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Jumping Nonlinearities and Mathematical Models of Suspension Bridge

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Abstract. In this paper we study the mathematical model of suspension bridge proposed by Lazer and McKenna. It is one-dimensional nonlinear beam equation with damping. We use the previous results of Fučík and Krejčí concerning the boundary value problems with jumping nonlinearities in order to explain large oscillations of the bridge.

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(Dedicated to the memory of Svatopluk Fučík)

1 Introduction

Let us consider the *periodic-boundary value problem* for the *beam equation*

$$\begin{cases} u_{tt} + EI\ u_{xxxx} + \delta u_t = -ku^+ + W(x) + \varepsilon f(x,\, t), \\ u\ (0,\, t) = u\ (L,\, t) = u_{xx}(0,\, t) = u_{xx}(L,\, t) = 0, \\ u(x,\, t + 2\pi) = u(x,\, t), \ -\infty < t < \infty,\, x \in (0,\, L). \end{cases} \tag{1.1}$$

This problem can be regarded as a *model* of *suspension bridge* under the following simplifying assumptions: We assume that the bridge is one-dimensional, vibrating beam, supported above by cables whose restoring force due to elasticity is proportional to $u^+ = \max\{u,\, 0\}$. Here $u = u(x,\, t)$ is the displacement at a point at distance x from one end of the bridge at time t and u is measured in the downward direction. Simultaneously, we assume that a cable does not exert a restoring force if compressed. The meaning of the given constants and functions is the following:

E	Young's moduls
I	moment of inertia of the cross section
δ	friction coefficient
k	elastic coefficient of the cable
W	weight per unit length of the bridge pushing it down
ϵf	external time-periodic forcing term (due to the wind)
L	length of the bridge

This model was introduced in the work of Lazer and McKenna [8] and then it has been studied in several papers (see e. g. Lazer and McKenna [9], [10]], Glover, Lazer and McKenna [6], McKenna and Walter [11], Fonda, Schneider and Zanolin

[4]). It should be emphasized that the problem (1.1) *does not describe the complex behavior* of the bridge: the motion of the main cable and the towers is ignored, the coupling of the main cable and the roadbed is neglected and also the torsional oscillations of the roadbed are not considered.

The purpose of this paper is to explain on the simple model (1.1) some unexpected phenomena occuring in connection with the collapse of the Tacoma Narrows suppension bridge. The novelty with respect to the previous papers mentioned above consists in the fact that we consider here the beam equation (1.1) with nonzero friction coefficient $\delta > 0$ and concerning the existence result we do not assume any kind of symmetries (like in [11]). It fits very well with reality that we obtain unique solution in the case $\varepsilon = 0$ (there is no wind) and multiple solutions for special $W(x)$ and $f(x, t)$, whenever $\epsilon \neq 0$. More general functions $W(x)$ and $f(x, t)$ are considered in the forthcomming paper Berkovits, Drábek and Mustonen [2].

Acknowledgement. This paper is dedicated to the memory of professor Svatopluk Fučík who died 15 years ago. The notion of jumping nonlinearity introduced and studied by Fučík is one of the most important tools in dealing with this simple but instructive model. The author is proud to be a pupil of professor Fučí k and he is grateful for everything what professor Fučí k tought him during the studies at Charles University in Prague.

2 Preliminaries

By making a change of scale of variable x we can transform the periodic-boundary value problem (1.1) to

$$\begin{cases} u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + k\ u^+ = W(x) + \varepsilon f(x,\, t), \\ u(0,\, t) = u\ (\pi,\, t) = u_{xx}(0,\, t) = u_{xx}(\pi,\, t) = 0, \\ u(x,\, t+2\pi) = u\ (x,\, t), \ -\infty < t < \infty,\ x \in (0, \pi) \end{cases} \tag{2.1}$$

with $\alpha \neq 0$, $\beta > 0$. (We write again k, W, ε and f for rescaled $\hat{k}$, $\hat{W}$, $\hat{\varepsilon}$ and $\hat{f}$.) We will work with the generalized solution of (2.1).

Let $H = L^2([0,\, \pi] \times [0,\, 2\pi])$ be the usual Hilbert space and let $\mathcal{D}$ stands for all smooth functions $v : [0,\, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfying the boundary conditions in (2.1). Let ψ be a continuous real function defined on $\mathbb{R}$ and suppose that there exist a_1, $a_2 \geq 0$ such that

$$|\psi(\xi)| \leq a_1 + a_2|\xi|,\ \xi \in \mathbb{R}.$$

Let $h \in H$. A function $u \in H$ is called a *generalized solution* (GS) of the beam equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + \psi(u) = h$$

(with periodic-boundary conditions from (2.1)) if and only if the integral identity

$$\int_0^{2\pi}\int_0^{\pi} u\ (v_{tt} + \alpha^2 v_{xxxx} - \beta v_t) \mathrm{d}x\mathrm{d}t =$$

$$= \int_0^{2\pi} \int_0^{\pi} [h - \psi(u)]v \, dxdt$$

holds for all $v \in \mathcal{D}$.

The system

$$\{e^{int} \sin mx; n \in \mathbb{Z}, \, m \in \mathbb{N}\}$$

forms a complete orthogonal system in $\tilde{H} = H + iH$ and each $u \in H$ has a representation

$$u(x, t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} u_{nm} e^{int} \sin mx,$$

where the convergence of the series is considered in the space $\tilde{H}$. Here

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} |u_{nm}|^2 < \infty, \; u_{-nm} = \bar{u}_{nm}$$

(see e. g. Berkovits, Mustonen [1]). Let $p, \, r \in \mathbb{N} \cup \{0\}$. Define

$$H^{p,\,r} = \{h \in H; \; \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} (n^{2r} + m^{2p})|h_{nm}|^2 < \infty\}.$$

Then $H^{p,\,r}$ with the norm

$$||h||_{H^{p,\,r}} = (\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} (n^{2r} + m^{2p})|h_{nm}|^2)^{\frac{1}{2}}$$

is the Sobolev space (see e. g. Vejvoda [12]).

The starting point are some basic properties of the linear beam equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t - \lambda u \; = h. \tag{2.2}$$

If $\{u_{nm}\}$ and $\{h_{nm}\}$ are Fourier coefficients corresponding to u and h, respectively, then u is a GS of (2.2) if and only if

$$(i\beta n + \alpha^2 m^4 - n^2 - \lambda)u_{nm} = h_{nm} \tag{2.3}$$

holds for all $n \in \mathbb{Z}, \, m \in \mathbb{N}$.

Put

$$\mathcal{N}_\lambda = \{(m, \, n) \in \mathbb{N} \times \mathbb{Z}; \alpha^2 m^4 - n^2 - \lambda = 0\},$$
$$\mathcal{S} = \{\lambda \in \mathbb{R}; \mathcal{N}_\lambda \neq \phi\},$$
$$\sigma = \{\lambda \in \mathbb{R}; \lambda = \alpha^2 q^4, \, q \in \mathbb{N}\}.$$

Then $\sigma \subset \mathcal{S}$ and some important properties of (2.2) can be proved.

Proposition 2.1. *Let $\lambda \in \mathbb{R}$. Then the equation* (2.2) *has for arbitrary $h \in H$ a unique* GS *$u \in H$ if and only if $\lambda \notin \sigma$. If $\lambda \notin \sigma$ then the mapping*

$$T_\lambda : H \to H, T_\lambda : h \to u,$$

where u is the unique GS *of* (2.2) *with the right hand side $h \in H$, has the following properties:*

(i) T_λ *is linear and* $\operatorname{Im} T_\lambda \subset C([0, \pi] \times [0, 2\pi])$;

(ii) $T_\lambda : H \to H$ *is compact and for its norm* $\|T_\lambda\|$ *we have*

$$\|T_\lambda\| \leq \frac{1}{\max\{dist(\lambda, \mathcal{S}),\ \min\{\beta, dist(\lambda, \sigma)\}\}} =$$
$$= \frac{1}{\min\{dist(\lambda, \sigma),\ \max\{\beta, dist(\lambda, \mathcal{S})\}\}};$$

(iii) *The mappings*
mb $T_\lambda : H \to C([0, \pi] \times [0, 2\pi])$, $T_\lambda|_{C([0,\pi]\times[0,2\pi])} : C([0, \pi] \times [0, 2\pi]) \to C([0, \pi] \times [0, 2\pi])$
are compact to.

(iv) *If* $p, r \in \mathbb{N} \cup \{0\}$ *then* $T_\lambda(H^{p,r}) \subset H^{p+2, r+1}$.

The proof follows the lines of the proof of Theorem (2.4) in Fučík [5] (see also [2]).

Following the ideas from Propositions 2.1 and 2.2 in [5] we get the "regularity property" of $u \in H^{2,1}$.

Lemma 2.1. *Let* $u \in H^{2,1}$. *Then*

$$(\mu u^+ - \nu u^-) \in H^{1,1}$$

for any real numbers μ *and* ν.

The following assertion justifies the reduction of the beam equation to the fourth order ODE in some special cases.

Proposition 2.2. *Let* $(\mu, \nu) \in \mathbb{R}^2$, $u \in H$ *and* $h \in H^{1,1}$, h *independent of* t. *Then* u *is* GS *of*

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t = \mu u^+ - \nu u^- + h(x)$$

if and only if the function u *is independent of the variable* t *and* $\tilde{u}(x) = u\ (x, t)$ *is a classical solution of the boundary value problem*

$$\begin{cases} \alpha^2 u^{(4)} = \mu u^+ - \nu u^- + h & in \quad (0, \pi), \\ u\ (0) = u\ (\pi) = u''(0) = u''(\pi) = 0. \end{cases}$$

(For the proof see [5] or [2].)

Combining Proposition 2.2 and Theorem 3.8 from Krejčí [7] we get the following characterization of the Fučík's spectrum of damped beam operator.

Proposition 2.3. *The set of all* $(\mu, \nu) \in \mathbb{R}^2$ *such that there exists a nontrivial* GS u *of the equation*

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t = \mu u^+ - \nu u^-$$

is a system of continuous curves $\{S_i^+, S_i^-; i \in \mathbb{N}\}$ *such that*

(i) $S_1^+ = \{(\alpha^2, \nu); \nu \in \mathbb{R}\}$, $S_1^- = \{(\mu, \alpha^2); \mu \in \mathbb{R}\}$;

(ii) $S_i^+, S_i^- \subset (0, \infty) \times (0, \infty)$, $i > 1$;

(iii) S_i^+ *is symmetrical to* S_i^- *with respect to the straight line* $\mu = \nu$. *If* i *is even then* $S_i^+ = S_i^-$;

(iv) *for each* $i \in \mathbb{N}$ *we have*

$$(S_i^+ \cup S_i^-) \cap (S_{i+1}^+ \cup S_{i+1}^-) = \phi.$$

Remark 2.1. In particular, this means that the equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t = \mu u^+$$

has only trivial GS for any $\mu \in \mathbb{R}$.

3 Main results

We have the following basic existence result.

Theorem 3.1. *Let* $\varepsilon \in \mathbb{R}$, $k > 0$, $W \in L^2(0, \pi)$, $f \in H$. *Then* (2.1) *hat at least one* GS $u \in H$.

PROOF: The generalized solvability of (2.1) is equivalent to the existence of $u \in H$ solving the operator equation

$$u - T_0(k\ u^+) - T_0(W + \varepsilon f) = 0 \tag{3.1}$$

with T_0 from Proposition 2.1. Hence, to prove the assertion, it is sufficient to show that

$$deg[G; B_R(0), 0] \neq 0, \tag{3.2}$$

where "*deg*" denotes the Leray-Schauder degree, $G : H \to H$ is defined by

$$G(u) = u - T_0(k\ u^+) - T_0(W + \varepsilon f),$$

and $B_R(0)$ is the ball in H centered at the origin with sufficiently large radius $R > 0$. To show (3.2), consider the homotopy

$$\mathcal{H}(\tau, u) = u - T_0(\tau k\ u^+) - \tau T_0(W + \varepsilon f),$$

$u \in H$, $\tau \in [0, 1]$. Assume that there are $\{u_n\} \subset H$ and $\{\tau_n\} \subset [0, 1]$ such that $||u_n||_H \to \infty$ and

$$\mathcal{H}(\tau_n, u_n) = 0. \tag{3.3}$$

Passing to suitable subsequences, we may assume $\tau_n \to \tau_0 \in [0, 1]$, $v_n = \frac{u_n}{\|u_n\|_H} \rightharpoonup v_0 \in H$ (*weakly*). Due to the compactness of $T_0 : H \to H$, we have $v_n \to v_0$ (*strongly*) and after the limiting process we get from (3.3) that τ_0 and v_0 verify

$$v_0 - T_0(\tau_0 k\ v_0^+) = 0.$$

This is equivalent to the fact that v_0 is GS of the equation

$$v_{tt} + \alpha^2 v_{xxxx} + \beta v_t = -\tau_0 k\ v^+.$$

But this contradicts Proposition 2.3 (see Remark 2.1). □

Theorem 3.2. *Let $k > 0$, $\varepsilon = 0$ and $W \in H^{1,1}$, W independent of t. Then* (2.1) *has unique* GS $u_0 \in H$ *which is independent of t.*

PROOF: Due to Proposition 2.2 any GS of (2.1) with $\epsilon = 0$ is independent of time t and $\tilde{u}_0(x) = u_0(x, t)$ is a classical solution of

$$\begin{cases} \alpha^2 u^{(4)} = -ku^+ + W & \text{in } (0, \pi), \\ u(0) = u\ (\pi) = u''(0) = u''(\pi) = 0. \end{cases} \tag{3.4}$$

The uniqueness of this solution follows from the monotonicity of the operator

$$u \mapsto \alpha^2 u^{(4)} + k\ u^+.$$

□

Corollary. *Assume in* Theorem 3.2 *that $W(x) = W_0$ (= nonzero constant). Then the corresponding* GS $u_0 \in H$ *is positive, symmetric with respect to the line $x = \frac{\pi}{2}$, and satisfies*

$$(u_0)_x(0, t) > 0,\ (u_0)_x(\pi, t) < 0. \tag{3.5}$$

PROOF: Since u_0 is a classical solution of (3.4), we can apply Lemma 5 from McKenna and Walter [11] together with a simple shift of variable $x(x := x + \frac{\pi}{2})$. □

Theorem 3.3. *Let $\varepsilon \in \mathbb{R}$, $k > 0$, $W(x) = W_0$, $f \in H^{1,2}$. Then there is $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the problem* (2.1) *has positive* GS $u \in H^{3,3}$ *which is "close" to u_0 from the* Corollary.

PROOF: Let us go back to the assertion (iv) of Proposition 2.1. Let us assume that $h \in H^{p,r}$ and $u = T_\lambda h$ with $\lambda \notin \sigma$. Then due to (2.3) we have $u \in H^{p+2,r+1}$ and

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} (n^{2r+2} + m^{2p+4})|u_{nm}|^2 =$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(n^{2r+2}+m^{2r+4})}{\beta^2 n^2 + (\alpha^2 m^4 - n^2 - \lambda)^2} |h_{nm}|^2. \tag{3.6}$$

For $\alpha^2 \geq 1$ we have

$$(m^{2p+4} + n^{2r+2}) \leq (m^4 + n^2)(m^{2p} + n^{2r}) \leq \\ \leq (\alpha^2 m^4 + n^2)(m^{2p} + n^{2r})$$

and for $\alpha^2 < 1$ we have

$$(m^{2p+4} + n^{2r+2}) \leq \frac{1}{\alpha^2}(\alpha^2 m^4 + n^2)(m^{2p} + n^{2r}).$$

Hence it follows from here and from (3.6) that

$$||u||^2_{H^{p+2,\, r+1}} \leq \\ \leq \gamma \textstyle\sum_{n=-\infty}^{\infty} \sum\limits_{m=1}^{\infty} \frac{\alpha^2 m^4 + n^2}{\beta^2 n^2 + (\alpha^2 m^4 - n^2 - \lambda)^2} (m^{2p} + n^{2r})|h_{nm}|^2, \tag{3.7}$$

where $\gamma = 1$ for $\alpha^2 \geq 1$ and $\gamma = \frac{1}{\alpha^2}$ for $\alpha^2 < 1$. Let us denote

$$a_{nm} = \frac{\alpha^2 m^4 + n^2}{\beta^2 n^2 + (\alpha^2 m^4 - n^2 - \lambda)^2},\ d = dist(\lambda, \sigma).$$

Then we get

(i) for $n = 0 : a_{nm} = \frac{\alpha^2 m^4 - \lambda + \lambda}{(\alpha^2 m^4 - \lambda)^2} \leq \frac{1}{d} + \frac{|\lambda|}{d^2}$;

(ii) for $n \neq 0$ and $|\alpha^2 m^4 - n^2 - \lambda| \leq 1$:

$$a_{nm} \leq \frac{2n^2 + |\lambda| + 1}{\beta^2 n^2} \leq \frac{|\lambda| + 3}{\beta^2};$$

(iii) for $n \neq 0$ and $|\alpha^2 m^4 - n^2 - \lambda| > 1$:

$$a_{nm} \leq \frac{2n^2 + \lambda}{\beta^2 n^2} + \frac{1}{\alpha^2 m^4 - n^2 - \lambda} \leq \frac{|\lambda| + 2}{\beta^2} + 1.$$

It follows from (3.7) and (i)-(iii) that there is a constant $c > 0$ independent of u and h such that

$$||u||_{H^{p+2,\, r+1}} \leq c||h||_{H^{p,\, r}}. \tag{3.8}$$

Let us denote $u_\varepsilon = T_k(\varepsilon f)$, for given $f \in H^{1,\,2}$. We have $u_\varepsilon \in H^{3,\,3}$ and if $|\varepsilon| < \varepsilon_0$ (with sufficiently small $\varepsilon_0 > 0$) then due to (3.5), (3.8) and the imbedding

$$H^{3,\,3} \hookrightarrow C^{1,\,1}([0,\, \pi] \times [0,\, 2\pi])$$

(see Vejvoda [12]) we obtain that $u = u_0 + u_\varepsilon$ is positive in $(0,\, \pi) \times (0,\, 2\pi)$. But this implies that $u \in H^{3,\,3}$ is GS of (2.1). □

Remark 3.1. Let us assume the same as in Theorem 3.3. If there is also some other GS $u_1 \in H^{3,3}$, $u_1 \neq u$, then this solution must assume negative values due to Proposition 2.3. Really, if u_1 was positive too, then $u_1 - u$ is nontrivial GS of the equation

$$v_{tt} + \alpha^2 v_{xxxx} + \beta v_t + k\ v\ = 0$$

which is a contradiction. The question of the multiplicity result of this kind will be studied in the forthcomming paper [2].

Let us assume, now, that $W \in H^{1,1}$ from Theorem 3.2 is of the form

$$W(x) = W_0 \sin x,\ \ x \in [0,\ \pi].$$

Then the corresponding unique GS $u_0 \in H$ is of the form

$$u_0(x) = \frac{W_0}{\alpha^2 + k} \sin x, \tag{3.9}$$

i.e. u_0 is positive in $(0,\ \pi)$ and satisfies (3.5). Hence we get the following variant of Theorem 3.3.

Theorem 3.4. *Let $\varepsilon \in \mathbb{R}$, $k\ > 0$, $W(x) = W_0 \sin x$, $f \in H^{1,2}$. Then there is $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the problem* (2.1) *has positive* GS $u \in H^{3,3}$ *which is "close" to u_0 defined by* (3.9).

The proof is the same as that of Theorem 3.3.

Remark 3.2. In the forthcomming considerations we will show that under some special *additional assumptions* in Theorem 3.4 we have the existence of *another solution* which has to *assume* also *negative values* due to Remark 3.1.

Let us suppose that the assumptions of Theorem 3.4 are satisfied, and

$$f(x,\ t) = \sin x \cdot \sin(t + \rho)$$

with some $\rho \in \mathbb{R}$. Looking for GS of (2.1) in the form $u(x,\ t) = v(t) \cdot \sin x$, we arrive at the following equation for v:

$$v'' + \beta v' + (\alpha^2 + k)v^+ - \alpha^2 v^- = W_0 + \varepsilon \sin(t + \rho). \tag{3.10}$$

Then the results of [6] show that if $\varepsilon > 0$,

$$\frac{2}{\sqrt{\alpha^2 + k}} < 2 < \frac{1}{\alpha} + \frac{1}{\sqrt{\alpha^2 + k}}$$

and the ratio $\frac{\varepsilon}{W_0}$ and $\beta > 0$ are sufficiently small then the equation with jumping nonlinearity (3.10) will have *exactly two stable 2π-periodic solutions v_i*, $i = 1, 2$, one of which (say v_1) is "close" to the constant $\frac{W_0}{\alpha^2+k}$ and the other (v_2) is "close" to a nonconstant 2π -periodic solution of

$$v'' + (\alpha^2 + k)v^+ - \alpha^2 v^- = W_0.$$

Hence $u_1(x, t) = v_1(t) \sin x$ is GS of (2.1) the existence of which is guaranteed by Theorem 3.4. On the other hand $u_2(x, t) = v_2(t) \sin x$ is the other GS which assumes also negative values due to Remark 3.1.

4 Concluding remarks and discussions.

Remark 4.1. Similarly as in the papers mentioned in the references, we were motivated by an unexpected behavior of the Tacoma Narrows bridge. We consider the model which includes nonzero damping term and use some results of Krejčí concerning the Fučík's spectrum for the fourth order equation. It seems that this result published more that 10 years ago was not very well known.

Remark 4.2. Let us remark that this paper offers a unified view on the behaviour of the model (1.1) in certain sence. As it was already pointed out this model was proposed by Lazer and McKenna [8] and it has been investigated in a series of forthcomming papers. The results showing the existence of large scale oscillations of the bridge were proved considering the beam equation with or/and without damping. In the case of damped equation the existence of multiple solutions considered in Remark 3.2 were known (see e. g. [9,10]) but there was no uniqueness result (as in Theorem 3.2) or general existence result (as in Theorem 3.1). Note that always a special form of $W(x) = W_0 \sin x$ was considered. On the other hand, in the case of the equation without damping, the multiplicity result was proved in [11] under more realistic assumption $W(x) \equiv W_0$. However, the multiple solutions persisted also if $\varepsilon = 0$ ("if there is no wind") in this case and it is not "natural".
Interpretation of the main results. The existence result of Theorem 3.1 justifies that the problem is well-prosed. Theorem 3.2 asserts that if there is no external disturbance (e. g. there is no wind) then the bridge achieves unique position determined only by its weight $W(x)$ per unit length. Under some special assumptions on $W(x)$ Theorems 3.3, 3.4 show that in the case of small external disturbances there is always the solution "near" to the position when the bridge is not disturbed. Considerations in Remark 3.2 imply that if $W(x) = W_0 \sin x$ and externally imposed periodic function $f(x, t)$ is of special from then there is another solution which is "far" from the position of the non-disturbed bridge.

It is clear that there are still many open questions. Note that the open problems posed in [10] are not answered yet. It should be interesting to derive the same multiplicity result as in Remark 3.2 but considering $W(x) \equiv W_0$, etc.

Note also that the methods of this paper allow to deal with damped beam equation without "separation of variables" and the restriction to the second order equation (see [6,8,9,10]), and also without the restriction to the spaces of symmetric functions (see [11]).

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