A Graph Method for Markov Models Solving

JAROSLAV MARKL

Abstract. In this note a useful tool for Markov models solving is presented. A simple method for finding stabilizied probabilities is stated and proved. The method is based on graph representations of Markov processes and gives analytical solutions in a non-algebraic way.

Keywords:

1991 Mathematics Subject Classification: 60J10, 60J27, 68R10

1. Problem statement

We only consider *stable Markov processes* with finite state sets. Recall that Markov process is said to be stable if a steady-state distribution (stabilizied probabilities of states) exists and does not depend on the initial distribution (the initial probabilities of states).

Let $S = \{1, 2, ..., n\}$ be the state set. Then the steady-state distribution is given by a stochastic vector

$$p = (p_1, p_2, \dots, p_n)^T \tag{1.1}$$

which coordinates fulfil the conditions

$$0 \le p_i \le 1, \quad i \in S, \tag{1.2}$$

$$\sum_{i=1}^{n} p_i = 1. \tag{1.3}$$

The steady-state distribution is uniquely determined by the transition rates matrix $Q = (q_{ij})$, $i, j \in S$ (for stable Markov processes with continuous time) or by the transition probabilities matrix $P = (p_{ij})$, $i, j \in S$ (for stable Markov processes with discrete time).

The problem under discussion is to find vector p if matrix Q (or matrix P) is given.

Elements of the Markov process theory can be found e.g.in [1]. They include the standard solution sketched in Section 2. The nonstandard (graph) solution is described in Section 3.

2. Standard algebraic solution

Let us consider the case of continuous time first. The elements of transition rates matrix Q fulfil the conditions

$$q_{ij} \geq 0, \qquad i,j \in S,\ i \neq j, \tag{2.1}$$

$$\sum_{j=1}^{n} q_{ij} = 0, \qquad i \in S. \tag{2.2}$$

Condition (2.2) can be rewritten as follows

$$q_{ii} = -\sum_{j=1, j\neq i}^{n} q_{ij}, \qquad i \in S. \tag{2.3}$$

The steady-state distribution p can be obtained as the solution of the linear equations system

$$Q^T.p = \mathbf{0} \tag{2.4}$$

together with the normalization condition (1.3). From (2.2) it follows that the system (2.4) is linearly dependent and also that arbitrary one equation can be omitted. Let us omit the last equation and replace it by equation (1.3). We get, using (2.3), the following system of n linear equations of n unknown quantities p_i.

$$\begin{pmatrix} -\sum_{j\neq 1} q_{1,j} & q_{2,1} & \cdots & q_{n-1,1} & q_{n,1} \\ q_{1,2} & -\sum_{j\neq 2} q_{2,j} & \cdots & q_{n-1,2} & q_{n,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{1,n-1} & q_{2,n-1} & \cdots & -\sum_{j\neq n-1} q_{n-1,j} & q_{n,n-1} \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} . \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_{n-1} \\ p_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

Let us denote this system by

$$A.p = b. \tag{2.5}$$

From the assumed stability of Markov processes it follows that system (2.5) has exactly one solution. Consequently, $det(A) \neq 0$ and the solution of (2.5) is given by the formulas

$$p_i = det(A_i)/det(A), \qquad i \in S, \tag{2.6}$$

where A_i is the matrix obtained from matrix A by replacing the i-th column by the column-vector b. Notice that (1.3),(2.6) imply

$$det(A) = \sum_{i=1}^{n} det(A_i). \tag{2.7}$$

Thus, $det(A)$ need not be independently computed.

Let us now consider the case of discrete time (Markov chains). The elements of transition probabilities matrix P fulfil the conditions

$$0 \leq p_{ij} \leq 1, \qquad i,j \in S, \tag{2.8}$$

$$\sum_{j=1}^{n} p_{ij} = 1, \qquad i \in S. \tag{2.9}$$

Condition (2.9) can be rewritten as follows

$$p_{ii} = 1 - \sum_{j=1, j\neq i}^{n} p_{ij}, \qquad i \in S. \tag{2.10}$$

The steady-state distribution given by vector p can be found as the solution of the linear equations system

$$P^T.p = p \tag{2.11}$$

completed by the normalization equation (1.3). Due to (2.9), system (2.11) is linearly dependent and arbitrary one equation can be omitted. The system (2.11) can be rewritten as

$$(P^T - I).p = \mathbf{0}. \tag{2.12}$$

Replacing the last equation of (2.12) with the equation (1.3) we get - using (2.10) - a system of n linear equations of n unknown quantities p_i which is formally quite identical with the system (2.5). The only one difference is that the rates q_{ij} are replaced by the probabilities p_{ij}. The assumed stability of Markov chain guarantees the existence and uniqueness of the solution.

3. Graph solution

The non-standard method is based on usual graph representations of Markov processes. With any stable Markov process we associate a weighted directed graph $< V, E, f >$, where V is a set of vertices, E is a set of edges (a subset of cartesian product $V \times V$) and f is a mapping from E into the set of positive real numbers (weights of edges). The graph $< V_Q, E_Q, f_Q >$ associated with a Markov process with transition rates matrix $Q = (q_{ij})$, $i,j \in S$, is defined as follows:

$$V_Q = S = \{1, 2, ..., n\},$$
$$E_Q = \{(i,j) : i,j \in S \wedge i \neq j \wedge q_{ij} > 0\},$$
$$f_Q((i,j)) = q_{ij} \quad \text{for all}(i,j) \in E_Q.$$

Similarly we define the graph $\langle V_P, E_P, f_P \rangle$ associated with a Markov chain with transition probabilities matrix $P = (p_{ij})$, $i,j \in S$:

$$V_P = S = \{1, 2, ..., n\},$$
$$E_P = \{(i,j) : i,j \in S \wedge i \neq j \wedge p_{ij} > 0\},$$
$$f_P((i,j)) = p_{ij} \quad \text{for all } (i,j) \in E_P.$$

Notice that the assumed stabilitity of a Markov process implies connectivity of its graph representation (otherwise there exist at least two different steady-state distributions).

In what follows we use the graph theory notion of the *tree*. By a tree we always understand a directed (rooted) tree where edges are directed to the root. A *spanning tree* of a given graph is a (directed) tree containing all the vertices. By the *weight* of such a tree we mean the product of weights of all edges belonging to the tree. More about basic notions of the graph theory can be found e.g. in [2].

The essence of the graph method for Markov models solving (finding vector p if matrix Q, resp. P is given) is expressed in the following theorem:

Theorem. *Let G be the graph associated with a stable Markov process (it means that $G =< V_Q, E_Q, f_Q >$ or $G =< V_P, E_P, f_P >$). Then the stabilizied probabilities of the process are given by formulas*

$$p_i = B_i / \sum_{j \in S} B_j, \qquad i \in S, \tag{3.1}$$

where B_i ($i \in S$) is the sum of weights of all the spanning trees of G which have their roots in vertex i.

PROOF: The assumed stability of the Markov process implies that there exists at least one spanning tree of G (namely for every state $i \in S$ with property $p_i \neq 0$ there is a spanning tree with the root in vertex i). Weights of all edges are positive real numbers and therefore the weight of any spanning tree is positive too. Consequently, the denominator of (3.1) has a non-zero value and the quantities p_i, $i \in S$, always exist. From the formulas (3.1) it is also clear that quantities p_i fulfil the conditions (1.2), (1.3).

To prove the theorem we have to show that quantities p_i given by (3.1) are the solutions of the linear equations system (2.5). The fulfilling of the last equation of (2.5) follows immediately from (3.1). It remains to prove validity of the first $n-1$ equations of (2.5). The j-th equation ($j = 1, 2, ..., n-1$) is of the form

$$(-\sum_{k=1,\ k \neq j}^{n} q_{jk}).p_j + \sum_{i=1,\ i \neq j}^{n} q_{ij}.p_i = 0. \tag{3.2}$$

Substituting (3.1) into (3.2) we get

$$\sum_{i=1,\ i \neq j}^{n} B_i.q_{ij} = B_j. \sum_{k=1,\ k \neq j}^{n} q_{jk}. \tag{3.3}$$

The left side of (3.3) is a sum of products of the form $b_i.q_{ij} (i \in S, i \neq j)$ where b_i is the weight of some spanning tree of G having the root in vertex i. Similarly, the right side of (3.3) is a sum of products $b_j.q_{jk} (j \neq k)$ where b_j is the weight of some spanning tree of G having the root in vertex j. Recall that any such b_i, resp.b_j , is also a product (of weights of all edges that belong to the relevant tree). Hence the expressions $b_i.q_{ij}$, resp.$b_j.q_{jk}$, are elementary addends of the left, resp. the right side of (3.3).

The theorem will be proved if we show that every elementary addend from the left side occurs also on the right side and vice versa. With every addend from the both sides we can associate a spanning tree with an additional edge from the root to another vertex; let us call it a "shorting edge" (it causes a cycle in the graph).

In Fig.1 a diagram corresponding to an expression $b_i.q_{ij}$ from the left side is sketched; (i,j) is the shorting edge. The unique successor of vertex j is denoted by k. Notice that the path from vertex k to vertex i can content several vertices and edges ($(k,i) \in E$ or $k = i$ being special cases). The graph depicted in Fig.1. can be interpreted in the way depicted in Fig.2. – as a spanning tree having the root in vertex j with the additional shorting edge (j,k), i.e. as a graph corresponding to some expression $b_j.q_{jk}$ from the right side of (3.3). It means that every elementary addend from the left side of equation (3.3) occurs also on the right side of this equation.

In a similar way we can prove the inverse. Every elementary addend $b_j.q_{jk}$ from the right side of (3.3) corresponds to a graph like the one sketched in Fig.2. By i we denote the immediate predecessor of the root j that lies on the unique branch from vertex k to the root j. The graph depicted on Fig.2. can now be interpreted in the sense of Fig.1. – as a spanning tree having root in vertex i with the shorting edge (i,j), i.e. as a graph corresponding to some expression $b_i.q_{ij}$ from the left side of (3.3). The proof of the theorem is completed. □

The just proved theorem has a simple natural meaning: the steady-state (stabilizied) probabilities are proportional to overall transitions rates (measured by the quantities B_i) directed to the corresponding states:

$$p_1 : p_2 : \ldots : p_n = B_1 : B_2 : \ldots : B_n.$$

4. Examples

Let us give some simple examples to illustrate the graph method.

Example 1. For the Markov process represented by the diagram in Fig.3. it holds

$$B_i = a_1 a_2 \ldots a_{i-1} b_{i+1} b_{i+2} \ldots b_n$$

for every state $i \in S = \{1, 2, \ldots, n\}$. For every vertex there exists only one spanning tree with root in that vertex. In the special case, when $a_i = a$, $b_i = b$ for every $i \in S$, we get $B_i = a^{i-1}b^{n-i}$ and hence we obtain for the stabilizied probabilities

$$p_i = (a^{i-1}b^{n-i}) / \sum_{j=1}^{n} (a^{j-1}b^{n-j}), \quad i \in S.$$

Example 2. For Markov chain defined by the graph depicted in Fig.4. it holds

$$B_1 = p_{32}.p_{21} + p_{31}.p_{21} + p_{23}.p_{31}$$
$$B_2 = p_{13}.p_{32} + p_{12}.p_{32} + p_{31}.p_{12}$$
$$B_3 = p_{21}.p_{13} + p_{23}.p_{13} + p_{12}.p_{23}$$

For every vertex there exist exactly three different spanning trees that have root in that vertex. The stabilized probabilities of states are:

$$p_i = B_i/(B_1 + B_2 + B_3), \qquad i = 1,2,3.$$

Example 3. Consider the Markov process presented by the graph diagram in Fig.5. The solution is as follows:

$$B_1 = b.d.h.g + b.d.h.f = b.d.h.(g + f)$$
$$B_2 = a.d.h.g + a.d.h.f = a.d.h.(g + f)$$
$$B_3 = a.c.h.g + a.c.h.f + a.e.g.h = a.h.(c.g + c.f + e.g)$$
$$B_4 = a.d.e.h$$
$$B_5 = a.d.e.g$$

There are two spanning trees for every vertex 1 and 2, three spanning trees for vertex 3 and one spanning tree for every vertex 4 and 5. The steady-state probabilities are:

$$p_i = B_i/(B_1 + B_2 + ... + B_5), \qquad i = 1,2,...,5.$$

The solutions of all three examples can be read directly from the graph diagrams. For more complex Markov processes (with many states and many possible transitions between them) it is not so easy to find all spanning trees by only viewing the diagrams. There is a serious danger that some of spanning trees will be omitted. In this case, we must use an algorithm, that systematically generates all spanning trees for given finite graph and given vertex of it (formulation and proof of such an algorithm is behind the scope of this paper).

Finally, let us note that an analytic solution, independently of whether it was obtained by the standard method or by the graph method, is valuable only if it is not too complex. Only in this case the solution can be effectively analyzed by mathematical tools. Otherwise we have to content ourselves with numerical solutions and to use simulations techniques for analysis.

5. Conclusion

In this note an original graph method for Markov models solving was presented. Most of Markov models widely used in reliability theory, renewal theory, queue theory and many other related theories can be solved by this method much more easily and quickly than by the standard algebraical methods.

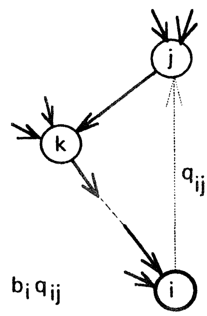

Fig.1.

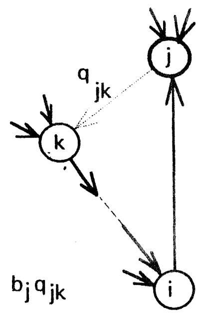

Fig.2.

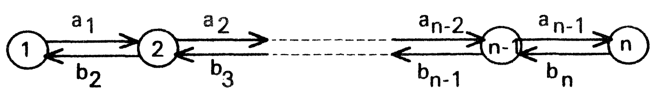

Fig.3.

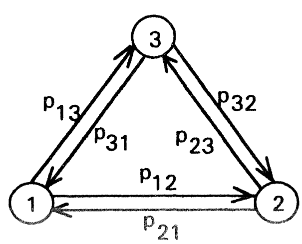

Fig.4.

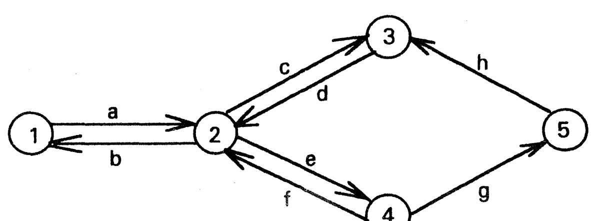

Fig.5.

References

[1] Bharucha-Reid, A.T., *Elements of the Theory of Markov Processes and their Applications*, New York, J.Willey and Sons, 1960.

[2] Ore O., *Theory of graphs*, American Mathematical Society, Rhode Island, 1962.

Address: Dept. of Computer Science, Faculty of Electrical Engineering, Technical University of Ostrava, 17.listopadu, 708 33 Ostrava, Czech Republic

(*Received* May 28, 1993)