^{q'}$ $\left(\frac{1}{q}+\frac{1}{q'}=1\right)$ which is given by the function

$$\xi \mapsto |\xi|^{q-2}\xi.$$

Concerning the compactness of G we argue in a similar way: It follows from the continuity of the Nemytskii operator from $L^q \times L^r$ into $L^{q'}$ which is given by the function

$$(\xi,\eta) \mapsto g(.,\xi,\eta).$$

□

Lemma 2.3. *The operator J is one-to-one and, moreover, J and J^{-1} are continuous.*

PROOF: It follows from the properties of the real function $\xi \mapsto |\xi|^{p-2}\xi$ that J is *strictly monotone,* i. e.

$$(J(u)-J(v),u-v)>0$$

holds for any $u,v \in X$ such that $u \neq v$. The *continuity* of J follows from the continuity of the Nemytskii operator given by

$$\xi \mapsto a(.)|\xi|^{p-2}\xi$$

and acting between $L^p(a)$ and $(L^p(a))^*$ (see [4]). It follows directly form the the definition that J is *coercive.* Applying the theory of monotone operators (see e. g. [2]) we get that J is *onto* X^*, J^{-1} is *strictly monotone, bounded* (i. e., J^{-1} maps bounded sets onto bounded sets) and *demicontinuous* (i. e. J^{-1} maps strongly convergent sequences onto weakly convergent sequences). We prove that it follows already from here that J^{-1} is continuous. Let $\{w_n\} \subset X^*$ be a sequence which converges to some element $w \in X^*$. Denote $u_n = J^{-1}(w_n)$, $u = J^{-1}(w)$. From the definition of J we obtain

$$(J(u_n)-J(u),u_n-u) \geq (||u_n||_X^{p-1}-||u||_X^{p-1})(||u_n||_X-||u||_X),$$

i. e.

$$\begin{aligned}||w_n-w||_{X^*}||J^{-1}(w_n)-J^{-1}(w)||_X &\geq (w_n-w,J^{-1}(w_n)-J^{-1}(w)) \geq \\ &\geq \left[||J^{-1}(w_n)||_X^{p-1}-||J^{-1}(w)||_X^{p-1}\right]\left[||J^{-1}(w_n)||_X-||J^{-1}(w)||_X\right].\end{aligned}$$

The left hand side approaches zero due to the convergence $w_n \to w$ in X^* and the boundedness of J^{-1}. Hence

$$||J^{-1}(w_n)||_X \to ||J^{-1}(w)||_X.$$

This convergence together with the demicontinuity of J^{-1} and the uniform convexity of X imply that

$$J^{-1}(w_n) \to J^{-1}(w)$$

(strongly) in X (see [2]). □

Let us define the operator $T : X \to X^*$ by the relation

$$T(w) := w + S(J^{-1}(w)) - G(J^{-1}(w)),$$

$w \in X^*$. It follows from Lemmas 2.2, 2.3 that T is a compact perturbation of the identity on X^* and hence the *Leray-Schauder degree theory* can be applied to T. Due to Lemma 2.1 the operator

$$w \mapsto S(J^{-1}(w))$$

is *odd* and the operator

$$w \mapsto G(J^{-1}(w))$$

is *bounded.*

Theorem 2.1. *The equation* (2.6) *has at least one solution.*

PROOF: Let us define the homotopy of compact perturbations of the identity in the following way:

$$\mathcal{H}(w, \lambda) = w + S(J^{-1}(w)) - \lambda G(J^{-1}(w))$$

for $w \in X^*$, $\lambda \in [0, 1]$. Let us prove that $\mathcal{H}$ is an *admissible homotopy,* i.e. there exists $R > 0$ such that

$$\mathcal{H}(w, \lambda) \neq 0 \tag{2.7}$$

for any $\lambda \in [0, 1]$ and for any $w \in X^*$ satisfying $\|w\|_{X^*} = R$. For a given $w \in X^*$ denote $u = J^{-1}(w)$. Then we get

$$(\mathcal{H}(w, \lambda), J^{-1}(w)) = (J(u), u) + (S(u), u) - \tag{2.8}$$

$$-\lambda(G(u), u) \geq \|u\|_X^p - c\|u\|_X, \tag{2.9}$$

where the constant $c > 0$ is independent of $\lambda \in [0, 1]$. It is easy to see that

$$\|w\|_{X^*} = \|u\|_X^{p-1} (= \|J^{-1}(w)\|_X^{p-1})$$

for any $w \in X^*$ and hence (2.7) follows from (2.8) for sufficiently large $R > 0$.

Let us denote by $B_R(0)$ the ball in X^* centered at the origin and with the radius $R > 0$. It follows from (2.7) and from the homotopy invariance property of the Leray-Schauder degree that

$$\deg[T; B_R(0), 0] = \deg[\mathcal{H}(., 1); B_R(0), 0] =$$

$$= \deg[\mathcal{H}(., 0); B_R(0), 0] = \deg[I + S \circ J^{-1}; B_R(0), 0].$$

Since the last degree is equal to an odd number, by the Borsuk theorem, we get

$$\deg[T; B_R(0), 0] \neq 0.$$

The basic property of the degree implies the existence of at least one $w \in B_R(0)$ such that

$$T(w) = 0.$$

But then $u = J^{-1}(w)$ is the solution of the operator equation (2.4). □

Remark 2.1. It follows from the considerations at the beginning of this section that the element $u \in X$ the existence of which is guaranteed by Theorem 2.1 is the weak solution of the Dirichlet boundary value problem (2.1).

Remark 2.2. Let us note that for $q \geq p$ and $r \geq p$ the compactness of the imbeddings (2.3) is guaranteed if the following conditions are satisfied:

$$\lim_{t \to 0_+} B_i(t) = \lim_{t \to 1_-} B_i(t) = 0, \quad i = 1, 2, 3,$$

where

$$B_1(t) = (1-t)^{1-\frac{1}{q}} \left(\int_0^t a^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$
$$B_2(t) = (1-t)^{\frac{1}{q}} \left(\int_0^t (t-s)^{p'} a^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$
$$B_3(t) = (1-t)^{\frac{1}{r}} \left(\int_0^t a^{1-p'}(s)ds \right)^{\frac{1}{p'}}.$$

(For details, see Section 3.)

3 The compactness of the imbedding (1.9)

First, let us introduce two operators, namely the operator M_w of *pointwise multiplication* (by a weight function w):

$$(M_w f)(t) = w(t)f(t), \quad t \in (0,1), \tag{3.1}$$

and the *integral operator* T:

$$(Tf)(x) = \int_0^1 K(x,t)f(t)\mathrm{d}t. \tag{3.2}$$

Proposition 3.1. *Let $s > 1$ and w, v be weight functions on $(0,1)$. Then the operator M_w is an isometric isomorphism,*

$$M_w : L^s(v) \to L^s(w^{-s}v) \tag{3.3}$$

with $\|M_w\| = 1$, and $(M_w)^{-1} = M_{\frac{1}{w}}$.

Proposition 3.2. *Let $p, q > 1$ and suppose that the kernel $K(x,t)$ satisfies*

$$\left(\int_0^1 \left(\int_0^1 |K(x,t)|^{p'} \mathrm{d}t\right)^{\frac{q}{p'}} \mathrm{d}x\right)^{\frac{1}{q}} < \infty. \tag{3.4}$$

Then the operator T from (3.2) *maps L^p compactly into L^q.*

The proof of Proposition 3.1 is straightforward, the proof of Proposition 3.2 can be found in [1]. Using these two results, we are able to prove the following assertion.

Theorem 3.1. *Let $p, q > 1$. Let v_0, v_k be weight functions on* $(0,1)$ *and suppose that the kernel $K(x,t)$ satisfies*

$$\left(\int_0^1 \left(\int_0^1 |K(x,t)|^{p'} v_k^{1-p'}(t) \mathrm{d}t\right)^{\frac{q}{p'}} v_0(x) \mathrm{d}x\right)^{\frac{1}{q}} < \infty. \tag{3.5}$$

Then the operator T from (3.2) *maps $L^p(v_k)$ compactly into $L^q(v_0)$.*

PROOF: If we denote

$$\tilde{K}\ (x,t) = K(x,t) v_k^{-\frac{1}{p}}(t) v_0^{\frac{1}{q}}(x)$$

then condition (3.5) implies that, in view of Proposition 3.2, the operator $\tilde{T}$ defined by

$$(\tilde{T}h)(x) = \int_0^1 \tilde{K}(x,t) h(t) \mathrm{d}t$$

maps L^p compactly into L^q. Hence

$$\begin{aligned} (Tf)(x) &= \textstyle\int_0^1 K(x,t) f(t) \mathrm{d}t = \\ &= \textstyle\int_0^1 \tilde{K}(x,t) v_0^{-\frac{1}{q}}(x) v_k^{\frac{1}{p}}(t) f(t) \mathrm{d}t = \\ &= v_0^{-\frac{1}{q}}(x) \textstyle\int_0^1 \tilde{K}(x,t) v_k^{\frac{1}{p}}(t) f(t) \mathrm{d}t, \end{aligned}$$

i. e.

$$T = M_{v_0^{-\frac{1}{q}}} \circ \tilde{T} \circ M_{v_k^{\frac{1}{p}}}.$$

Since

$$\tilde{T} : L^p \to L^q$$

and (due to Proposition 3.1)

$$M_{v_k^{\frac{1}{p}}} : L^p(v_k) \to L^p, \qquad M_{v_0^{-\frac{1}{q}}} : L^q \to L^q(v_0),$$

we have

$$L : L^p(v_k) \to L^q(v_0).$$

Moreover, T is compact as a composition of the compact operator $\tilde{T}$ and two bounded operators of the type M_w. □

In [5], the investigation of the Hardy inequality (1.8) for $u \in W^{k,p}(M_0, M_1, v_k)$ is reduced to the study of the inequality

$$\left(\int_0^1 |(Tf)(x)|^q v_0(x)\mathrm{d}x\right)^{\frac{1}{q}} \le c\left(\int_0^1 f^p(x)v_k(x)\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.6}$$

where T is an integral operator of the form (3.2), defined on the set of all measurable nonnegative functions f on $(0,1)$. More precisely, the kernel $K(x,t)$ in (3.2) is the *Green function* of the boundary value problem

$$\begin{cases} u^{(k)} = f & \text{on } (0,1), \\ u^{(i)}(0) = 0 & \text{for } i \in M_0, \quad u^{(j)}(1) = 0 \quad \text{for } j \in M_1. \end{cases} \tag{3.7}$$

It can be shown – supposing that the boundary value problem (3.7) is uniquely solvable – that $K(x,t)$ is given by

$$K(x,t) = \begin{cases} K_1(x,t) & \text{for } 0 < t < x < 1, \\ K_2(x,t) & \text{for } 0 < x < t < 1. \end{cases} \tag{3.8}$$

Consequently, we have that

$$(Tf)(x) = \int_0^x K_1(x,t)f(t)\mathrm{d}t + \int_x^1 K_2(x,t)f(t)\mathrm{d}t \tag{3.9}$$

or

$$Tf = T_1 f + T_2 f,$$

where

$$(T_1 f)(x) = \int_0^x K_1(x,t)f(t)\mathrm{d}t, \quad (T_2 f)(x) = \int_x^1 K_2(x,t)f(t)\mathrm{d}t. \tag{3.10}$$

In order to investigate the compactness of the imbedding (1.9), i.e.

$$W^{k,p}(M_0, M_1, v_k) \hookrightarrow\hookrightarrow L^q(v_0), \tag{3.11}$$

it suffices to investigate the compactness of the operator T from (3.9) – or the two operators T_1, T_2 from (3.10) – as mappings from $L^p(v_k)$ into $L^q(v_0)$. The reason is that the identity operator

$$I : W^{k,p}(M_0, M_1, v_k) \to L^q(v_0)$$

can be considered as the composition

$$I = T \circ D_k \quad (\text{or} \quad I = T_1 \circ D_k + T_2 \circ D_k),$$

where the operator $D_k : W^{k,p}(M_0, M_1, v_k) \to L^p(v_k)$ is defined by

$$D_k u = u^{(k)}.$$

Consequently, we will use the following assertion (see also the proof of Lemma 7.12 in [8]).

Lemma 3.1. *Let $p, q > 1$ and suppose that the operator T (or the operators T_1, T_2): $L^p(v_k) \to L^q(v_0)$ is (are) compact. Then the imbedding* (3.11) *is compact, too.*

So, we can concentrate on the compactness of the integral operators T_1, T_2. For the kernel K from (3.8) the condition (3.5) reads as follows

$$\left(\int_0^1 \left(\int_0^x |K_1(x,t)|^{p'} v_k^{1-p'}(t)\mathrm{d}t + \int_x^1 |K_2(x,t)|^{p'} v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{q}{p'}} v_0(x)\mathrm{d}x\right)^{\frac{1}{q}} < \infty.$$

This condition can be replaced, in general, by a *pair* of conditions

$$\left\{\begin{array}{ll} \left(\int_0^1 \left(\int_0^x |K_1(x,t)|^{p'} v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{q}{p'}} v_0(x)\mathrm{d}x\right)^{\frac{1}{q}} & < \infty, \\ \left(\int_0^1 \left(\int_x^1 |K_2(x,t)|^{p'} v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{q}{p'}} v_0(x)\mathrm{d}x\right)^{\frac{1}{q}} & < \infty. \end{array}\right. \tag{3.12}$$

From Theorem 3.1, we immediately obtain.

Theorem 3.2. *Let $p, q > 1$ and suppose that the kernels K_1, K_2 satisfy the conditions* (3.12). *Then the operator T from* (3.9) *maps $L^p(v_k)$ compactly into $L^q(v_0)$.*

Remark 3.1. Since the kernel $K(x,t)$ from (3.8) is connected with the boundary value problem (3.7), the kernels K_1, K_2 have a specific form, which allows to derive – at least in some cases – conditions, which are necessary and sufficient.

First, let us consider the case

$$M_0 = \{0, 1, \ldots, k-1\}, \quad M_1 = \emptyset.$$

In this case, the boundary value problem (3.7) is in fact the Cauchy problem

$$u^{(k)} = f \text{ in } (0,1), \quad u(0) = u'(0) = \ldots = u^{(k-1)}(0) = 0,$$

and its solution $u = Tf$ is given by

$$u(x) = (Tf)(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)\mathrm{d}t. \tag{3.13}$$

So we have (3.9) with

$$K_1(x,t) = \frac{1}{(k-1)!}(x-t)^{k-1}, \quad K_2(x,t) \equiv 0$$

and the following assertion holds.

Theorem 3.3. *Let $1 < p \le q < \infty$ and let v_0, v_k be weight functions on $(0,1)$. Then the operator T from (3.13) maps $L^p(v_k)$ compactly into $L^q(v_0)$ if and only if the following conditions are satisfied:*

$$\lim_{x\to 0_+} B_i(x) = \lim_{x\to 1_-} B_i(x) = 0, \quad i = 1,2, \tag{3.14}$$

where

$$\begin{cases} B_1(x) = \left(\int_x^1 (t-x)^{(k-1)q} v_0(t)\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^x v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}}, \\ B_2(x) = \left(\int_x^1 v_0(t)\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^x (x-t)^{(k-1)p'} v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}}. \end{cases} \tag{3.15}$$

Instead of giving a proof let us only mention that the conditions (3.14) are in fact the conditions derived by Stepanov [9]. He considered the operator T and the corresponding weighted Lebesgue spaces on $(0,\infty)$ instead of $(0,1)$ but his approach can be used almost literally also in our case.

Remark 3.2. If $k=1$, then the functions $B_1(x), B_2(x)$ *coincide* and the condition (3.14) is exactly the condition for the *Hardy operator* H_1,

$$(H_1 f)(x) = \int_0^x f(t)\mathrm{d}t \tag{3.16}$$

to map $L^p(v_k)$ *compactly* into $L^q(v_0)$. See [8], Theorem 7.3.

In what follows, we shall consider kernels $K(x,t)$ of the type

$$K(x,t) = w(x)W(t)$$

with w, W weight functions. In [5], it is shown (for $M_0 \cap M_1 = \emptyset$) or at least conjectured (for the remaining choice of M_0, M_1) that the kernels $K_i(x,t)$, $i = 1,2$, appearing in the boundary value problem (3.7), are in the corresponding triangles $\Delta_1 = \{(x,t) : 0 < t < x < 1\}$; $\Delta_2 = \{(x,t); 0 < x < t < 1\}$ equivalent to special products $w_i(x)W_i(t)$ with $w_i(x) = x^{\alpha_i}(1-x)^{\beta_i}$, $W_i(t) = t^{\gamma_i}(1-t)^{\delta_i}$. The nonnegative integers $\alpha_i, \beta_i, \gamma_i, \delta_i$ depend on M_0, M_1. In some cases, these kernels are not only equivalent but, moreover, *equal* to such products: e.g., for $k=2$ and $M_0 = M_1 = \{0\}$, it is $K_1(x,t) = (1-x)t$ and $K_2(x,t) = x(1-t)$.

For kernels K_i of the form

$$K_i(x,t) = w_i(x)W_i(t) \tag{3.17}$$

with *general* weight functions w_i, W_i on $(0,1)$, we can give again necessary and sufficient conditions for the compactness of the corresponding operators T_i from (3.10). For this purpose, let us consider, besides the Hardy operator H_1 from (3.16), its counterpart H_2 :

$$(H_2 f)(x) = \int_x^1 f(t)\mathrm{d}t. \tag{3.18}$$

Since necessary and sufficient conditions, under which H_i maps $L^p(v_k)$ *compactly* into $L^p(v_0)$, are already known (see [8], Section 7), we immediately obtain the following assertion.

Theorem 3.4. *Let $p, q > 1$ and let $v_0, v_k, w_i, W_i (i = 1,2)$ be weight functions on $(0,1)$. Let*

$$(T_1 f)(x) \quad = w_1(x) \int_0^x W_1(t) f(t)\mathrm{d}t, \tag{3.19}$$

$$(T_2 f)(x) \quad = w_2(x) \int_x^1 W_2(t) f(t)\mathrm{d}t \tag{3.20}$$

[*i. e.* T_1 *and* T_2 *are the operators from* (3.10) *with kernels* K_1, K_2 *of the form* (3.17)]. *Denote*

$$\begin{cases} B_1(x) = \left(\int_x^1 w_1^q(t) v_0(t)\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^x W_1^{p'}(t) v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}}, \\ B_2(x) = \left(\int_0^x w_2^q(t) v_0(t)\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_x^1 W_2^{p'}(t) v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}}, \end{cases} \tag{3.21}$$

$$\begin{cases} A_1 = \left(\int_0^1 \left(\int_x^1 w_1^q(t) v_0(t)\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_0^x W_1^{p'}(t) v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{r}{q'}} W_1^{p'}(x) v_k^{1-p'}(x)\mathrm{d}x\right)^{\frac{1}{r}}, \\ A_2 = \left(\int_0^1 \left(\int_0^x w_2^q(t) v_C(t)\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_x^1 W_2^{p'}(t) v_k^{1-p'}(t)\mathrm{d}t\right)^{\frac{r}{q'}} W_2^{p'}(x) v_k^{1-p'}(x)\mathrm{d}x\right)^{\frac{1}{r}}, \end{cases} \tag{3.22}$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Then T_i maps the space $L^p(v_k)$ compactly into $L^q(v_0)$ if and only if the following conditions are fulfilled:

(i) For $1 < p \leq q < \infty$, the functions $B_i(x)$ are bounded on $(0,1)$ and

$$\lim_{x \to 0_+} B_i(x) = \lim_{x \to 1_-} B_i(x) = 0 \quad (i = 1,2). \tag{3.23}$$

(ii) For $1 < q < p < \infty$, it is

$$A_i < \infty \quad (i = 1,2).$$

PROOF: It follows from (3.19) that

$$T_i = M_{w_i} \circ H_i \circ M_{W_i}, \quad i = 1,2, \tag{3.24}$$

where H_i are the Hardy operators (3.16), (3.18) and M_{w_i}, M_{W_i} are the operators from (3.1). Due to Proposition 3.1 we have

$$\begin{aligned} M_{W_i}: \quad L^p(v_k) &\to L^p(W_i^{-p} v_k), \\ M_{w_i}: L^q(w_i^q v_0) &\to L^p(w_i^{-q} w_i^q v_0) = L^q(v_0). \end{aligned}$$

The boundedness of the function $B_i(x)$ from (3.21) together with (3.23) (for $p \leq q$) or the finiteness of the number A_i from (3.22) (for $p > q$) are the necessary and sufficient conditions for the Hardy operator H_i to be compact as

$$H_i : L^p(W_i^{-p} v_k) \to L^q(w_i^q v_0)$$

(see [8], Section 7). Now, the assertion follows from (3.24) due to the boundedness of M_{W_i} and M_{w_i}. □

Remark 3.3. The functions $B_i, i = 1, 2$, from Remark 2.2 are the functions (3.15) from Theorem 3.3 for the special case $k = 2$ and $v_0(x) \equiv 1$.

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Address: Pavel Drábek, Department of Mathematics, University of West Bohemia, Americká 42, 306 14 Plzeň, Czech Republic, E-mail: pdrabek@zcu.cz
Alois Kufner, Mathematical Institute of the Academy of Sciences, Žitná 25, 115 67 Praha, Czech Republic, E-mail: kufner@csearn.bitnet, kufner@earn.cvut.cz

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