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On the Notions of Characteristics and Type for Modules and their Applications

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Abstract. We introduce notions of characteristic and type for valuation rings and modules over such rings. These notions are exceedingly useful and fruitful. We use them here in connection to filter semigroups, as well as initiate studies on homogeneous and separable modules, from this point of view. Balanced exact sequences and balanced projective dimension are other notions we define and explore more fully elsewhere (see [5]). The paper also provides a basis for interesting results on kinds of dualities between Hom_R and $\otimes_R$ (as explored in [6]).

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Introduction

This paper is devoted to the study of torsion free modules over valuation domains and is intended as a reference to a number of results that rely on it. It is elementary in nature and it introduces indispensable notions of characteristics and type for modules which prove to be very powerful tools in studying modules and rings. We show how the notions are comfortably used in work on homogeneous and separable modules, filter semigroups, results on Hom_R and $\otimes_R$ and the balanced projective dimension.

For an R-module M and $a \in M$, the set

$$\chi_M(a) = \{r \in R : a = ta_1 \text{ for some } a_1 \in M\}$$

is called the characteristic of a in M. Characteristics $\chi_M(a)$ and $\chi_M(b)$ are equivalent if there is an $r \in R$ with either $\chi(r)\chi_M(b) = \chi_M(a)$ or $\chi(r)\chi_M(b) = \chi_M(b)$. The equivalence class of this relation is the type $t_M(a)$ of a. The Baerlike theorem holds: Two rank one modules are isomorphic if and only if they are of the same type and every type is the type of a rank one module. Note that the authors of [8] have constructed the notions of heights and indicators for modules over valuation domains, but that these notions have no natural application in torsion free modules over valuation rings. The correct notions to use are exactly those of characteristics and types introduced in the present paper.

The filter semigroup $\mathcal{F}(R)$ of a valuation ring R consists of subsets F of R with the property that $1 \in F$ and $\forall r \in F \forall s \in R$, $s \mid r$ implies $s \in F$, with the natural operation of multiplication of filters. The principal filter semigroup $\mathcal{F}^\circ(R)$ is a subsemigroup consisting of the principal filters $\chi_R(r) = \chi(r)$. The role of filter semigroups is rather important in the multiplicative structure of valuation rings. Among other uses, we point out the representation of general segmental semigroups via the principal filter semigroup.

A module is t-homogeneous if all its elements have the same type, or equivalently, if all $\langle a\rangle_*$ ($a \in M$) are mutually isomorphic. Of special interest are homogeneous modules that are separable at the same time. We introduce a notion of complete greatest common divisor ideals ($CGCD$- ideals) in order to study the products of homogeneous modules.

At the end we show how characteristics can be used to suitably define the notions of balanced exact sequences and balanced projective dimension and leave it for a deeper study to develope the notions further in [5].

1. $\sharp$ and $\flat$ Calculus

Throughout the paper we will assume that R is a commutative valuation ring (in fact a domain most of the time) with 1, not a field, and all R-modules will be unitary and torsion free, although, upon close inspection of definitions and proofs, one can see that some of the results remain valid for wider class of rings as well as modules. Since rank one torsion free modules are isomorphic either to Q — the quotient field of R or to an ideal of R, we will identify such modules with submodules of Q. Moreover, our ideals I of R would often include the cases $I = R$ or $I = Q$. The maximal ideal of R will be denoted by P and the monoid of units U. $N \leq_* M$ will mean that N is a pure submodule of M. $\mathcal{M}(\rho)$ will denote the category of ρ-modules over a commutative ring ρ. For unexplained terminology and theoretical background we refer to the standard reference on modules over valuation domains [8].

For an R-module M define

$$M^\flat = \{r \in R \mid rM = M\} \qquad \text{(read : "M flat").}$$

This is apparently a multiplicatively closed subset of R which contains units and it measures how near is a module to a divisible module. Closely related to this set is

$$M^\sharp = \{r \in R \mid rM < M\} = R \setminus M^\flat \qquad \text{(read : "M sharp")}$$

which is consequently a prime ideal of R (enabling localizations of R).

We state the following lemma (Lemma 4.5, p. 15, [8]), as we will use its statements, without a special reference :

Lemma 1.1. *If I and J are non-trivial ideals of R, then :*
(a) *$I^\sharp$ is a prime ideal containing I;*
(b) *$I^\sharp = I$ if and only if I is prime;*
(c) *$I \cong J$ implies $I^\sharp = J^\sharp$;*
(d) *$(I^\sharp)^\sharp = I^\sharp$.*

Note that last two parts of the previous lemma are true in a wider setting : if M, N are torsion free R-modules, then $M \cong N$ implies $M^\sharp = N^\sharp$ and $(M^\sharp)^\sharp = M^\sharp$ (or $(M^\sharp)^\flat = M^\flat$).

We also need a lemma relating the sharp of the Hom_R and $\otimes_R$ functors with the sharps of its components :

Lemma 1.2. *If M and N are torsion free modules, then :*
(a) $N^\sharp \cap M^\sharp \geq (\mathrm{Hom}_R(N, M))^\sharp$;
(b) $N^\sharp \cap M^\sharp \geq (N \otimes_R M)^\sharp$;
(c) *If* $\mathrm{Hom}_R(N, M) \cong M$ $(\cong N)$, *then* $M^\sharp \leq N^\sharp$ $(N^\sharp \leq M^\sharp)$;
(d) *If* $N \otimes_R M \cong M$, *then* $M^\sharp \leq N^\sharp$.

PROOF: In fact we will prove the corresponding inequalities for the flats:(a) if $r \in N\flat$ and $\alpha \in \mathrm{Hom}_R(N, ,M)$, then, for every $n \in N$ there is a unique $n_1 \in N$ such that $rn_1 = n$; define $\beta : N \longrightarrow M$ by $\beta(n) = \alpha(n_1)$. β is a homomorphism such that $r\beta = \alpha$ hence $r \in (\mathrm{Hom}_R(N, M))^\flat$. If $r \in M^\flat$ and $\alpha \in \mathrm{Hom}_R(N, M)$, then, for every $n \in N$ there is a unique $m \in M$ such that $\alpha(n) = rm$; $\beta : n \mapsto m$ is a homomorphism such that $r\beta = \alpha$, hence $r \in (\mathrm{Hom}_R(N, M))^\flat$;
(b) Trivial, as for every $r \in R$, $r(N \otimes_R M) = (rN) \otimes_R M$;
(c) By the comment after Lemma 1.1, $(\mathrm{Hom}_R(N, M))^\sharp = M^\sharp$ $(= N^\sharp)$, which in view of (a) gives the desired result;
(d) Same proof as for (c), with the aid of (b). □

Notice that the converse of, say (d), in the last lemma need not be true: If P is the maximal ideal of R, then $P^\sharp = R^\sharp = P$, but $P \otimes_R R \not\cong R$ for non-principal P.
It turns out that sharps and flats play significant role in determining homomorphism of rank one modules, as shown in

Proposition 1.3. (a) $\mathrm{End}_R I \cong R_{I^\sharp}$;
(b) $\mathrm{Aut}_R I \cong I^\flat \cup (I^\flat)^{-1}$;
(c) $\mathrm{End}_R I \cong \mathrm{End}_R J$ *if and only if* $I^\sharp = J^\sharp$;
(d) $(R_{I^\sharp})^\sharp = I^\sharp$.

PROOF: (a) and (b) are in [8][Proposition 4.6, p. 15] (see also [11][Theorem 2.3], (c) is in [4] [Proposition 3] and (d) follows from Lemma 1.2(a), for $M = N = I$, (d) of the same lemma and the first part of Remark 1.5 below. □

Only a couple more facts are needed in the sequel and we state them here :

Lemma 1.4. *Let M, J be torsion free R-modules, where J is of rank one. Then every element in $M \otimes_R J$ is of the form $a \otimes r$, for some $a \in M$, $r \in J$.*

PROOF: Every $u \in M \otimes_R J$ is of the form $u = \sum_{i=1}^n (a_i \otimes r_i$, for some $a_i \in M$, $r_i \in J$, $i = 1, \ldots, n$. We induct on n: for $n = 1$ there is nothing to prove. The essential step reduces to the case $n = 2$: if $u = a_1 \otimes r_1 + a_2 \otimes r_2$, then, without loss of generality, there is an $s \in R$ such that $r_2 = sr_1$ and consequently $u = (a_1 + sa_2) \otimes r_1$. □

Remark 1.5. We can therefore assume that tensoring with rank one modules is like multiplying by them, as $M \otimes_R I \cong MI$.Thus, we may identify $M \otimes_R I$ with the submodule MI of M, whenever convenience dictates so. We point out that an R-module M is also an $R_{M^\sharp}$-module, as $M = R_{M^\sharp} \cdot M \cong R_{M^\sharp} \otimes_R M$ (indeed,

$1 \in R_{M^\sharp}$, hence $M \leq R_{M^\sharp} \cdot M$; on the other hand if $m_0 = \frac{r}{s} m \in R_{M^\sharp} \cdot M$, $s \in M^\flat$, then $s \mid m$ and therefore $m_0 \in M$). Specially $R_{I^\sharp} \otimes_R R_{I^\sharp} \cong R_{I^\sharp}$ (use Proposition 1.3 (d))

2. Filters, characteristics, types

A subset F of a ring R is called a *filter of* R if $1 \in F$ and $\forall r \in F \forall s \in R$, $s \mid r$ implies $s \in F$. In case of valuation rings filters are linearly ordered. The minimal filter is obviously U — the multiplicative monoid of units, and the largest is $R \setminus 0$, but we will assume that $R = \chi(0)$ is a filter too (an "infinite" or zero filter). Notice that $0 \in F$ if and only if $F = R$. If M is an R-module, then $M^\flat$ is an R-filter.

Filters of a ring are as important as the ideals, for they are precisely the complements of ideals :

Lemma 2.1. (a) *If* $I \neq R$ *is an ideal of* R, *then* $F = R \setminus F$ *is a filter.*
(b) *If* F *is a filter of* R, *then* $I = R \setminus F$ *is an ideal.*
(c) *Multiplicatively closed filters are precisely the complements of prime ideals.*

A product of two filters F and G is defined to be $F * G = F \cdot G = \{fg : f \in F, g \in G$, for $0 \notin F \cdot G$ and $= R$, otherwise. In case R is a valuation ring, the product of two filters is again a filter. Indeed, assume $0 \notin FG$ and let $fg \in FG$. If $st = fg$, we need to show that $s \in FG$. If T is a unit, there is nothing to prove, so assume that is not a unit. If $s \mid f$, then we again get a clear case. Now assume $s = fs'$. Then $fs't = fg$ i.e. $f(s't - g) = 0$ and we infer that $s' \mid g$ (if, on the contrary, $s' = gs''$, then $fg(s''t - 1) = 0$ and since t is not a unit, $fg = 0$ — a contradiction). This concludes the proof.

Note that when R is a valuation domain, then $F * G = R$ if and only if $F = R$ or $G = R$; in general, there are genuine zero divisors.

Lemma 2.2. *If* R *is a domain and* I *its proper ideal, then* $\chi(r) * (R \setminus I) = R \setminus rI$, *for every* $r \in R$.

Proof: Let first $s \in \chi(r)$, i.e. $r = sr'$ and $t \in R \setminus I$; we need $st \in R \setminus rI$. If, on the contrary,$st = ri$, for some $i \in I$, then $s(t - r'i) = 0$. As $t \notin I$, $i = tw$, for some non- unit $w \in R$ and $st(1 - r'w) = 0$, which gives $st = 0$ — a contradiction since R is a domain. Conversely, if $t \in R \setminus rI$, then if $t = rp$, for some $p \notin I$, then $t \in \chi(r) * (R \setminus I)$; if $r = tp$, then $t \in \chi(r)$, and again $t \in \chi(r) * (R \setminus I)$. □

If we denote by $\mathcal{F}(R)$ the set of all filters of R, then we have the following :

Proposition 2.3. $(\mathcal{F}(R), *)$ *is a semigroup with zero* (R) *and the unity* (U). *We call it the filter semigroup of* R.

If M is an R-module and $a \in M$, then the set

$$\chi_M(a) = \{r \in R : a = ra_1 \text{ for some } a_1 \in M\}$$

is called the *characteristic of* a *in* M. When the index in the symbol for the characteristic is omitted, we assume that it is the largest ambient module (set) in question containing a. Characteristics of elements of R will be called the *principal filters semigroup* and denoted by $\mathcal{F}^{\circ}(R)$. The following result in effect states that $\mathcal{F}^{\circ}(R)$ is a subsemigroup of $\mathcal{F}^{\circ}(R)$:

Lemma 2.4. *For any r, $s \in R$, $\chi(r) * \chi(s) = \chi(rs)$.*

PROOF: The left inclusion is obvious, thus let us start with a $t \in \chi(rs)$, i.e. assume $rs = tx$, for some $x \in R$. If $t \in \chi(r)$, we are done, so assume that $t = rt'$, for some non-unit t'; then $r(s - t'x) = 0$. If $t' = sy$, for a non-unit $y \in R$, then $rs(1 - yx) = 0$, which implies that $\chi(rs) = R$. Also $0 = rs \in \chi(r) * \chi(s)$. □

The following list of properties of characteristics is easy to verify :

(1) $\chi_m(a) \cap \chi_M(b) \leq \chi_M(a + b)$;
(2) $\forall r \in R \chi_M(ra) = \chi(r)\chi_M(a)$;
(3) If $M = \oplus_{k\in K} A_k$ and $m = (a_k)_{k\in K} \in M$ (only finite number of a_k are non-zero), then

$$\chi_M(m) = \bigcap_{k\in K} \chi_{A_k}(a_k);$$

In fact the same formula is valid for the products.
(4) For every homomorphism $\alpha : A \to B$ and $a \in A$, $\chi_A(a) \leq \chi_B(\alpha a)$;
(5) $N \leq M$, $a \in N$ implies $\chi_N(a) \leq \chi_M(a)$;
(6) $N \leq_* M$ if and only if $\forall a \in N \chi_N(a) = \chi_M(a)$;
(7) If $N \leq M$, then $\chi_M(a) \leq \chi_{M/N}(a + N)$;
(8) If $a \in M$ and $N \leq_* M$, then $\chi_{M/N}(a + N) = \bigcup_{b\in a+N} \chi_M(b)$;
(9) $\chi_M(m) \cup \chi_N(n) \leq \chi_M(m) \cdot \chi_N(n) \leq \chi_{M\otimes_R N}(m \otimes_R n)$;
(10) If $a \in M$ and $i \in I$ such that $\chi_I(i) \leq \chi_M(a)$, then there is a homomorphism $\alpha : I \longrightarrow M$ such that $\alpha(i) = \alpha$;
(11) $M^b = \bigcap_{m\in M} \chi_M(m)$;
(12) If $f : A \longrightarrow B$ is an isomorphism such that $fa = b$, then $\chi_A(a) = \chi_B(b)$.

If F is a filter of R, we will denote by $\frac{R}{F}$ the R-submodule of Q generated by the set $\{\frac{1}{f} \mid f \in F\}$. It equals to $\{\frac{r}{f} \mid r \in R, f \in F\}$.
Strictly speaking, $\frac{R}{F}$ may be identified with the set $(R\times 1)\cup(1\times(F\cap P))$ of ordered pairs representing reduced fractions, with the operations defined appropriately. $\frac{R}{F}$ is not a ring in general, but when we need it we will localize R with respect to F which is the same as localizing it in the multiplicative closure $\overline{F}$ of F.

Every characteristic is a filter and rank one modules may be expressed via characteristics :

Lemma 2.5. (a) *Every R-filter F is equal to $\chi_{\frac{R}{F}}(1)$.*
(b) *For any $a \in M$, $\langle a\rangle_* = \frac{R_a}{\chi_M(a)}$.*

Another notion fundamental for the theory of modules is that of *type*. First we will say that R-filters F_1 and F_2 are equivalent if there is an $r \in R$ such that either $F_1 = \chi(r)F_2$ or $F_2 = \chi(r)F_1$. This definition is equivalent to the condition that there are r and s in R such that $\chi(r)F_1 = \chi(s)F_2$ and the relation $(\sim)$ is a congruence relation on $(\mathcal{F}(R), *)$. The equivalence classes of filters will be called *types*. For an R-module M, the *type of an element* $a \in M$ (denoted by $t_M(a)$) will be the equivalence class of $\chi_M(a)$. We will sometimes speak of a filter as a type meaning that the filter is a representative of the type in question. We would be in fact dealing with the *semigroups of types of* R : $\mathcal{T}(\mathcal{R}) = \mathcal{F}(\mathcal{R})/\sim$.

If all the elements of an R-module M have the same type, we will speak of the type of the module and denote it by tM. For instance, characteristics of all the elements in a torsion free rank one module belong to the same type.

The following theorem is fundamental and shows that there is one to one correspondence between the set of types and torsion free rank one modules.

Theorem 2.6. (a) *For rank one torsion free R-modules M and N, $tM = tN$ if and only if $M \cong N$.*
(b) *Every type $t = t(F)$ is the type of a rank one torsion free R-module.*

PROOF: (a) Let $a \in M$ and $b \in N$. Then $M = \langle a\rangle_*$ and $N = \langle b\rangle_*$. If $tM = tN$, then, w. l. o. g. $\chi_M(a) = \chi_N(rb)$, for an $r \in R$ and $M = \langle a\rangle_* = \frac{Ra}{\chi_M(a)} \cong \frac{R}{\chi_N(rb)} \cong \frac{Rrb}{\chi_N(rb)} = \langle rb\rangle_{*N} = \langle b\rangle_*$; this means that $M \cong N$. Conversely, if we assume the latter isomorphism, then, w. l. o. g. $\chi_M(a) = \chi_N(b)$, hence $tM = tN$.
(b) This is in fact consequence of Lemma 2.5(a). □

Corollary 2.7. *The number of non-isomorphic rank one torsion free R-modules is at most $2^{|R|}$. This bound is attainable just like the lower bound which equals 3.*

3. Filter semigroups

In this section we do not necessarily assume in advance that the valuation rings are domains. We will assume that all the (multiplicative) semigroups will have a 0, after adjoining it if necessary.

Every filter F has a representation in terms of principal filters : $F = \bigcup_{f\in F} \chi(f)$.

If $\alpha : R \longrightarrow$ is a ring homomorphism (injection, surjection), then $\overline{\alpha} : \mathcal{F}(R) \longrightarrow \mathcal{F}(S)$, defined via $\overline{\alpha}(F) = \bigcup_{f\in F} \chi_S(\alpha f)$ is a (filter) semigroup homomorphism (injection, surjection). The proof that $\overline{\alpha}(F * G) = (\overline{\alpha}F) * (\overline{\alpha}G)$ is done first for the case when F and G are principal filters and then with the aid of the fact that $F * G = \bigcup_{f\in F,\ g\in G} \chi(f) * \chi(g)$. Lemma 2.4 is instrumental in the proof.

The filter semigroups are rather useful in representation of so called general segmental semigroups. We first list a few intermediate results.

Lemma 3.1. *Let $(\mathcal{F}, *)$ be a semigroup and $\mathcal{I}$ its ideal. Define $(\mathcal{S}, \star, \infty)$ to be the set $(\mathcal{F} \setminus \mathcal{I}) \cup \{\infty\}$ with the operation*

$$F \star G = \begin{cases} F \star G, & \textit{if } F \star G \notin \mathcal{I}; \\ \infty, & \textit{if } F \star F \in \mathcal{I}. \end{cases}$$

Then the following hold:
1) $(S, \star, \infty)$ is a semigroup (with the zero element ∞) which is called the Rees quotient of $\mathcal{F}$ and is denoted by $\mathcal{F} \setminus \mathcal{I}$ (see [9]).
2) The mapping $q : \mathcal{F} \longrightarrow S$ defined by

$$qF = \begin{cases} F, & \textit{for } F \notin \mathcal{I}; \\ \infty, & \textit{for } F \in \mathcal{I} \end{cases}$$

is a homomorphism of semigroups such that $q\mathcal{I} = \infty$ and q is an epimorphism.
3) If $(S', *', \infty')$ is a semigroup with a zero element ∞', and $q' : \mathcal{F} \longrightarrow S'$ is a homomorphism of semigroups such that $q'\mathcal{I} = \infty'$, then there is a unique semigroup homomorphism $\alpha : S \longrightarrow S'$, such that $\alpha q = q'$ (and $\alpha\infty = \infty'$).

The proof offers no surprises.

Lemma 3.2. *Let $0 \longrightarrow I \longrightarrow D \xrightarrow{q} R \longrightarrow 0$ be an exact sequence of rings, where D is a valuation domain. Then, for all $d \in D$, $\chi_R(qd) = q(\chi_D(d))$.*

PROOF: It is sufficient to give a proof for $d \in D \setminus I$. Assume that $qd' \in q(\chi_D d)$; then for some t, $d = d't$ and $qd = qd'qt$, hence $qd' \in \chi_R(qd)$ and this proves $\chi_R(qd) \supseteq q(\chi_D d)$. For the reverse, let $r \mid qd$; $r = qd'$, as q is onto, thus $qd'qt = qd$. If $d = d't'$, then $qd' \in q(\chi_D d)$. If $d' = ds$ for a non-unit s, then since $d't - d \in I$, $dst - d = i \in I$ and $d(st-1) = i$. Since $st - 1$ is a unit, $d = i(st-1)^{-1} \in I$, which is contrary to our assumption, thus indeed $\chi_R(qd) \subseteq q(\chi_D d)$. □

Lemma 3.3. *If $0 \longrightarrow I \longrightarrow D \xrightarrow{q} R \longrightarrow 0$ is an exact sequence of valuation rings, where D is a domain, then $0 \longrightarrow \mathcal{F}^\circ(I) \longrightarrow \mathcal{F}^\circ(D) \xrightarrow{\overline{q}} \mathcal{F}^\circ(R) \longrightarrow 0$ is an exact sequence of semigroups, and $\mathcal{F}^\circ(R)$ is isomorphic to the Rees quotient $\mathcal{F}^\circ(D) \setminus \mathcal{F}^\circ(I)$.*

PROOF: Define $\phi : \mathcal{F}^\circ(D) \setminus \mathcal{F}^\circ(I) \longrightarrow \mathcal{F}^\circ(R)$ by $\phi(\infty) = R$ and $\phi(\chi_D(d)) = \overline{q}(\chi_D(d)) = \chi_R(qd)$ (this is made possible by Lemma 3.2). It is not difficult to check that ϕ is a well designed homomorphism which is injective and surjective. □

Lemma 3.4. *If $0 \longrightarrow I \longrightarrow D \xrightarrow{q} R \longrightarrow 0$ is an exact sequence of valuation rings, $0 \longrightarrow \mathcal{F}(I) \longrightarrow \mathcal{F}(D) \xrightarrow{\overline{q}} \mathcal{F}(R) \longrightarrow 0$ is an exact sequence of semigroups, where $\mathcal{F}(R)$ is isomorpic to the Rees quotient $\mathcal{F}(D) \setminus \mathcal{F}(I)$.*

PROOF: Define $\phi : \mathcal{F}(D) \setminus \mathcal{F}(I) \longrightarrow \mathcal{F}(R)$ with $\phi(\infty) = R$ and $\phi(F) = \overline{q}(F)$, for $F \in \mathcal{F}(D) - \mathcal{F}(I)$. ϕ is a homomorphism as we proved at the begining of this section that $\overline{q}$ is. $\overline{q}(F) = \bigcup_{f \in F} \chi_R(gf)$. Let $G \in \mathcal{F}(R)$. Then $G = \bigcup_{g \in G} \chi_R(g)$. For every $q \in Q$, there is an $f \in D$ with $qf = g$. Let $F = \bigcup_{f \in q^{-1}G} \chi_D(f) = q^{-1}G$. F is a filter of D and it is obvious that $\overline{q}F = G$. To prove injectivity of ϕ, assume that $\phi(F) = \phi(G)$. We have $\bigcup_{f \in F} \chi_R(qf) = \bigcup_{g \in G} \chi_R(qg)$ and we want $F = \bigcup_{f \in F} \chi_R(f) = \bigcup_{g \in G} \chi_R(g) = G$. Because of symmetry, it is sufficent to prove that $F \subseteq G$. Thus let $f \in F$. Then there is a $g \in G$ such that $qf \in \chi_R(qg)$. We infer that $f \in \chi_D(g) \subseteq G$, for if $f = gt$, for a non-unit t, then $qf = qgqt \mid qg$, thus $qg = qxqgqt$, for some x and $qg(1 - qxqt) = 0$. $1 - qxqt$ is a unit, because t is not, thus $qg = 0$ and $\phi(G) = \phi(F) = R$, hence $F = G$. □

Call a commutative semigroup S a *general segment* if, 1) S is weakly cancellative i.e. $\forall a, b \in S$, if $ab = b$ then $b = 0$ and 2) S is naturally totally ordered by the divisibility relation, i.e. $\forall a, b \in S$, either $a = bc$ or $b = ac'$, for some $c, c' \in S$, and if both relations hold, then $a = b$ (the last part can be omitted, however as it follows from weak cancellation). Note that a general segment does not contain a unity; this is the reason that we need a notion of a *1-general segment*: a commutative semigroup such that $S \setminus \{1\}$ is a general segment. We will follow here terminology and results of [2], if somewhat modified and generalized.

For example, $\mathcal{F}^\circ(R)$ is an 1-general segment, for a valuation ring R. For if $\chi(r), \chi(s) \in \mathcal{F}^\circ(R) \setminus U$ and $\chi(r) * \chi(s) = \chi(s)$, then by Lemma 2.4, $\chi(rs) = \chi(s)$. This implies $rst = s$, for some $t \in R$ and since r is not a unit, this implies $s = 0$, which establishes weak cancellation.

One of the consequences is that $\mathcal{F}^\circ : ValRings \longrightarrow 1 - GenSegm$ is a functor between categories of valuation rings and 1-general segments; it is defined on objects by $\mathcal{F}^\circ R = \mathcal{F}^\circ(R)$ and on mappings by $\mathcal{F}^\circ(\alpha : R \longrightarrow S) = \mathcal{F}^\circ(\alpha)$, where $\mathcal{F}^\circ(\alpha)(\chi_R(r)) = \chi_S(\alpha r)$. We also have the functor $\mathcal{F} : ValRings \longrightarrow FilterSemi$ between the categories of valuation rings and filter semigroups, where $\mathcal{F}R = \mathcal{F}(R)$ on objects and for mappings, $\mathcal{F}(\alpha : R \longrightarrow S) = \mathcal{F}(\alpha)$, where $\mathcal{F}(\alpha)(F) = \mathcal{F}(\alpha)(\bigcup_{f\in F} \chi_R(f)) = \bigcup_{f \in F} \chi_S(\alpha f)$.

Let S be a general segment and denore by Φ the free semigroup over $S \setminus 0$, i.e. the set of all finite words $a_1 \dots a_m$, $(a_i \in S \setminus 0)$ with the operation of concatenation. $\pi : \Phi \longrightarrow S$ will denote the homomorphism $\pi(a_1 \dots a_m) = a_1 \cdot \ldots \cdot a_m$. We will say that $\gamma \in \Phi$ is a refinement of $\alpha = a_1 \dots a_m \in \Phi$, if there are $\gamma_1, \dots, \gamma_m \in \Phi$, such that $\gamma = \gamma_1 \dots \gamma_m$ and $\pi(\gamma_i) = a_i$, for $i = 1, \dots, m$. We now define a relation $\sim$ on Φ by stipulating that $\alpha \sim \beta$, if α and β have a common refinement. $\sim$ is a cancellative congruence and for $T = \Phi/\sim$ and $T_0 = \{\tilde{\alpha} \in T : \pi(\alpha) = 0\}$, the following hold: a) T is commutative, naturally totally ordered, cancellative semigroup (without a unity); b) $\xi : S \longrightarrow T \setminus T_0$ (the Rees quotient), defined by $\xi(\tilde{\alpha}) = \pi\alpha$, if $\tilde{\alpha} \neq \infty$ and $= 0$, if $\tilde{\alpha} = \infty$, is an isomorphism of semigroups; c) $T - T_0$ generates T; d) T is uniquely determined in the sense that if T' is another semigroup with the properties a) and c) and an isomorphism $\phi : S - 0 \longrightarrow T' - T_0{}'$, then there is an isomorphism $T \longrightarrow T'$.

Theorem 3.5. *The following are equivalent for a semigroup S :*
1) S is a 1-general segment;
2) $S \cong \mathcal{F}^\circ(R)$, for a valuation ring R which is the quotient of a valuation domain;
3) $S \cong T \setminus T_0$ (the Rees quotient), where T is extended (i. e. $\infty \in T$) positive cone of a totally ordered abelian group G and T_0 is an ideal of T.

PROOF: (1)$\Longrightarrow$(3): Disregarding the trivial case $S = \{0, 1\}$, let $S' = S - \{1\}$. Use the construction above to get $T' = \Phi/\sim$. Then $S' \cong T' \setminus T_0{}'$ (Rees), where T' is commutative, cancellative naturally totally ordered semigroup, hence it can be embedded in its group of quotients $T' \hookrightarrow G$ (and identified with its image). Let $P = T' \cup \{1\} \subseteq G$. P can be viewed as a positive cone, thus making G into a naturally totally ordered abelian group. We define the extended positive cone of G to be $T = P \cup \{\infty\}$ and $T_0 = T_0{}' \cup \{\infty\}$, whence we have $S \cong T \setminus T_0$ (Rees).

(3)$\Longrightarrow$(2): By a known result of Krull, there is a valuation $v : Q \longrightarrow G \cup \{\infty\}$ and the valuation domain $V = \{x \in Q : v(x) \geq 0\}$ with the value group G. $I = v^{-1}(T_0)$ is an ideal and we set $R = V/I$. Claim that $\mathcal{F}^\circ(R) \cong T \setminus T_0 \cong S$. By Lemma 3.3, $\mathcal{F}^\circ(R) = \mathcal{F}^\circ(V/I) \cong \mathcal{F}^\circ(V) \backslash \mathcal{F}^\circ(I)$ (Rees) and it suffices to define $\phi : \mathcal{F}^\circ(V) \backslash \mathcal{F}^\circ(I) \longrightarrow T \setminus T_0)$; the rule is $\phi(\chi_V(r)) = v(r)$ and $\phi(\infty) = \infty$. ϕ is a homomorphism, because v is. ϕ is surjective, because $v : V \longrightarrow T$ is surjective. ϕ is injective because G is naturally totally ordered and hence v is injective.

(2)$\Longrightarrow$(1): This has been already proved. □

We note that a similar result has been established in [10].

4. Homogeneous and Separable Modules

An R-module M is homogeneous of type t if, for every $a \in M$, $t_M(a) = t$. By Theorem 2.5, we can interchangeably use the following equivalent definition: If $I \leq Q$, then an R-module M is called *I- homogeneous* or *homogeneous of type I* if, for every $a \in M$, the purification $\langle a \rangle_* \cong I$. Obviously $tI = t$. $\mathcal{H}(\mathcal{I})$ (or $\mathcal{H}(\sqcup)$, if $tI = t$) will denote the category of I-homogeneous (or t- homogeneous) R-modules.

A torsion free R-module M is said to be *separable*, if every finite set of elements of M is contained in a completely decomposable direct summand of M, or, equivalently, if for every $a \in M$, $\langle a \rangle_*$ is a direct summand of M.

Example 4.1. As the direct decomposition into rank one modules are unique for modules over valuation domains, $\oplus I$ is easily seen to be I-homogeneous, for an arbitrary ideal I; pure submodules of I- homogeneous modules are I-homogeneous. Completely decomposable modules are separable and in the case of *countable* rank these are the only separable modules. Pure and fully invariant submodules of separable modules are again separable (see [8],[Theorem 2.9, p. 276] and [4],[Lemma 8]).

Example 4.2. Any direct sum of I-homogeneous modules is I- homogeneous and direct sums of separable modules are separable. Indeed, let $M = \oplus M_k$ be an arbitrary direct sum. If $a = a_1 + \ldots + a_n \in M$, where $a_i \in M_i$ $(i = 1, \ldots, n)$, then $\langle a \rangle_* \leq_* \langle a_1 \rangle_{*M_1} \oplus \ldots \oplus \langle a_n \rangle_{*M_n} = C$, where C is completely decomposable pure submodule of M and, depending on whether all M_k are I-homogeneous or separable, either $C \cong \oplus I$ or C is a direct summand of M, which in either case proves the claim.

Example 4.3. If either $gl.d.R > 2$ or $pdQ > 1$, then there are separable not completely decomposable R-modules of any uncountable rank, namely pure submodules of free modules (see [7],[Theorem 9]). Moreover, under the same conditions, there are separable, $\aleph_1$ generated $\aleph_1$-free modules that are not completely decomposable ([3],[Lemma 7]).

If both $gl.d.R \leq 2$ and $pdQ = 1$, then the example in [7], p. 97 provides a separable not completely decomposable $\aleph_1$ generated, $\aleph_1$-free R-module.

Lemma 4.4. *If M is I-homogeneous, then, for all $a \in M$, $(\langle a \rangle_*)^\sharp = M^\sharp = I^\sharp$. The converse is not true, i. e. $(\langle a \rangle_*)^\sharp = M^\sharp = I^\sharp$ need not imply I-homogeneity of M.*

PROOF: As usual, we prove the corresponding equalities for flats: if $s \in (\langle a \rangle_*)^\flat$ and $m \in M$, then, by homogeneity, $s \in (\langle b \rangle_*)^\flat$, hence $m \in sM$ and $s \in M^\flat$.

Conversely, if $s \in M^b$ and $b \in \langle a\rangle_*$, then $b = sm$, for some $m \in M$ and thus $m \in \langle a\rangle_*$, therefore $s \in (\langle a\rangle_*)^b$. If $I = P$, $M = R$, then $M^\sharp = I^\sharp = P$, but M is not I-homogeneous for non-principal P. □

The lemma that follows describes pure rank one submodules of tensor products and Homs and is useful in studying homogeneity and separability of the functors $\otimes_R$ and Hom_R, the facts reflected in Proposition 4.6.

Lemma 4.5. *If I, M and N are torsion free modules, where I is of rank one, then*
(a) *$a \otimes b \in M \otimes_R N$, implies $\langle a \otimes b\rangle_* = \langle a\rangle_* \otimes_R \langle b\rangle_*$;*
(b) *$\alpha \in \mathrm{Hom}_R(I,\, M)$, implies $\langle \alpha\rangle_* = \mathrm{Hom}_R(I,\, \langle m_0\rangle_*)$, where $m_0 = \alpha i_0$, for some $i_0 \in I$.*

PROOF: (a) Clearly $\langle a\rangle_* \otimes_R \langle b\rangle_* \le \langle a \otimes b\rangle_*$. On the other hand, after tensoring by $\langle a\rangle_*$, the pure exact sequence $0 \longrightarrow \langle b\rangle_* \longrightarrow N$ becomes the pure exact sequence $0 \longrightarrow \langle a\rangle_* \otimes_R \langle b\rangle_* \longrightarrow \langle a\rangle_* \otimes_R N$. Similarly, tensoring $0 \longrightarrow \langle a\rangle_* \longrightarrow M$ we get the pure exact sequence $0 \longrightarrow \langle a\rangle_* \otimes_R N \longrightarrow M \otimes_R N$and we may infer that $\langle a\rangle_* \otimes_R \langle b\rangle_* \le_* M \otimes_R N$. On the other hand, $a \otimes b \in \langle a\rangle_* \otimes_R \langle b\rangle_*$ implies $\langle a \otimes b\rangle_* \le \langle a\rangle_* \otimes_R \langle b\rangle_*$ and we have the desired equality.

(b) If $\beta \in \mathrm{Hom}_R(I,\, \langle m_0\rangle_*)$, then $\beta_{i_0} \in \langle m_0\rangle_*$ i.e. either $s\alpha i_0 = \beta i_0$ or $\alpha i_0 = s\beta i_0$, for some $s \in R$, therefore either $\beta = s\alpha$ or $\alpha = s\beta$, and so $\beta \in \langle \alpha\rangle_*$, which proves $\mathrm{Hom}_R(I,\, \langle m_0\rangle_*) \le \langle \alpha\rangle_*$. On the other hand every $s\alpha : I \longrightarrow \langle m_0\rangle_*$, and if $\beta \in \langle \alpha\rangle_*$ is such that $\alpha = s\beta$, $\alpha i_0 = s\beta i_0$, then $\beta_{i_0} \in \langle m_0\rangle_*$, which proves that $\mathrm{Hom}_R(I,\, \langle m_0\rangle_*) \le \langle \alpha\rangle_*$. □

Proposition 4.6. *Let M, N, I, J be torsion free R-modules, where I and J are of rank one.*
(a) *If M is J-homogeneous, then $I \otimes_R M$ is IJ-homogeneous and $\mathrm{Hom}_R(I,\, M)$ is $\mathrm{Hom}_R(I,\, J) \cong (J : I)$-homogeneous;*
(b) *If M is separable, then $\mathrm{Hom}_R(I,\, M)$ is separable;*
(c) *If M and N are separable, then so is $M \otimes_R N$.*

PROOF: (a) This is a straightforward consequence of Lemma 4.5 (a).
(b) If $\alpha \in \mathrm{Hom}_R(I,\, M)$, where $\alpha i_0 = m_0$, $i_0 \in I$ then by Lemma 4.5(b), $\langle \alpha\rangle_* = \mathrm{Hom}_R(I,\, \langle m_0\rangle_*)$. $\mathrm{Hom}_R(I,\, M) = \mathrm{Hom}_R(I,\, M') \oplus \mathrm{Hom}_R(I,\, \langle m_0\rangle_* = \mathrm{Hom}_R(I,\, M') \oplus \langle \alpha\rangle_*$, because $M = M' \oplus \langle m_0\rangle_*$; thus $\mathrm{Hom}_R(I,\, M)$ is separable. (c) Let $x = m_1 \otimes n_1 + \ldots + m_t \otimes n_t \in M \otimes_R N$. There is a finite rank completely decomposable direct summand $B = \oplus\langle b_k\rangle_*$ of M, containing all the elements m_k, $k = 1, \ldots, t$ and a finite rank completely decomposable direct summand $C = \oplus\langle c_k\rangle_*$ of N, containing all the elements n_k, $k = 1, \ldots, t$. The module $B \otimes_R C = \oplus(\langle b_k\rangle_* \otimes_R \langle c_l\rangle_*)$ is a finite rank completely decomposable direct summand of $M \otimes_R N$, containing the element X, which is enough for separability. □

A rather challenging question are those of homogeneity and separability of the products of homogeneous and separable modules respectively. We initiate the answers by stating a result for a power of R and give more complete answers in [6] to the former.

To this end, let us define, for a given cardinal κ, an ideal I of R to be a κ-*greatest common divisor* ideal (or abbreviated : κ—GCD), if for every set $\{i_\alpha\}$ of elements of I, of cardinality κ, there is an $i_0 \in I$ such that

$$\bigcap_{\alpha\in\kappa} \chi_I(i_\alpha) = \chi_I(i_0).$$

If I is κ-GCD, for finite κ, we call it a GCD- ideal and if the property is satisfied for all κ, I will be called a *complete greatest common divisor* ideal (or a $CGCD$-ideal). If the ring itself satisfies the properties, it will be given the appropriate names. Notice that every Bezout domain is a GCD ring while every factorial domain is a $CGCD$ ring. GCD as well as $CGCD$ rings are completely integrally closed. It is strightforward to show that if R has the property that every κ-generated ideal is principal, then R is κ-GCD, hence the discrete valuation domains are $CGCD$-rings.

To justify the terminology and give it purpose we prove the following

Proposition 4.7. *The following statements are equivalent :*
(a) *R is a κ-GCD ring;*
(b) *Every subset $\{r_\alpha\}$ of R of cardinality κ, possesses a greatest common divisor $r \in R$ i. e. an element $r \in R$ such that $r \mid r_\alpha$ for all α and for any other elements $s \in R$ with this property, $s \mid r$.*
(c) *$\prod_{\alpha\in\kappa} R$ is an R-homogeneous module.*

PROOF: (a) implies (b): For a subset $\{r_\alpha\}$ of R, of cardinality κ, let $\bigcap_{\alpha\in\kappa}\chi(r_\alpha) = \chi(r)$, for some $r \in R$. Then, certainly, r divides all r_α. If an $s \in R$ also divides all r_α, then $s \in \chi(r)$ and hence r is the greatest common divisor of the given family.
(b) implies (a): If r is the greatest common divisor of the family $\{r_\alpha\}$, then $\bigcap_{\alpha\in\kappa}\chi(r_\alpha) = \chi(r)$. (a) implies (c): Let $a = (r_\alpha)_{\alpha\leq\kappa} \in \prod_{\alpha\leq\kappa}\chi(r_\alpha) = \chi(r)$. But $\chi(1)$ is equivalent to $\chi(r)$, for every $r \in R$, thus $ta = tR$, i. e. the product is R-homogeneous.
(c) implies (a): For $\{r_\alpha\}_{\alpha\leq\kappa} \subseteq R$, let $a = (r_\alpha)_{\alpha\leq\kappa} \in \prod_{\alpha\leq\kappa} R$. There exists an $r \in R$ such that either $\bigcap_{\alpha\leq\kappa}\chi(r_\alpha) = \chi(r)\chi(1)$ or $\chi(r)\bigcap_{\alpha\leq\kappa}\chi(r_\alpha) = \chi(1)$ (because of R-homogeneity, $ta = t1$). In the former case $\bigcap_{\alpha\leq\kappa}\chi(r_\alpha) = \chi(r)$ and in the latter $\bigcap_{\alpha\leq\kappa}\chi(r_\alpha) = \chi(1)$, thus the condition for a κ-GCD ring is satisfied in both alternatives. □

In fact the GCD and $CGCD$ rings have been invented before under the names "pseudo-bezoutien" and "pseudo-principal" domains via the characterization in part (b) of the previous proposition (refer the Exercise 21, p. 86 in [1]).

5. Balanced exact sequences

We would like to acknowledge a question posed by L. Fuchs; while looking for its answer the author was lead to the present definition of a balanced exact sequence. We recast some of the known results in the new light and leave more in depth applications for [5] and [6].

A pure exact sequence $E : 0 \longrightarrow A \xrightarrow{*} B \xrightarrow{q} C \longrightarrow 0$, of torsion free R-modules is called *balanced* if for every $c \in C$, there is a $b \in B$ such that $qb = c$ and $\chi_C(c) = \chi_B(b)$. We will also say that A is a *balanced submodule* of B (see also [8], p. 247).

It is always useful to have in hand several characterizations of a notion and this why we start with a

Proposition 5.1. *For the pure exact sequence E the following are equivalent:*
(a) *E is balanced;*
(b) *Every torsion free rank one module I has the projective property with respect to E;*
(c) *For every torsion free rank one module I the sequence $0 \longrightarrow \mathrm{Hom}_R(I, A) \longrightarrow \mathrm{Hom}_R(I, B) \xrightarrow{\overline{q}} \mathrm{Hom}_R(I, C) \longrightarrow 0$ is pure exact.*

PROOF: Assume (a), fix $i_0 \in I$ and let for $f : I \longrightarrow C$, $fi_0 = c$. Then there is a b with $qb = c$ and $\chi_B(b) = \chi_C(c)$. Define $g(i_0) = b$; g is well defined homomorphism for $\chi_I(i_0) \leq \chi_C(c) = \chi_B(b)$. Clearly $qgi_0 = qb = c = fi_0$, and thus (b) holds.
Assume (b). We need only point out that $\overline{q}$ is defined in a way as to coincide with the commuting triangle condition in (b) : $\overline{q}(g : I \longrightarrow B) = qg : I \longrightarrow C$. This is a pure exact sequence since E is such. In fact (b) and (c) are equivalent.
If (c) is satisfied, let $c \in C$ and set $I = \langle c \rangle_*$ with the inclusion $f \in \mathrm{Hom}_R(I, C)$. There is a $g \in \mathrm{Hom}_R(I, B)$ such that $f = qg$. Setting $b = g(c)$, we have the inequalities $\chi_C(c) \geq \chi_B(b) \geq \chi_C(c)$ i. e. (a). □

The set $\mathrm{Bext}_R(A, B)$ of all balanced exact sequences forms a submodule of $\mathrm{Ext}_R(A, B)$. *Balanced projectives* are modules with the projective property with respect to balanced exact sequences.

Proposition 5.2. *There are enough balanced projectives i. e. for every torsion free R-module M, there is a balanced exact sequence $C \longrightarrow M \longrightarrow 0$, where C is balanced projective. Balanced projectives are exactly the completely decomposable modules.*

PROOF: Let $E : 0 \longrightarrow A \xrightarrow{*} B \xrightarrow{q} C \longrightarrow 0$ be a balanced exact sequence. We prove first that completely decomposable modules $M = \oplus M_k$ (M_k rank one) are balanced projectives. Indeed, if $f : M \longrightarrow C$ is a homomorphism, then for the restrictions $f \mid M_k = f_k : M_k \longrightarrow C$, there are homomorphisms $\phi_k : M_k \longrightarrow B$ such that $q\phi_k = f_k$ (by Proposition 5.1(b)). If we set $\phi = \oplus\phi_k : \phi_k \oplus M_k \longrightarrow M$, then we get the wanted commutative triangle $q\phi = f$. In fact direct summands of balanced projectives are balanced projective, but in the case of modules over valuation domains, they are again completely decomposable.

For a given torsion free module M, define $C = \oplus_{a \in M} \langle a \rangle_*$ and let $q(\oplus a) = \sum a$ for $\oplus a \in M$. This defines an epimorphism onto M. If $a \in M$, its characteristics in M and in $\langle a \rangle_*$ are equal and this is sufficient for balanceness of the sequence $C \longrightarrow M \longrightarrow 0$.

If M is balanced projective and we use the balanced projective cover $C \xrightarrow{q} M \longrightarrow 0$, then the projective property for the identity homomorphism $id : M \longrightarrow M$ ensures the existence of a homomorphism $\phi : M \longrightarrow C$ with $q\phi = id$, which gives the splitting. Thus, M is a direct summand of a completely decomposable module and therefore is itself completely decomposable. $\square$

It makes sense to define the *balanced projective dimension* of a module as the length of its balanced projective resolution. We leave this however for a separate study [5]. Specific dualities between Hom_R and $\otimes_R$ are investigated in [6].

Remark. The first version of the paper and that of [6] date back prior to mid 1989, when they were presented at a conference on abelian groups, in Oberwolfach.

References

[1] Bourbaki, N., *Eléments de mathématique the, Algébre commutative*, Chapitre 7, 86, Hermann, 1965.

[2] Clifford, A. H., *Naturally totally ordered commutative semigroups*, Amer. J. Math **76** (1954), .631–646

[3] Dimitrić, R., *On chains of free modules over valuation domains*, Czechoslovak Math. J. **37(112)** (1987), 400–406.

[4] Dimitrić, R., *A note on isomorphism of endomorphism algebras of modules over valuation domains*, Arch. Math. **51** (1988), 419–424.

[5] Dimitrić, R., *Balanced projective dimension of modules over valuation domains*, Preprint, 1991.

[6] Dimitrić, R., *Dualities between* Hom_R *and* $\otimes_R$ *and their applications*, Kobe J. Math. **9** (1992), .

[7] Dimitrić, R., Fuchs, L., *On torsion-free modules over valuation domains*, Publ. Math. Debrecen **34** (1987), 91–97.

[8] Fuchs, L., Salce, L., *"Modules Over Valuation Domains"*, Lecture Notes in Pure and Applied Mathematics, 97, Marcel Dekker, Inc., New York and Basel, 1985.

[9] Rees, D., *On semi-groups*, Proc. Cambridge Phil. Soc. **36** (1940), 387–400.

[10] Shores, T. S., *On generalized valuation rings*, Mich. Math. J. **21** (1974), 405–409.

[11] Shores, T. S., Lewis, W. J., *Serial modules and endomorphism rings*, Duke Math. J. **41** (1974), 889–909.

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