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Divisor Class Groups of Ordered Subgroups

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Abstract. We show that if a po-group G admits a theory of quasi-divisors (strong theory of quasi-divisors, respectively), then the factor po-group G/H has the same property if H is an o-ideal of G. We introduce a notion of a *divisor class group* C of an ordered subgroup G of an l-group Γ and we show some relationships between properties of C and conditions under which the inclusion $G \subseteq \Gamma$ is a strong theory of quasi-divisors. Finally, we present some examples of *po*-groups with a strong theory of quasi-divisors.

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1. Introduction

In [13] we introduced the notion of a *po*-group G which admits a strong theory of quasi-divisors and we investigated some relationships between the existence of this strong theory of quasi-divisors and the existence of some approximation theorems for t-valuations of G. Recall that a directed *po*-group $(G, ., \leq)$ has a *theory of quasi-divisors* if there exists an l-group $(\Gamma, .)$ and a map $h : G \longrightarrow \Gamma$ such that

(i) h is an order isomorphism from G into Γ.

(ii) $(\forall \alpha \in \Gamma_+)(\exists g_1, \dots, g_n \in G_+)\alpha = h(g_1) \wedge \dots \wedge h(g_n)$.

Moreover, we say that G has a *strong theory of quasi-divisors* if there exists an o-isomorphism h from G into an l-group Γ such that

(iii) $(\forall \alpha, \beta \in \Gamma_+)(\exists \gamma \in \Gamma_+)\alpha.\gamma \in h(G), \beta \wedge \gamma = 1$.

In the theory of *po*-groups with a theory of quasi-divisors (or, equivalently t-Prüfer *po*-groups, see [9]) an important role has a localization of an r-system. We recall very roughly this construction (see [3],[13]).

Let (G, x) be a *po*-group with an r-system x of a finite character (for the notion of an r-system see e.g. [9]). Let H be an o-ideal of G, i.e. H is a directed convex subgroup of G, and let $\varphi : G \longrightarrow G/H$ be a canonical homomorphism. Then for any lower bounded subset $\mathcal{A} \subseteq G/H$ we may find a lower bounded subset $A \subseteq G$ such that $\{aH : a \in A\} = \mathcal{A}$ (see [13]). Then we set $\mathcal{A}_{x_H} := A_x/H$. According to [13];2.1, x_H is an r-system on G/H of a finite character. We also introduced a notion of an x-local o-ideal, where H is such o-ideal if x_H is a local r-system, i.e. in $(G/H)_+$ there exists the unique maximal x_H-ideal.

In this paper we show at first that if G is a *po*-group with a theory of quasi-divisors (strong theory of quasi-divisors, respectively), then the same property has the factor *po*-group G/H, where H is an *o*-ideal. Then for a *po*-group G and its *o*-isomorphism h into an *l*-group Γ we introduce a notion of a divisor class group $\mathcal{C}_h$ of $h : G \longrightarrow \Gamma$ and we show some relationships between properties of $\mathcal{C}_h$ and conditions under which the inclusion $G \subseteq \Gamma$ is a strong theory of quasi-divisors. Finally, using these results we present a method for constructing examples of *po*-groups with strong theory of quasi-divisors by using the restricted Hahn group $H(\Delta, \mathbf{Z})$, where Δ is some root system.

2. Divisor class groups

We start this section with investigation of some structural properties of *po*-groups G and G/H, where H is an *o*-ideal of G. Recall that if G is a directed *po*-group, then a t-ideal generated by a lower directed subset $X \subseteq G$ is defined

$$X_t = \bigcup_{Y \subseteq X,\ Y \text{ finite}} Y_v \qquad \text{where } Y_v = \bigcap_{a \in G,\ Y \subseteq (a)_v} (a)_v$$

where $(a)_v = (a)_t = \{g \in G : g \geq a\}$.

Lemma 2.1. *Let H be an o-ideal of a directed po-group (G, x) with an r-system x of a finite character and let $\mathcal{P}_{x_H}$ be a proper x_H-ideal of $(G/H)_+$. Then the following statements are equivalent:*

(1) $\mathcal{P}_{x_H}$ is a prime x_H-ideal.

(2) There exists a prime x-ideal Q of G_+ such that $Q/H = \mathcal{P}_{x_H}$.

PROOF: (1)$\Longrightarrow$(2). Let P be a lower directed subset of G which represents $\mathcal{P}$. Then from a definition of a localization we have $\mathcal{P}_{x_H} = P_x/H$. Since $\mathcal{P}_{x_H}$ is a prime x_H-ideal, according to [13];2.4, we obtain that $\mathcal{H} = [(G/H_+ \setminus \mathcal{P}_{x_H}]$ is an x_h-local *o*-ideal of G/H, where $[X]$ is a subgroup generated by X. Then there exists an *o*-ideal T of G such that $\mathcal{H} = T/H$, $H \subseteq T$. In what follows we may identify $(G/H)/\mathcal{H}$ and G/T (under the map $(aH)\mathcal{H} \mapsto a.T$). Then $P_x \cap T = \emptyset$ as follows from the maximality of $\mathcal{P}_{x_H}/\mathcal{H} = P_x/T$. According to [9], there exists a prime x-ideal Q of G_+ such that $P_x \subseteq Q$, $Q \cap T = \emptyset$. We show that $\mathcal{P}_{x_H} = Q/H$. In fact, since Q/T is a proper $(x_H)_\mathcal{H}$-ideal in G/T, we have

$$\mathcal{P}_{x_H}/\mathcal{H} = P_x/T = (P_x/H)/\mathcal{H} \subseteq (Q/H)/\mathcal{H} \subseteq \mathcal{P}_{x_H}/\mathcal{H}$$

and $\mathcal{P}_{x_H}/\mathcal{H} = (Q/H)/\mathcal{H}$, hence $P_x/T = Q/T$. We show that $Q = G_+ \setminus T$. In fact, if $a \in T_+$, then from $a \in Q$ it follows that $T = aT \in Q/T = P_x/T$, a contradiction. Let $a \in G_+ \setminus Q$ and let us assume that $a \notin T$. Then $aT > T$ and since $P_x/T = Q/T$ is the unique maximal x_H-ideal in G/H, we have $aT \in Q/T$ (see [9]). Since T is an *o*-ideal, there exist $q \in Q, h_1, h_2 \in T_+$ such that $ah_1 = qh_2$. Since $ah_1 \geq q$, we have $ah_1 \in Q, h_1 \notin Q$ and it follows that $a \in Q$, a contradiction. Therefore, $\mathcal{P}_{x_H} = Q/H$.

(2)$\Longrightarrow$(1). Let Q be a prime x-ideal of G_+ such that $q/H = \mathcal{P}_{x_H}$. Then $Q \cap H = \emptyset$ and if $aH, bH \geq H$, $abH \in Q/H$, we have $(t_1.a).(t_2.b)h_1 = h_2p$

for some $t_i, h_i \in H_+$, $p \in Q$ and $at_1, bt_2 \geq 1$. Then since $h_1 \notin Q$, we have $(t_1a)(t_2b) \in Q$ and it follows that $aH \in \mathcal{P}_{x_H}$, or $bH \in \mathcal{P}_{x_H}$. □

In what follows, we denote by $\mathcal{H}_H(G,x)$ the set of all x-local o-ideals T of a po-group G with an r-system x such that $H \subseteq T$. If $H = \{1\}$, we write simply $\mathcal{H}(G,x)$.

Proposition 2.2. *Let (G,x) be a directed po-group with an r-system x of a finite character and let H_0 be an o-ideal of G. Then there exists a bijection φ between $\mathcal{H}_{H_0}(G,x)$ and $\mathcal{H}(G/H_0, x_{H_0})$ such that $G/H \cong (G/H_0)/\varphi(H)$ for any $H \in \mathcal{H}_{H_0}(G,x)$.*

PROOF: Let $\mathcal{T} \in \mathcal{H}(G/H_0, x_{H_0})$. Then $\mathcal{T} = T/H_0$, where T is an o-ideal of G, $H_0 \subseteq T$. We show that $T \in \mathcal{H}_{H_0}(G,x)$. According to [13]; 24, the set $(G/H)_+ \setminus \mathcal{T}$ is a prime x_H-ideal $\mathcal{P}_{x_H}$ in $(G/H_0)_+$. Then according to 2.1, there exists a prime x-ideal Q in G such that $Q/H_0 = \mathcal{P}_{x_H}$. Then $Q = G_+ \setminus T$ and according to [13], 2.4, and according to the proof of 2.1, we have $\varphi(\mathcal{T}) := T \in \mathcal{H}_{H_0}(G,x)$.

Hence, we defined a map $\varphi : \mathcal{H}(G/H_0, x_{H_0}) \longrightarrow \mathcal{H}_{H_0}(G,x)$. Conversely, if $H \in \mathcal{H}_{H_0}(G,x)$, then H/H_0 is an o-ideal of G/H. Again, $G_+ \setminus H = Q$ is a prime x-ideal of G and according to 2.1, Q/H_0 is a prime x_H-ideal of G/H_0. Then $(G/H_0)_+ \setminus Q/H_0 = (H/H_0)_+$ and it follows that H/H_0 is x_H-local. Hence, $\psi(H) = H/H_0$ is the inverse of φ. □

Proposition 2.3. *Let (G,x) be an x-Prüfer directed po-group such that x is of a finite character and let H be an o-ideal of G. Then G/H is a x_H-Prüfer po-group.*

PROOF: Let $\mathcal{T} \in \mathcal{H}(G/H, x_H)$. Then according to 2.2, there exists $T \in \mathcal{H}_H(G,x)$ such that $(G/H)/\mathcal{T} \cong_o G/T$. Then the proposition follows from [2];Th.8. □

Proposition 2.4. *Let G be a directed po-group with a theory of quasi-divisors and let H be an o-ideal of G. Then G/H has a theory of quasi-divisors.*

PROOF: Since G has a theory of quasi-divisors, it is a Prüfer t-group according to [2]. Then according to 2.3, G/H is a t_H-Prüfer group and since $t_H \leq t$ in G/H, then according to [2];Th.1, G/H is a t-Prüfer group as well. Hence, G/H has a theory of quasi-divisors. □

Proposition 2.5. *Let G be a directed po-group with a strong theory of quasi-divisors and let H be an o-ideal of G. Then G/H has a strong theory of quasi-divisors.*

PROOF: Let $h : G \longrightarrow \Gamma$ be a strong theory of quasi-divisors. Since h is a theory of quasi-divisors as well, (see [13]),Γ may be identified with the Lorenzen t-group $\Lambda_t(G)$ of G and we may assume that $h : G \longrightarrow \Lambda_t(G)$ is an inclusion $x \mapsto (x)_t$. If H is an o-ideal, then according to 2.4, G/H admits a theory od quasi-divisors which then may be identified with the inclusion $h_H : G/H \longrightarrow \Lambda_t(G/H)$. Since $t_H \leq t$ on G/H, then the composition φ of morphisms $(G,t) \longrightarrow (G/H, t_H) \longrightarrow (G/GH, t)$ is a (t,t)-morphism. Hence, according to [2];Th.1, there exists an l-epimorphism $\hat{\varphi}$ such that the diagram

$$\begin{array}{ccc} (G,t) & \xrightarrow{\quad h \quad} & \Lambda_i(G) \\ \Big\downarrow \varphi & & \Big\downarrow \hat{\varphi} \\ (G/H,t) & \xrightarrow{\quad h_H \quad} & \Lambda_i(G/H) \end{array}$$

commutes. The proposition then follows from the fact that h is a strong theory of quasi-divisors and $\hat{\varphi}$ is an l-epimorphism. □

Now, let G and Γ be ordered groups and let $h : G \longrightarrow \Gamma$ be an o-isomorphism from G into Γ. Then the factor group $\mathcal{C}_h = \Gamma/h(G)$ is called a *divisor class group of* h. We show at first that the construction of $\mathcal{C}_h$ has some functorial character.

Proposition 2.6. *Let G admits a theory of quasi-divisors $h : G \longrightarrow \Gamma$ and let H be an o-ideal of G. Let $h_H : G/H \longrightarrow \widehat{\Gamma}$ be a theory of quasi-divisors. Then there exists an o-epimorphism $\hat{\psi} : \Gamma \longrightarrow \widehat{\Gamma}$ and epimorphism $\sigma : \mathcal{C}_h \longrightarrow \mathcal{C}_{h_H}$ such that the diagram*

$$\begin{array}{ccccc} G & \xrightarrow{\quad h \quad} & \Gamma & \xrightarrow{\quad \varphi \quad} & \mathcal{C}_h \\ \Big\downarrow \psi & & \Big\downarrow \hat{\psi} & & \Big\downarrow \sigma \\ G/H & \xrightarrow{\quad h_H \quad} & \widehat{\Gamma} & \xrightarrow{\quad \hat{\varphi} \quad} & \mathcal{C}_{h_H} \end{array}$$

commutes.

PROOF: Since G admits a theory of quasi-divisors, G is a t-Prüfer group and we may identify Γ with the group of finitely generated t-ideals of G. Analogously, $\widehat{\Gamma}$ may be identified with the group of finitely generated t-ideals of G/H. Since the canonical map ψ is a (t, t_H)-morphism and $t_H \le t$, ψ is a (t,t)-morphism as well and according to [9];Th.1, there exists an o-epimorphism $\hat{\psi} : \Gamma \longrightarrow \widehat{\Gamma}$ such that $\hat{\psi}.h = h_H.\psi$. Let $\alpha = A_t \in \Gamma$, where A is a finite set in G. We set $\sigma(\varphi(A_t)) = \hat{\varphi}((\psi(A)_t)$. This definition is correct. In fact, let $A_t, B_t \in \Gamma$ be such that $\varphi(A_t) = \varphi(B_t)$. Then there exists $g \in G$ such that $A_t = (gB)_t$. Let f be a bijection between $\mathcal{H}(G/H, t_H)$ and $\mathcal{H}_H(G,t)$ (see 2.2). Then $(\psi(A))_{t_H} \cong A_t/f(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{H}(G/H, t_H)$. Hence,

$$(\psi(A)_{t_H})/\mathcal{T} \cong A_t/f(\mathcal{T}) = (gB)_t/f(\mathcal{T}) \cong (\psi(g).\psi(B))_{t_H}/\mathcal{T}.$$

Thus, according to [13];2.8, we obtain $(\psi(A))_t = \psi(g).(\psi(B))_t$.
The rest is obvious. □

It should be observed that

$$\ker\sigma = \{\varphi(A_t) \in \mathcal{C}_h : \text{ there exists } \inf_{G/H}(\psi(A))\}.$$

In fact, if $\varphi(A_t) \in \ker\sigma$, then $\sigma(\varphi(A_t)) = \hat{\varphi}((\psi(A))_t) = 0$. Then there exists $g \in G$ such that $(\psi(A))_t = (\psi(g))$ and it follows that $\psi(g) = \inf(\psi(A))$.

Lemma 2.7. *Let h be an o-isomorphism from a directed po-group G into an l-group Γ. Then the following statements are equivalent:*

(1) h is a theory of quasi-divisors.

(2) $(\forall\alpha \in \Gamma_+)\alpha = \inf_\Gamma(h(G) \cap (\alpha)_t)$.

PROOF: (1)$\Longrightarrow$(2). Let $\alpha \in \Gamma_+$. Since h is a theory of quasi-divisors, there exist $g_1, \ldots, g_n \in h(G) \cap (\alpha)_t$ such that $\alpha = h(g_1) \wedge \ldots \wedge h(g_n)$ in Γ. Let $\beta \in \Gamma$ be a lower boud of elements from $h(G) \cap (\alpha)_t$. Then $h(G) \cap (\alpha)_t \subseteq (\beta)_t$ and it follows that

$$(\alpha)_t = (h(g_1) \wedge \ldots \wedge h(g_n))_t = (h(g_1), \ldots, h(g_n))_t \subseteq (\beta)_t.$$

Hence, $\alpha \geq \beta$ and $\alpha = \inf(h(G) \cap (\alpha)_t)$.

(2)$\Longrightarrow$(1). Let $\alpha \in \Gamma_+$. Since $h(G) \cap (\alpha)_t$ is lower bounded, we have $(h(G) \cap (\alpha)_t)_t = (\alpha)_t$. Hence, $(h(G) \cap (\alpha)_t)_t$ is a t-invertible t-ideal and since t is an r-system of a finite character, $(h(G) \cap (\alpha)_t)_t$ is finitely generated and its generators could be coosen from the set $h(G) \cap (\alpha)_t$ (see [9]). Hence, there exist $h(g_1), \ldots, h(g_n) \in h(G) \cap (\alpha)_t$ such that $\alpha = h(g_1) \wedge \ldots \wedge h(g_n)$ and it follows that h is a theory of quasi-divisors. □

Proposition 2.8. *Let $h : G \longrightarrow \Gamma$ be a theory of quasi-divisors of a directed po-group G, let C_h be a divisor class group of h and let $\varphi : \Gamma \longrightarrow C_h$ be a canonical map. Then for any $\alpha \in \Gamma$, $\alpha > 1$, we have*

$$\varphi(\Gamma_+ \setminus (\alpha)_t) = C_h.$$

PROOF: Since h is a theory of quasi-divisors, for any $\alpha \in \Gamma$ there exists $\beta \in \Gamma_+$ such that $\varphi(\alpha) = \varphi(\beta)$. Hence, $C_h = \varphi(\Gamma_+)$. Let $\alpha \in \Gamma, \alpha > 1$ and let $\beta \in \Gamma_+$. Then there exists $\gamma \in \Gamma_+ \setminus (\alpha)_t$ such that $\beta.\gamma \in h(G)$. In fact, let us assume at first that α is incomparable with β or $\alpha > \beta$. Then $(h(G) \cap (\beta)_t) \setminus (h(G) \cap (\alpha)_t) \neq \emptyset$ as follows from 2.7. Let $h(g)$ be an element of this nonempty set. Then $h(g) = \gamma.\beta$, where $\gamma \geq 1$ and $\gamma \in \Gamma_+ \setminus (\alpha)_t$.

Let $\alpha \leq \beta$. Then $\alpha.\beta > \beta \geq \alpha$ and it follows that $(h(G) \cap (\alpha.\beta)_t) \subset (h(G) \cap (\beta)_t)$. Let $h(g) \in h(G) \cap (\beta)_t$ be such that $h(g) \notin (\alpha.\beta)_t$. Then $h(g) = \gamma_1.\beta$, where $\gamma_1 \geq 1$. If $\gamma_1 \geq \alpha$, then $h(g) = \beta.\gamma_1 \geq \beta.\alpha$, a contradiction. Hence, we proved that $\varphi(\Gamma_+) \subseteq \varphi(\Gamma_+ \setminus (\alpha)_t)$. □

Now, we say that an l-group Γ is *finitely atomic*, if for any element $\alpha \in \Gamma, \alpha > 1$, the set of all atoms $\sigma \in \Gamma_+$ such that $\sigma \leq \alpha$ is nonempty and finite. A trivial example of a finitely atomic l-group is a group $\mathbf{Z}^{(P)}$.

Theorem 2.9. *Let h be an o-isomorphism from a directed po-group G into an l-group Γ, let C_h be a divisor class group of h and let $\varphi : \Gamma \longrightarrow C_h$ be a canonical map. Let us consider the following statements:*

(1) h is a strong theory of quasi-divisors.

(2) If $\alpha_1, \ldots, \alpha_n$ are elements of Γ such that $\alpha_i > 1$ for all i, then $\varphi(\Gamma_+ \setminus \{\alpha_1, \ldots, \alpha_n\}_t) = \mathcal{C}_($.

(3) If $\alpha_1, \ldots, \alpha_n$ are atoms in Γ_+, then $\varphi(\Gamma_+ \setminus \{\alpha_1, \ldots, \alpha_n\}_t) = \mathcal{C}_h$.

Then (1)$\Longrightarrow$(2)$\Longrightarrow$(3). If Γ is finitely atomic, then all the statements are equivalent.

PROOF: (1)$\Longrightarrow$(2). Let $\alpha_1, \ldots, \alpha_n \in \Gamma, \alpha_i > 1$ for all i. Let $\varphi(\delta) \in \mathcal{C}_h$. Then there exists $\alpha \in \Gamma_+$ such that $\delta.\alpha \in h(G)$. Let $\beta = \alpha_1 \ldots \alpha_n$. Then there exists $\gamma \geq 1$ such that $\beta \wedge \gamma = 1$ and $\alpha.\gamma \in h(G)$. Hence, $\varphi(\alpha) + \varphi(\gamma) = 0 = \varphi(\delta) + \varphi(\alpha)$ and $\varphi(\gamma) = \varphi(\delta)$. If $\gamma \notin \bigcap_i(\Gamma_+ \setminus (\alpha_i)_t)$, then there exists i such that $\gamma \geq \alpha_i$. But, in this case we have $\gamma \wedge \beta \geq \alpha_i > 1$, a contradiction.

(2)$\Longrightarrow$(3). Trivial.

Now, let us assume that that Γ is finitely atomic and let (3) hold. Let $\alpha, \beta \in \Gamma_+, \alpha \notin h(G)$. Since $\mathcal{C}_h = \varphi(\Gamma_+)$, we have $-\varphi(\alpha) \in \varphi(\Gamma_+)$ and there exists $\delta \geq 1$ such that $-\varphi(\alpha) = \varphi(\delta)$. Hence, $\alpha.\delta \in h(G)$. Now, according to the assumption we have $\{\sigma : \sigma \text{ is an atom in } \Gamma_+, \sigma \leq \beta\} = \{\sigma_1, \ldots, \sigma_n\}$ and according to (3) we have $\varphi(\bigcap_i(\Gamma_+ \setminus (\sigma_i)_t)) = \mathcal{C}_h$. Then there exists $\gamma \in \bigcap_i(\Gamma_+ \setminus (\sigma_i)_t)$ such that $\varphi(\gamma) = \varphi(\delta)$. If $\gamma \wedge \beta > 1$ then there exists an atom σ such that $\sigma \leq \beta \wedge \gamma \leq \beta, \gamma$ and it follows that $\sigma = \sigma_i$ for some i, a contradiction with $\gamma \not\geq \sigma_i$. Hence, $\beta \wedge \gamma = 1$ and $\alpha.\gamma \in h(G)$. Therefore, h is a strong theory of quasi-divisors. □

3. Examples

In this part of the paper we should like to present a method for constructing examples of *po*-groups with a strong theory of quasi-divisors. This methos uis based on application of Theorem 2.9 onto a special l-group, the restricted Hahn group $H(\Delta, \mathbf{Z})$ and this method generalizes in some sense a method of constructing examples of groups with divisors theory presented by L. Skula [17].

Recall that if Δ is a *root system* (i.e. $(\Delta, \leq)$ is a partly ordered set for which $\{\alpha \in \Delta : \alpha \geq \gamma\}$ is totally ordered for any $\gamma \in \Delta$), then the *restricted Hahn group* $H(\Delta, \mathbf{Z})$ on Δ is the group $\mathbf{Z}^{(\Delta)}$ ordered in a following way:

$$a \in H(\Delta, \mathbf{Z}), a \geq 0 \Leftrightarrow a_\alpha > 0 \text{ for all } \alpha \in \mathrm{ms}(a),$$

where ms(a) is the *maximal support of* a, i.e. the set of all maximal elements in $\mathrm{supp}(a) = \{\alpha \in \Delta : a_\alpha \neq 0\}$. Then $H(\Delta, \mathbf{Z})$ is an l-group (see e.g. [2]).

Now, let Δ_0 be the set of all minimal elements of Δ. We say that Δ is *atomic* if for any element $\alpha \in \Delta$ there exists $\beta \in \Delta_0$ such that $\alpha \geq \beta$. Moreover, we say that Δ is *finitely atomic* if for any $\alpha \in \Delta$, the set $\{\sigma \in \Delta_0 : \sigma \leq \alpha\}$ is nonempty and finite. Finally, let $\alpha \in \Delta$. Then by a^α we denote the element of $H(\Delta, \mathbf{Z})$ such that

$$a^\alpha_\beta = \begin{cases} 1, & \text{if } \beta = \alpha \\ 0, & \text{otherwise.} \end{cases}$$

In the following lemma we summarize some properties of $H(\Delta, \mathbf{Z})$ which would be of interest for our examples of groups with a strong theory of quasi-divisors.

Lemma 3.1. *Let Δ be a root system.*

(1) Let Δ be atomic and let $\alpha \in \Delta_0, b \in H(\Delta, \mathbf{Z})_+$. Then $b \geq a^\alpha$ if and only if there exists $\beta \in ms(b)$ such that $\beta \geq \alpha$.

(2) If Δ is atomic, then $a \in H(\Delta, \mathbf{Z})$ is an atom if and only if $a = a^\alpha$ for some $\alpha \in \Delta_0$.

(3) If Δ is finitely atomic, then $H(\Delta, \mathbf{Z})$ is finitely atomic.

PROOF: (1). Let $b \geq a^\alpha$ for some $\alpha \in \Delta_0$. If $b = a^\alpha$, then $\alpha \in \mathrm{ms}(b)$. Let $b > a^\alpha$. Then $\mathrm{supp}(b - a^\alpha) \setminus \{\alpha\} \subseteq \mathrm{supp}(b)$ and $\alpha \in \mathrm{supp}(b)$. In fact, if $b_\alpha = 0$, then there exists $\beta \in \mathrm{ms}(b - a^\alpha)$ such that $\alpha \leq \beta$. If $\alpha = \beta$ then $\alpha \in \mathrm{ms}(b - a^\alpha)$ and it follows that $-1 = (b = a^\alpha)_\alpha > 0$, a contradiction. Hence, $\alpha < \beta$ and $\beta \in \mathrm{supp}(b)$. Then there exists $\gamma \in \mathrm{ms}(b)$ such that $\alpha < \beta \leq \gamma$.

Conversely, let $\beta \in \mathrm{ms}(b)$ be such that $\beta \geq \alpha$. Let $\beta > \alpha$ firstly and let $\gamma \in \mathrm{ms}(b - a^\alpha)$. Let us consider the two only possible cases.

(a) $\gamma = \alpha$. Since $b_\beta > 0$ and $a^\alpha_\beta = 0$ we have $\beta \in \mathrm{supp}(b - a^\alpha)$, a contradiction with the maximality of γ.

(b) $\gamma \neq \alpha$. Then $\gamma \neq \beta$ and it follows that $\gamma \in \mathrm{ms}(b)$ as follows from the minimality of α. Then $b_\gamma - a^\alpha_\gamma = b_\gamma > 0$. Hence, if $\beta > \alpha$, we proved that $b \geq a^\alpha$.

Now,let $\beta = \alpha$ and let $\gamma \in \mathrm{ms}(b - a^\alpha)$. Let us consider again the two only possible cases.

(a)$\gamma = \alpha$. Since $b_\alpha - 1 \neq 0$ and $b_\alpha > 0$, we have $b_\alpha \geq 2$ and it follows that $(b - a^\alpha)_\alpha > 0$.

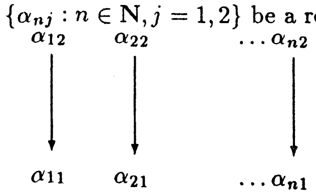
(b) $\gamma \neq \alpha = \beta$. Then from the minimality of α it follows that $\gamma \in \mathrm{ms}(b)$ and we have $(b - a^\alpha)_\gamma = b_\gamma > 0$. Therefore, $b \geq a^\alpha$ in this case.

(2) Let $\alpha \in \Delta_0$ and let us assume that $b \in H(\Delta, \mathbf{Z})$ be such that $a^\alpha \geq b > 0$. Then it may be proved easily that $\mathrm{ms}(a^\alpha - b) \subseteq \{\alpha\}$. Now, if $a^\alpha > b$, we have $\mathrm{ms}(a^\alpha - b) = \{\alpha\}$. Let $\beta \in \mathrm{supp}(b)$. Then it follows easily that $\beta \leq \alpha$. Thus, $\beta = \alpha$, a contradiction. Therefore, $a^\alpha = b$ and a^α is an atom. Conversely, let $b \in H(\Delta, \mathbf{Z})_+$ be an atom. Then $\mathrm{ms}(b) \neq \emptyset$ and for $\beta \in \mathrm{ms}(b)$ there exists an atom $\alpha \in \Delta_0$ such that $\alpha \leq \beta$. From (1) it follows that $b = a^\alpha$.

(3) Let $b \in H(\Delta, \mathbf{Z})_+$, $b > 0$. Then $\mathrm{ms}(b)$ is a finite set and according to (2) and (1), the set $\{a \in H(\Delta, \mathbf{Z})_+ : a \text{ is an atom and } a \leq b\}$ equals to the set $\{a^\alpha : \alpha \in \mathrm{ms}(b)\}$ which is nonempty and finite.
Hence, $H(\Delta, \mathbf{Z})$ is finitely atomic. □

Now, using the *l*-group $H(\Delta, \mathbf{Z})$, where Δ is a finitely atomic root system, we may derive examples of *po*-groups with a strong theory of quasi-divisors. Let us consider the following example.

Example 3.2. Let $\Delta = \{\alpha_{nj} : n \in \mathbf{N}, j = 1, 2\}$ be a root system such that



Let us consider a map $\varphi : H(\Delta, \mathbf{Z}) \longrightarrow \mathbf{Z}$ such that

$$\varphi(a) = \sum_{n\in\mathbf{N}, j=1,2} a_{\alpha_{nj}}.(-1)^n$$

Then φ is a group homomorphism and $H(\Delta, \mathbf{Z})$ is finitely atomic (see 3.1). Let $b_1, \dots, b_n$ be atoms in $H(\Delta, \mathbf{Z})_+$. Then $\varphi((\bigcap_{i=1}^n (H(\Delta, \mathbf{Z}) \setminus (b_i)_t)) = \mathbf{Z}$. In fact, according to 3.1, we may assume that

$$b_i(\alpha) = \begin{cases} 1, & \text{if } \alpha = \alpha_{i1} \\ 0, & \text{otherwise} \end{cases}$$

Let $m \in \mathbf{Z}$. If $m > 0$, then there exists $\alpha_{i1}, i > n$, and i is even. We set

$$a(\alpha) = \begin{cases} m, & \text{if } \alpha = \alpha_{i1} \\ 0, & \text{otherwise.} \end{cases}$$

Then according to 3.1, $a \in \bigcap_{i=1}^n (H)\Delta, \mathbf{Z})_+ \setminus (b_i)_t)$ and $\varphi(a) = m.(-1)^i = m$. If $m < 0$, then there exists α_{i1} such that $i > n$ and i is odd. We then set

$$a(\alpha) = \begin{cases} -m, & \text{if } \alpha = \alpha_{i1} \\ 0, & \text{otherwise.} \end{cases}$$

Then a is from ther same set as in previous case and $\varphi(a) = (-m).(-1)^i = m$. Hence,

$$\varphi(\bigcap_{i=1}^{n} (H(\Delta, \mathbf{Z})_+ \setminus (b_i)_t)) = \mathbf{Z}$$

and for the subgroup $G = \ker \varphi$ of $H(\Delta, \mathbf{Z})$ (with ordering induced from this group) the inclusion $G \hookrightarrow H(\Delta, \mathbf{Z})$ is a strong theory of quasi-divisors by 2.9.

This example may be modified in a following way.

Example 3.3. Let Δ be a finitely atomic root system such that card $(\Delta) = \aleph_0$ and let $\sigma : \Delta \longrightarrow \aleph_0$ be a bijection. Let $m \in \mathbf{Z}$ and let $\varphi_m : \mathbf{Z} \longrightarrow \mathbf{Z}/(m)$ be a canonical homomorphism. Then we may define a group homomorphism $\varphi : H(\Delta, \mathbf{Z}) \longrightarrow \mathbf{Z}/(m)$ such that

$$\varphi(a) = \sum_{\alpha\in\Delta} \varphi_m(a_\alpha).(-1)^{\sigma(\alpha)} \in \mathbf{Z}/(m).$$

Then $\mathbf{Z}/(m) = \varphi(\bigcap_{i=1}^n (H(\Delta, \mathbf{Z})_+ \setminus (b_i)_t))$ for any finite set $\{b_1, \dots, b_n\}$ of atoms in $H(\Delta, \mathbf{Z})$. In fact, according to 3.1, we may assume that there exist atoms $\alpha_1, \dots, \alpha_n$ in Δ such that

$$b_k(\alpha) = \begin{cases} 1, & \text{if } \alpha = \alpha_k \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi_m(s) \in \mathbf{Z}/(m)$. Then we may assume that $s \geq 0$ and then there exists $\alpha_0 \in \Delta_0 \setminus \{\alpha_1, \ldots, \alpha_n\}$ such that $\sigma(\alpha_0)$ is even. We then set

$$a(\alpha) = \begin{cases} s, & \text{if } \alpha = \alpha_0 \\ 0, & \text{otherwise}. \end{cases}$$

Then $a \in H(\Delta, \mathbf{Z})$ and according to 3.1, $a \not\geq b_k$, $k = 1, \ldots, n$. Moreover, $\varphi(a) = \varphi_m(a_{\alpha_0}).(-1)^{\sigma(\alpha_0)} = \varphi_m(s)$. Hence, $G = \ker\varphi \hookrightarrow H(\Delta, \mathbf{Z})$ is a strong theory of quasi-divisors.

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