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Division Space Axiomatics

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Abstract. The abstract Henstock integral is defined for real functions on a division space (Henstock [2], Muldowney [5], Ostaszewski [6]). We consider two sets of axioms for a division space, one due to Henstock, and another due to Thomson [7], [8]. We show and discuss their equivalence.

Keywords: Henstock integral, division space, derivation base

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Introduction. Kurzweil [4], and independently Henstock [1], introduced a generalization of the classical Riemann integral which is more powerful than the Lebesque integral, yet has all of its desired properties, especially with respect to limit theorems.

Their idea has been extended to an abstract setting — to what we will refer to as the *Henstock integral* of a real-valued function defined on a division space . We concentrate on the definition of that concept here.

1. Definition. Let X be a nonempty set, and Ψ a nonempty class of its subsets such that if I, $J \in \Psi$ then $I \cap J \in \Psi$, and $I \setminus J$ can be written as a finite union of elements of Ψ (this means that Ψ is a *set-theoretic semiring* in the sense of Kolmogoroff [3]). We will call the elements of Ψ *cells* (alternatively, they can be called *intervals*, as in Henstock [2], Muldowney [5], Ostaszewski [6], or Thomson [7], [8]). We also assume that $X = \bigcup_{I \in \Psi} I$.

Let Ψ_S denote the class of finite unions of elements of Ψ. Elements of Ψ_S are called *elementary sets.* If $E \in \Psi_S$ and $E = I_1 \cup I_2 \cup \ldots I_n$, where $I_1, I_2, \ldots, I_n$ are pairwise disjoint, $n \in \mathbf{N}$, then $\{I_1, I_2, \ldots, I_n\}$ is called a *partition* of E. Let $\{(x_i, I_1), (x_2, I_2), \ldots, (x_n, I_n)\}$, $n \in \mathbf{N}$, be a finite collection of pairs $(x_i, I_i) \in X \times \Psi$, $i = 1, 2, \ldots, n$, such that $\{I_1, I_2, \ldots, I_n\}$ is a partition of $E = I_1 \cup I_2 \cup \ldots \cup I_n$. Then $\{(x_1, I_1), (x_2, I_2), \ldots, (x_n, I_n)\}$ is called a *division* of E. If S is a subset of $X \times \Psi$ which contains a divison of an $E \in \Psi_S$, we say that S divides E.

If $S \subset X \times \Psi$, and $E \subset X$, we write

$$S(E) = \{(x, I) \in S : I \subset E\}$$

and

$$S(E) = \{(x, I) \in S : x \in E\}.$$

2. Definition. (Henstock [2]) A nonempty class Δ_H of subset of $X \times \Psi$, $\Delta_H \neq$

{0}, is called a *derivation base in the sence of Henstock* if it satisfies the following conditions:

(i) For every *E* G $5 there exists an S G A// such that S divides *E;*

(ii) If Si, S2 € A//, Si and S2 divide an *E* G ^5, then there exists an S3 such that S3 C Si fl S2 and S3 divides *E;*

(iii) If Ei, E_2 G *$s* are disjoint, and an S G A// divides *EiUE*$_2$ then S(f?i) € A// and *S(E\)* divides *E*

(iv) If JE7I, JE?2 € \ťs are disjoint, Si divides JE7i, Si divides ^2? then there exists an S € A// dividing £1 U E_2 with S C Si H S_2.

A base which satisfies (i),(ii), and (iii), but not (iv), is called a *nonadditive derivation base in the sense of Henstock.*

The definitions and notation follow generally, but not exactly, those of Henstock [2], and Muldowney [5]. DifFerences are an attemp to reconcile the above works with Thomson [7], [8], and Ostaszewski [6].

3. Definition. (Thomson [7], [8]) A nonempty class *AT* of subset of X x \ř, *AT* ^ {0}, is called a *derivation base in the sense of Thomson* if it satisfies the following two conditions:

(i) For every *E* G #s and S G A_T, S divides *E;*

(ii) If Si, S^ G Ar then there exists an S3 G AT such that S3 C Si fl S2.

If in addition to (i) and (ii) Ar satisfies the condition:

(iii) If JSi, E_2 G \ťs are disjoint, Si, S2 G Ar such that

$$S_3(ÍE;IU_JE;3)CSI(J5;_1)US_2(E2);$$

then Ar is called an *additive* derivation base in the sense of Thomson.

4. Definition. If A# is a derivation base in the sense of Henstock, the triple (X, ^, A//) is called a *division space.* If A// is the derivation base in the sense of Thomson, we will call the triple (X, \ř, *AT)* a *division space in the sense of Thomson.*

5. Definition. If *F* : X x $ —• R then its *lower,* and *upper limits* (respectively) at a point *xo* G X are defined as

$$(A) \liminf_{I\text{-}**o} F(x, I) = \sup_{S€A(*o,} \inf_{I)€S[\{ZO\}]} F(x_0, I),$$

$$(A) \limsup_{/\text{-}*o} F(:r, I) = \inf_{S€A(xo,} \sup_{i)es[\{x_0\}]} F(x_0, I).$$

The above definition does not extend to the degenerate case of $S \in \Delta$ for which $S[\{x_0\}] = \emptyset$, but that case is, generally speaking, not considered. Δ denotes a base in the sense of Henstock or in the sense of Thomson.

If both the upper and lower limits are equal, their commo value is reffered to as the *limit* of F at x_0, and denoted by $(\Delta)\lim_{I\to x_0} F(x, I)$,

6. Definition. (Henstock [2], Muldowney[5],Thomson [7], [8])

If $F : X \times \Psi \to \mathbf{R}$ and $G : X \times \Psi \to \mathbf{R}$, then the *upper*, *lower*, and *ordinary* (respectively) *derivates* of F with respect to G at $x_0 \in X$, with respect to the base Δ, are defined as:

$$\overline{D}_\Delta F_G(x_0) = (\Delta)\limsup_{I\to x_0} \frac{F(x, I)}{G(x, I)},$$

$$\underline{D}_\Delta F_G(x_0) = (\Delta)\liminf_{I\to x_0} \frac{F(x, I)}{G(x, I)},$$

$$D_\Delta F_G(x_0) = (\Delta)\lim_{I\to x_0} \frac{F(x, I)}{G(x, I)}.$$

7. Definition. (Henstock [2], Muldowney [5], Thomson [7], [8])

Let $F : X \times \Psi \to \mathbf{R}$ and $E \in \Psi_S$. The *upper* and *lower* (respectively) *integrals* of F over E, with respect to Δ, are defined as:

$$(\Delta)\overline{\int_E} F = \inf_{S\in\Delta} \sup_{\pi\subset S} \sum_{(x,\, I)\in\pi} F(x, I),$$

where $\pi \subset S$ are divisions of E, for a base in the sense of Henstock we consider *only* those S which divide E, and

$$(\Delta)\underline{\int_E} F = \sup_{S\in\Delta} \inf_{\pi\subset S} \sum_{(x,\, I)\in\pi} F(x, I),$$

and again $\pi \subset S$ are divisions of E, only S dividing E are considered if the Henstock's concept applies.

If the upper integral equals the lower one, their common value is referred to as the *Henstock integral* of F over E, written as

$$(\Delta)\int_E F.$$

8. Definition. If Δ_H is a base in the sense of Henstock, we define (this definition is being introduced here) its *Thomson analogue* as the class Δ_T

$$\Delta_T = \Big\{ \bigcup_{E\in\Psi_S} \alpha_E : \alpha_E \text{ is the coordinate of } \alpha \in \prod_{E\in\Psi_S} \Delta_{H,\,E} \text{ corresponding to } E \in \Psi_S \Big\},$$

where

$$\Delta_{H,\,E} = \{S(E) : S \in \Delta_H,\, S \text{ divides } E\}$$

for $E \in \Psi_S$, and the products in (1) is the standard Cartesian product.

If Δ_T is a base in the sense of Thomson then we define its Henstock analogue as

$$\Delta_H = \{S(E) : S \in \Delta_T,\, E \in \Psi_S\}.$$

9. Theorem. *(a) If Δ_H is a class of subsets of $X \times \Psi$ which satisfies the axioms* 2(i), 2(ii), *and* 2(iii) *then its Thomson analogue is a derivation base in the sense of Thomson.*

(b) *If Δ_H is a derivation base in the sense of Henstock then its Thomson analogue is an additive derivation base in the sense of Thomson.*

(c) *If Δ_T is a derivation base in the sense of Thomson then its Henstock analogue satisfies the axioms* 2(i), 2(ii) *and* 2(iii).

(d) *If Δ_T is an additive derivation base in the sense of Thomson then its Henstock analogue is a derivation base in the sense of Henstock.*

PROOF: (a) We will show that the axioms 2(i), 2(ii) and 2(iii) imply the axioms 3(i) and 3(ii). To show 3(i) note that if $S \in \Delta_T$, $E_0 \in \Psi_S$ then

$$S = \sum_{E\in\Psi_S} \alpha_E$$

for some

$$\alpha \in \prod_{E\in\Psi_S} \Delta_{H,\,E},$$

and α_{E_0} divides E_0 because $\alpha_{E_0} \in \Delta_{H,\,E_0}$; i.e., $\alpha_{E_0} = S(E_0)$ for some $S \in \Delta_H$ which divides E_0.

Now consider 3(ii). Let

$$S_1 = \sum_{E\in\Psi_S} \alpha'_E,\; S_2 = \sum_{E\in\Psi_S} \alpha''_E.$$

Then for each $E \in \Psi_S$

$$\alpha'_E = S'(E),\; \alpha''_E = S''(E),$$

where S'_E, $S''_E \in \Delta_H$ and both S'_E, S''_E divide E. By 1.2(ii) there exists an $S_E \in \Delta_H$ which divides E and such that $S_E \subset S'_E \cap S''_E$. Let

$$\alpha_E = S(E).$$

Define

$$S_3 = \bigcup_{E \in \Psi_S} \alpha_E.$$

Then $S_3 \in \Delta_T$ and $S_3 \subset S'_E \cap S''_E$, as desired.

(b) We will show that Δ_T, the Thomson analogue of Δ_H, satisfies the axiom 1.3(iii). Let E_1,E_2 $\in$ Ψ_S be disjoint, S_1,S_2 $\in$ Δ_T. Let $E \subset E_1 \cup E_2$ be an arbitrary elementary set. Define $E' = E \cap E_1$,$E'' = E \cap E_2$. Using the definition of $S_1 = \bigcup_{E \in \Psi_S} \alpha'$, $S_2 = \bigcup_{E \in \Psi_S} \alpha''$, choose $\alpha'_{E'}$, $\alpha''_{E''}$. Note that $\alpha'_{E'}$, $\alpha''_{E''} \in \Delta_H$ by 1.2(iii). Define $S' = \alpha'_{E'}$,$S'' = \alpha''_{E''}$. By 1.2(iv) there is an $S \in \Delta_H$ such that S divides $E' \cup E''$ and $S \subset S' \cup S''$. Let $\beta_E = S$ and for E not contained in $E_1 \cup E_2$ define β_E arbitrarily. Set $S_3 = \bigcup_{E \in \Psi_S} \beta_E$. Then $S_3 \in \Delta_T$ and

$$S_3(E_1 \cup E_3) \subset S_1(E_1) \cup S_2(E_2),$$

as desired.

(c) To prove that Δ_H satisfies 2(i) it suffices to note that every $S \in \Delta_T$ divides every $E \in \Psi_S$. Therefore, if $E \in \Psi_S$ then for $S \in \Delta_T$,$S(E) \in \Delta_H$ (the Henstock analogue),and $S(E)$ divides E.

To prove 2(ii) consider S_1,S_2 $\in$ Δ_H, both dividing an $E \in \Psi_S$. Then $S_1 = S'(E_1)$ for some $S' \in \Delta_T$ and $E_1 \in \Psi_S$. But since S_1 divides E, E must be a subset of E_1. Similarly, $S_2 = S''(E_2)$ with $S'' \in \Delta_T$, $E_2 \in \Psi_S$, $E \subset E_2$. By 3(ii) there exists an $S''' \in \Delta_T$ such that $S''' \subset S' \cap S''$. Let $S_3 = S'''(E)$. Then $S_3 \subset S_1 \cap S_2$ and S_3 divides E.

Now let us turn to 2(iii). This one is obvious. E_1, E_2 $\in$ Ψ_S are disjoint and $S \in \delta_H$ divides $E_1 \cup E_2$ then $S = S'(E)$ for some $S' \in \Delta_T$, $E \in \Psi_S$, $E \supset E_1 \cup E_2$. Then $S(E) = S'(E)(E_1) = S'(E_1) \in \Delta_H$ and $S(E_1)$ divides E_1 because S' does.

(d) We only need to show that the Henstock analogue Δ_H of an additive base Δ_T satisfies 2(iv). Let E_1, $E_2 \in \Psi_S$ be disjoint, and let $S_1 = S_1(E) \in \Delta_H$, where S_1 divides E_1, $S_2 = S_2(E_2) \in \Delta_H$, where S_2 divides E_2. By the definition of a Henstock analogue, we have $S_1 = S'(E_1)$, $S_2 = S''(E_2)$, with S', $S'' \in \Delta_T$. By 3(iii) there exists an $S''' \in \Delta_T$ such that

$$S'''(E_1 \cup E_2) \subset S'(E_1) \cup S''(E_2).$$

Let $S = S'''(E_1 \cup E_2)$. Then $S \in \Delta_H$, S divides $E_1 \cup E_2$, and

$$S \subset S'(E_1) \cup S''(E_2).$$

This completes the proof of part (d), and the entire Theorem 9. □

10. Definition. If Δ_H is a derivation base in the sense of Henstock such that

(v)(a) If $E \in \Psi_S$, $S_n \in \Delta_H$ and divides E for every $n \in \mathbf{N}$, and if $X_n \subset X$, $X_n \cap X_m = \emptyset$ for m,n $\in \mathbf{N}$, $m \neq n$, then there exists an $S \in \Delta_H$ dividing E such that

$$S[X_n] \subset S_n, \quad \text{for each } n \in \mathbf{N},$$

then (X , Ψ , Δ_H) is called a *decomposable division space in the sense of Henstock.*

If instead of (v)(a) we have

(v)(b) If $E \in \Psi_S$, $S_x \in \Delta_H$, where $x \in X$, and each S_x divides E, then there exists an $S \in \Delta_H$ dividing E such that

$$S[\{x\}] \subset S_x, \quad \text{for } x \in X,$$

then (X, Ψ, Δ_H) is called a *fully decomposable division space in the sense of Henstock.*

11. Definition. If Δ_T is a derivation base in the sense of Thomson, and

(v')(a) If $S_n \in \Delta_T$ for $n \in$N, $X_n \subset X$ with $X_n \cap X_m = \emptyset$ for $m \neq n, m$, $n \in$ N, then there exists an $S \in \Delta$ such that

$$S[X_n] \subset S_n, \quad \text{for } n \in \mathbf{N},$$

then (X, Ψ, Δ_T) is called a *decomposable division space in the sense of Thomson,* and $\Delta_T i$ is said to be of *σ-local character.*

If instead of (v')(a) Δ_T satisfied:

(v')(b) If $S_x \in \Delta_T$ for $x \in X$ then there exists an $S \in \Delta$ such that

$$S[\{x\}] \subset S_x, \quad \text{for } x \in X,$$

then (X, Ψ, Δ_T) is called a *fully decomposable division space in the sense of Thomson,* and Δ_T is said to be of *local character.*

12. Theorem. *(a) If (X, Ψ, Δ_H) is a decomposable division space in the sense of Henstock, and Δ_T is the Thomson analogue of Δ_H, then (X, Ψ, Δ_T) is a decomposable civision space in the sense of Thomson.*

(b) If (X, Ψ, Δ_H) is a fully decomposable division space in the sense of Henstock, and Δ_T is the Thomson analogue of Δ_H, then (X, Ψ, Δ_T) is a fully decomposable division space in the sense of Thomson.

(c) If (X, Ψ, Δ_T) is a decomposable division space in the sense of Thomson, and Δ_H in the Henstock analogue of Δ_T, then (X, Ψ, Δ_H) is a decomposable division space in the sense of Henstock.

(d) If (X, Ψ, Δ_T) is a fully decomposable division space in the sense of Thomson, and Δ_H is the Henstock analogue of Δ_T, then (X, Ψ, Δ_H) is a fully decomposable division space in the sense of Henstock.

PROOF: (a) Assume that $S_n \in \Delta_T$, $X_n \subset X$, with X_n 's pairwise disjoint, for $n \in$ N. Let $E \in \Psi_S$. Let

$$S_n = \bigcup_{E \in \Psi_S} \alpha_{E,\, n},$$

(choose $\alpha_{E,\, n}$ corresponding to that $E \in \Psi_S$ for each $n \in$ N). Because (X, Ψ, Δ_H) is decomposable, there exists an $\alpha_E \in \Delta_H$, dividing E, and such that

$$\alpha_E [X_n] \subset \alpha_{E,\, n}, \text{ for all } n \in \mathbf{N}.$$

Let $S = \cup_{E \in \Psi_S} \alpha_E$. Then for each $n \in \mathbf{N}$

$$S[X_n] = \bigcup_{E \in \Psi_S} \alpha_E[X_n] \subset \bigcup_{E \in \Psi_S} \alpha_{E,\, n} = S_n,$$

and this proves (a).

(b) Let $S_x \in \Delta_T$ for $x \in X$. For any $E \in \Psi_S$, since $S_x = \cup_{E \in \Psi_S} \alpha_{E,\, x}$, we can choose an $\alpha_{E,\, x} \in \Delta_H$ corresponding to that E, for every $x \in X$. Because (X, Ψ, Δ_H) is fully decomposable, there exists an $\alpha_E \in \Delta_H$, dividing E, and such that

$$\alpha_E[\{x\}] \subset \alpha_{E,\, x} \text{ for every } x \in X.$$

Define $S = \cup_{E \in \Psi_S} \alpha_E \in \Delta_T$. Then, for each $x \in X$,

$$S[\{x\}] = \bigcup_{E \in \Psi_S} \alpha_E[\{x\}] \subset \bigcup_{E \in \Psi_S} \alpha_{E,\, x} = S_x,$$

which proves (b).

(c) Let $E \in \Psi_S$, $S_n \in \Delta_H$, for $n \in \mathbf{N}$, and S_n divides E. Let $\{X_n\}$ be a sequence of disjoint subsets of X. Each S_n equals $S_{T,\, n}(E_n)$ for some $S_{T,\, n} \in \Delta_T$ and $E_n \in \Psi_S$. Since $S_n = S_{T,\, n}(E_n)$ divides E, E must be contained in E_n. Because (X, Ψ, Δ_T) is decomposable there exists an $S_T \in \Delta_T$ such that

$$S_T[X_n] \subset S_{T,\, n} \text{ for each } n \in \mathbf{N}.$$

Define $S = S_T(E)$. Then S divides E, $S \in \Delta_h$, and

$$\begin{aligned} S[X_n] &= S_T(E)[X_n] \subset S_T(E_n)[X_n] = \\ &= S_T[X_n](E_n) \subset S_{T,\, n}(E_n) = S_n, \end{aligned}$$

as desired.

(d) Let $E \in \Psi_S$, $S_x \in \Delta_h$ for $x \in X$ with each S_x dividing E. Every S_x equals some $S_{T,\, x}(E_x)$ where $S_{T,\, x} \in \Delta_T$ and $E_x \in \Psi_S$. Since $S_{T,\, x}(E_x)$ divides E, E must be contained in each E_x. Because (X, Ψ, Δ_T) is fully decomposable, there exists $S_T \in \Delta_T$ such that

$$S_T[\{x\}] \subset S_{T,\, x}$$

for every $x \in X$. Define $S = S_T(E)$. Then $S \in \Delta_H$, and for each $x \in X$

$$\begin{aligned} S[\{x\}] &= S_T(E)[\{x\}] \subset S_T(E_x)[\{x\}] = \\ &= S_T[\{x\}](E_x) \subset S_{T,\, x}(E_x) = S_x. \end{aligned}$$

This ends the proof of Theorem 12. □

13. Remark. The main purpose of this work is to investigate whether the Henstock and the Thomson axiomatics of a division space are merely two different perspectives on basically the same concept. We have been quite successful until now in showing just that. But the goal as stated above may be overly optimistic, as we will show starting with the following proposition (which requires an additional assumption for the perfect correspondence to be preserved).

14. Proposition . *Let Δ_T be a derivation base in the sense of Thomson, Δ_{TH} be its Thomson analogue, and Δ_{THT} be the Thomson analogue of the Henstock analogue. Assume that Δ_T has the property that if for every $E \in \Psi_S$, $S_E \in \Delta_T$ is chosen, then*

$$\bigcup_{E \in \Psi_S} S_E(E) \in \Delta.$$

Then $\Delta_{THT} = \Delta_T$.

PROOF: Recall that

$$\Delta_{TH} = \{S(E)\ S \in \Delta_T,\ E \in \Psi_S\}.$$

If $E' \in \Psi_S$ then

$$\begin{aligned} \Delta_{TH,\ E'} &= \{S(E')\ S \in \Delta_{TH},\ S \text{ divides } E'\} = \\ &= \{S(E)(E')\ S \in \Delta_T,\ S(E) \text{ divides } E',\ E \in \Psi_S\} = \\ &= \{S(E)(E')\ S \in \Delta_T,\ E' \subset E,\ E \in \Psi_S\} = \\ &= \{S(E')\ S \in \Delta_T\}. \end{aligned}$$

If $\alpha_{E'} \in \Delta_{TH,\ E'}$ for each E' then $\alpha_{E'} = S_{E'}(E')$, $E' \in \Psi_S$, $S_{E'} \in \Delta_T$. Every $S \in \Delta_{THT}$ is of the form

$$S = \bigcup_{E' \in \Psi_S} \alpha_{E'},$$

and by the above assumption, $S \in \Delta_T$. Thus $\Delta_{THT} \subset \Delta_H$. It is obvious that $\Delta_T \subset \Delta_{THT}$, as any $S \in \Delta_T$ equals $\cup_{E \in \Psi_S} S\ (\ E\ )$. This ends the proof. □

15. Example. The additional condition required in Proposition 14 is necessary. To illustrate that, consider the Thomson base generating the classical Riemann integral on closed subintervals of $\mathbf{R}$ (see Thomson [7]):

$$S_\delta = \{(x,\ [a,\ b)) : x \in [a,\ b] \subset (x - \delta,\ x + \delta)\},$$

where δ is a positive real number, and

$$\Delta_T = \{S_\delta : \delta \in (0, +\infty)\}.$$

It is easy to see that Δ_T is a derivation base in the sense of Thomson. For

$$E = [a_1,\ b_1) \cup [a_2,\ b_2) \cup \ldots \cup [a_n,\ b_n) \in \Psi_S$$

let

$$\delta_E = \max(|a_1|,\ |b_1|,\ |a_2|,\ |b_2|,\ \ldots,\ |a_n|,\ |b_n|),$$

and $S_E = S_{\delta_E}$. Then

$$\bigcup_{E \in \Psi_S} S_E(E)$$

belongs to Δ_{THT} but cannot be expressed as an $S \in \Delta_T$.

16. Proposition. *Let Δ_H be a derivation base in the sense of Henstock which has the following property: if $E \in \Psi_S$, and for each $E' \subset E$, $E' \in \Psi_S$, we have some $S_{E'} = S_{E'}(E') \in \Delta_H$ such that $S_{E'}$ divides E', then $\cup_{E' \subset E} S_{E'} \subset \Delta_H$. Let Δ_{HT} be its Thomson analogue nad Δ_{HTH} be the Henstock analogue of the Thomson analogue. We have then the equality $\Delta_{HTH} = \Delta_H$.*

PROOF: By definition, $S \in \Delta_{HT}$ if

$$S = \bigcup_{E \in \Psi_S} \alpha_E,$$

where $\alpha_E = \alpha_E(E) \in \Delta_H$, and α_E divides E. Let $S' \in \Delta_{HTH}$ be arbitrary. Then $S' = S(E')$ for some $E' \in \Psi_S$ and $S \in \Delta_{HT}$. But then

$$S' = \bigcup_{E \in \Psi_S} \alpha_E(E') = \bigcup_{E \in \Psi_S} \alpha_E(E' \cap E).$$

By the assumption, $S' \in \Delta_H$. Thus $\Delta_{HTH} \subset \Delta_H$. Since the inclusion $\Delta_H \subset \Delta_{HTH}$ is clear, we have the desired equality. □

17. Example. The additional assumption given in Proposition 16 is necessary. To show that, consider again the class

$$\Psi = \{[a,\, b) : a \in \mathbf{R},\, b \in \mathbf{R},\, a < b\}$$

on $\mathbf{R}$ (same as in Example 15). For an $E \in \Psi_S$ let

$$S_\delta = \{(x,\, [a,\, b)) : x \in [a,\, b] \subset (x - \delta,\, x + \delta) \subset E\}$$

for $\delta > 0$. If for $E_0 = [-1,\, 1)$, $E' \subset E_0$ we define

$$\delta_{E'} = \sup_{x \in E'} |x|,$$

then

$$\bigcup_{E' \subset E_0} S_{\delta_{E'}} \neq S_\delta,$$

for any $\delta > 0$, where S_δ is defined on E_0. For this base, $Delta_H$ is a proper subset of Δ_{HTH}.

Theorem. *(i) Let Δ_H be a derivation base in the sense of Henstock, $I_0 \in \Psi$, and $F : X \times \Psi \to \mathbf{R}$. Let Δ_{HT} be the Thomson analogue of Δ_H. Then then Henstock integrals of F obtained via Δ_H and Δ_{HT} are identical; i.e.,*

$$(\Delta_H) \underline{\int}_{I_0} F = (\Delta_{HT}) \underline{\int}_{I_0} F,$$

$$(\Delta_H) \overline{\int}_{I_0} F = (\Delta_{HT}) \overline{\int}_{I_0} F,$$

$$(\Delta_H) \int_{I_0} F = (\Delta_{HT}) \int_{I_0} F \quad \textit{(if the integral exists).}$$

(ii) Let Δ_T be a derivation base in the sense of Henstock, $I_0 \in \Psi$, and $F : X \times \Psi \to \mathbf{R}$. Let Δ_{TH} be the Henstock analogue of Δ_H. Then the Henstock integrals of F obtained via Δ_T and Δ_{TH} are identical; i.e.,

$$(\Delta_T)\underline{\int}_{I_0} F = (\Delta_{TH})\underline{\int}_{I_0} F,$$
$$(\Delta_T)\overline{\int}_{I_0} F = (\Delta_{TH})\overline{\int}_{I_0} F,$$
$$(\Delta_T)\int_{I_0} F = (\Delta_{TH})\int_{I_0} F \quad \textit{(if the integral exists).}$$

PROOF: Note that in the Definition 7 (of the Henstock integral), both in the case of using a base in the sense of Henstock and in the sense of Thomson, only divisions of I_0 are used, and only $S \subset \Delta$ dividing the entire I_0 (in the Henstock case), or $S(I_0)$, where $S \in \Delta_T$ (in the Thomson case), are used. This observation gives the desired equivalences of definitions immediately. □

19. Theorem. *(i) Let Δ_H be a derivation base in the sense of Henstock, and let $x_0 \in X$. Assume that Δ_H has the following property: if $(x, I) \in S \in \Delta_H$ then for every $E \in \Psi_S$ such that $I \cap E \neq \emptyset$ there is an $S \in \Delta_H$ dividing E such that for $(z, J) \in S$, $z \neq x$. Then for any $F, G : X \times \Psi \to \mathbf{R}$ the definitions of a limit of F at x_0, and of the derivative of F with respect to G with respect to Δ_H and its Thomson analogue Δ_{HT} are equivalent; i.e.,*

$$(\Delta_H)\liminf_{I\to x_0} F(x, I) = (\Delta_{HT})\liminf_{I\to x_0} F(x, I);$$
$$(\Delta_H)\limsup_{I\to x_0} F(x, I) = (\Delta_{HT})\limsup_{I\to x_0} F(x, I);$$
$$(\Delta_H)\lim_{I\to x_0} F(x, I) = (\Delta_{HT})\lim_{I\to x_0} F(x, I) \quad \textit{(if the limit exists);}$$
$$\underline{D}_{\Delta_H} F_G(x_0) = \underline{D}_{\Delta_{HT}} F_G(x_0);$$
$$\overline{D}_{\Delta_H} F_G(x_0) = \overline{D}_{\Delta_{HT}} F_G(x_0);$$
$$D_{\Delta_H} F_G(x_0) = D_{\Delta_{HT}} F_G(x_0) \quad \textit{(if the derivative exists).}$$

(ii) Let Δ_T be a derivation base in the sense of Thomson, and let $x_0 \in X$. Assume that Δ_T has the following property: if $(x_0, I) \in S \in \Delta_T$, $I \subset E \in \Psi_S$ then there is an $S_1 \in \Delta_T$ such that for $S_1 \subset S$, $(x_0, I) \in S_1$, and every $(x_0, J) \in S_1$ has $J \subset E$. Then for any $F, G : X \times \Psi \to \mathbf{R}$ the definitions of a limit of F at x_0, and of the derivative of F with respect to G, with respect to Δ_T and its Henstock analogue Δ_{TH} are equivalent; i.e.,

$$(\Delta_T)\liminf_{I\to x_0} F(x, I) = (\Delta_{TH})\liminf_{I\to x_0} F(x, I);$$
$$(\Delta_T)\limsup_{I\to x_0} F(x, I) = (\Delta_{TH})\limsup_{I\to x_0} F(x, I);$$
$$(\Delta_T)\lim_{I\to x_0} F(x, I) = (\Delta_{TH})\lim_{I\to x_0} F(x, I) \quad \textit{(if the limit exists);}$$
$$\underline{D}_{\Delta_T} F_G(x_0) = \underline{D}_{\Delta_{TH}} F_G(x_0);$$
$$\overline{D}_{\Delta_T} F_G(x_0) = \overline{D}_{\Delta_{TH}} F_G(x_0);$$
$$D_{\Delta_T} F_G(x_0) = D_{\Delta_{TH}} F_G(x_0) \quad \textit{(if the derivative exists).}$$

PROOF: If sufficies to prove the parts concerning limits, as derivatives are defined as limits. It is perfectly clear that the lower and upper limits with respect to the Thomson analogue (in the case (i)), or with respect to the base in the sense of Thomson (in the case (ii)) bound the upper and lower limits with respect to the base in the sense of Henstock (case (i)), or the Henstock analogue (case (ii)).

To finish the proof of (i) we will show that

$$(\Delta_H)\liminf_{I\to x_0} F(x,\, I) \le (\Delta_{HT})\liminf_{I\to x_0} F(x,\, I)\,.$$

Let ε be an arbitrary positive number. Let $s \in \Delta_H$ be such that for every $(x_0,\, I) \in S$

$$F(x_0,\, I) \ge (\Delta_H)\liminf_{I\to x_0} F(x,\, I) - \varepsilon.$$

Choose an $(x_0,\, I) \in S$. Let $E \in \Psi_S$ be arbitrary. Define $E' = E \cap I$, $E'' = E \setminus I$ (note that E', $E'' \in \Psi_S$). Let $S' = S(E')$, and let S'' be an element of Δ_H dividing E'' such that for $(z,\, J) \in S''$, $z \neq x_0$. By 2(iv) there exists an $S''' \in \Delta_H$, dividing E, such that $S''' \subset S' \cup S''$. We can assume that $S''' = S'''(E)$. Let $\beta_E = S'''$, and

$$S_T = \bigcup_{E\in\Psi_S} \beta_E \in \Delta_{HT}.$$

Then for every $(x_0,\, I) \in S_T$

$$F(x_0,\, I) \ge (\Delta_H)\liminf_{I\to x_0} F(x,\, I) - \varepsilon.$$

Thus

$$(\Delta_{HT})\liminf_{I\to x_0} F(x,\, I)) \ge (\Delta_H)\liminf_{I\to x_0} F(x,\, I)\,.$$

This proves that

$$(\Delta_{HT})\liminf_{I\to x_0} F(x,\, I) = (\Delta_H)\liminf_{I\to x_0} F(x,\, I)\,.$$

The equality of upper limits is proved similarly, and the rest of (i) follows easily.

To prove (ii) we only need to show that

$$(\Delta_{TH})\liminf_{I\to x_0} F(x,\, I) \le (\Delta_T)\liminf_{I\to x_0} F(x,\, I)\,.$$

Again, let ε be an arbitrary positive number. Let $S \in \Delta_{TH}$, $S = S_T(E)$, where $E \in \Psi_S$, $S_T \in \Delta_T$ be such that for every $(x_0,\, I) \in S$ we have

$$F(x_0,\, I) \ge (\Delta_{TH})\liminf_{I\to x_0} F(x,\, I) - \varepsilon.$$

Let $(x_0,\, I) \in S = S_T(E)$. Note that $I \subset E$. There is an $S_1 \in \Delta_T$ such that $S_1 \subset S_T$, $(x_0,\, I) \in S_1$, and every $(x_0,\, J) \in S_1$ has $J \subset E$. If $(x_0,\, J) \in S_1$ then $(x_0,\, J) \in S_T(E)$, so that

$$F(x_0,\, J) \ge (\Delta_{TH})\liminf_{I\to x_0} F(x,\, I) - \varepsilon.$$

Consequently

$$(\Delta_T)\liminf_{I\to x_0} F(x, I) \geq (\Delta_H)\liminf_{I\to x_0} F(x, I).$$

This proves that

$$(\Delta_T)\liminf_{I\to x_0} F(x, I) = (\Delta_H)\liminf_{I\to x_0} F(x, I).$$

The inequality concerning the upper limits can be proved similarly, and the rest of (ii) follows. This ends the proof of Theorem 19. □

20. Remark. The additional assumptions of Theorem 19 are not as extraordinary as they might appear. They are, in fact, satisfied by additive derivation bases used for the development of the generalized Riemann integral on **R** and in the plane (see Ostaszewski [6] and Thomson [7]).

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