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Projective Randers Change of *P-Finsler Spaces *

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Abstract

The purpose of the present paper to study a projective Randers change and *P-Finsler spaces, which are special Finsler spaces. The main result: Let $F^n = (M^n, L(x,y))$ and $\overline{F}^n = (M^n, L(x,y))$ be a *P-Finsler spaces. If there exists a projective Randers change between F^n and $\overline{F}^n$, then F^n is C-reducible if and only if $\overline{F}^n$ C-reducible, too.

Key words: Finsler space, projective Randers change, *P-Finsler space, C-reducible Finsler space, P-reducible Finsler space.

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1 Introduction

Let $F^n = \big(M^n, L(x,y)\big)$ be an n-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L(x,y)$, where $y^i = \dot{x}^i$ is the fundamental function defined on the manifold $TM\backslash 0$ of nonzero tangent vectors.

Definition 1 [6] A change of Finsler metric

$$F^n = \big(M^n, L(x,y)\big) \to \overline{F}^n = \big(M^n, \overline{L}(x,y)\big)$$

is called Randers change, if $\overline{L}(x,y) = L(x,y) + \rho(x,y)$, where $\rho(x,y) = \rho_i(x)y^i$ is a differential one-form on M^n.

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The notion of a Randers change has been proposed by M. Matsumoto, named by Hashiguchi–Ichijyo [1] and studied in detail Shibata [8].

The change is projective if and only if $\rho_i(x)$ is locally a gradient vector field.

Definition 2 [2] If a Finsler space satisfies the condition $P_{ijk} - \lambda C_{ijk} = 0$ the space is called a *P-Finsler space. Scalar function $\lambda(x, y)$ is given by $P_r C^r / C_r C^r$, where $P_r = P_r{}^s{}_s$, $C_r = C_r{}^s{}_s$, $C^r = C_s g^{sr}$, $P_{ijk} = C_{ijk|o}$, $2C_{ijk} = \partial g_{ij}/\partial y^k$.

2 The transformation of the tensor P_{ijk} under a projective Randers change

Now we restrict our consideration to special Randers changes, called projective changes, which preserve all the geodesic curves. According to Hashiguchi–Ichjyo [1], a Randers change is projective, if and only if $\rho_{i|j} - \rho_{j|i} = 0$, that is $\rho_i(x)$ is locally a gradient vector field and symbols "|" mean the covariant derivatives in F^n with respect to Berwald connection.

The transformation of $(v)hv$-torsion tensor $P_{ijk} = C_{ijk|o}$ has been studied by H. Matsumoto [6]. He considered: The $(v)hv$-torsion tensor $P_{ijk} = C_{ijk|o}$ of F^n is transformed to $\overline{P}_{ijk} = \overline{C}_{ijk|o}$ of the form (1) by projective Randers change $F^n \to \overline{F}^n$.

$$\overline{C}_{ijk|o} = tC_{ijk|o} + \frac{r_{00}}{2L} C_{ijk} + \frac{1}{2L}(h_{ij}q_k + h_{jk}q_i + h_{ki}q_j), \tag{1}$$

where

$$2C_{ijk} = \partial g_{ij}/\partial y^k, \quad h_{ij} = g_{ij} - l_i l_j, \quad l_i = \partial L/\partial y^i,$$

$$q_k = r_{0k} - \frac{r_{00}}{2L} + \{\rho_k + (1+t)l_k\}, \quad r_{ij} = \frac{1}{2}(\partial_j \rho_i + \partial_i \rho_j) - \rho_r F^r_{ij}, \quad t = \frac{\overline{L}}{L}.$$

We assume that

$$C_{ijk|o} = P_{ijk} = \lambda(x, y) C_{ijk}$$

and

$$\overline{C}_{ijk|o} = \overline{P}_{ijk} = \lambda(x, y)\overline{C}_{ijk},$$

that is F^n and $\overline{F}^n$ are *P-Finsler spaces.

Then we have

$$\lambda(x, y)\overline{C}_{ijk} = \left(t\lambda(x, y) + \frac{r_{00}}{2L}\right) C_{ijk} + \frac{1}{2L}(h_{ij}q_k + h_{jk}q_i + h_{ki}q_j). \tag{2}$$

For a projective Randers change the h_{ij} tensor is transformed as

$$\frac{\overline{h}_{ij}}{\overline{L}} = \frac{h_{ij}}{L}$$

which implies $\overline{L}\,\overline{h}^{ij} = L h^{ij}$, [1].

Using the Matsumoto paper [6] after transvecting by Lh^{ij} from right and $\overline{L}\,\overline{h}^{ij}$ from left we obtain

$$q_k = \frac{2}{n+1}\Big(\lambda(x,y)\overline{L}\,\overline{C}_k - L\Big(t\lambda(x,y) + \frac{r_{00}}{2L}\Big)C_k\Big) \tag{3}$$

Substituting (3) into (2) it follows that it follows that

$$\begin{aligned}\lambda(x,y) &= \overline{C}_{ijk}\Big(t\lambda(x,y) + \frac{r_{00}}{2L}\Big)C_{ijk} \\ &+ \frac{1}{L(n+1)}\Big[\lambda(x,y)\overline{L}(\overline{C}_k h_{ij} + \overline{C}_i h_{jk} + \overline{C}_j h_{ki}) - \\ &- L\Big(t\lambda(x,y) + \frac{r_{00}}{2L}\Big)\big(C_k h_{ij} + C_i h_{jk} + C_j h_{ki}\big)\Big].\end{aligned} \tag{4}$$

Secondly we deal with $\frac{\overline{L}}{L}h_{ij} = \overline{h}_{ij}$:

$$\frac{\lambda(x,y)}{n+1}[(n+1)\overline{C}_{ijk} - (\overline{C}_k\overline{h}_{ij} + \overline{C}_i\overline{h}_{jk} + \overline{C}_j\overline{h}_{ki})] =$$

$$= \frac{1}{n+1}\Big(t\lambda(x,y) + \frac{r_{00}}{2L}\Big)\big[(n+1)C_{ijk} - (C_k h_{ij} + C_i h_{jk} + C_j h_{ki})\big]. \tag{5}$$

From (5) we get the following

Theorem 1 *Let $F^n = \big(M^n, L(x,y)\big)$ and $\overline{F}^n = \big(M^n, \overline{L}(x,y)\big)$ be *P-Finsler spaces. If there exists a projective Randers change between F^n and $\overline{F}^n$, then F^n is C-reducible if and only if $\overline{F}^n$ C-reducible, too.*

3 Some remarks for projective Randers change of special Finsler spaces

Now we put that

$$C_{ijk|o} = P_{ijk} = \lambda(x,y)C_{ijk}$$

then we get

$$\overline{P}_{ijk} = t\lambda(x,y)C_{ijk} + \frac{r_{00}}{2L}C_{ijk} + \frac{1}{2L}(h_{ij}q_k + h_{jk}q_i h_{ki}q_j). \tag{6}$$

Using the Matsumoto paper [5] after transvecting by Lh^{ij} from left we obtain

$$\overline{L}\,\overline{P}_k = C_k\big(\overline{L}\lambda(x,y) + \frac{r_{00}}{2}\big) + \frac{n+1}{2}q_k$$

so we have

$$q_k = \frac{2}{n+1}\overline{L}\,\overline{P}_k - \frac{2}{n+1}\Big(\overline{L}\lambda(x,y) + \frac{r_{00}}{2}\Big)C_k \tag{7}$$

Substituting (7) into (6) it follows that

$$\overline{P}_{ijk} = \frac{1}{n+1}\frac{\overline{L}}{L}(\overline{P}_k h_{ij} + \overline{P}_i h_{jk} + \overline{P}_j h_{ki})$$

$$+\frac{1}{n+1}\Big(t\lambda(x,y) + \frac{r_{00}}{2L}\Big)\big\{(n+1)C_{ijk} - (C_k h_{ij} + C_i h_{jk} + C_j h_{ki})\big\}. \qquad (8)$$

Applying $h_{ij} = L\frac{\overline{h}_{ij}}{\overline{L}}$ the above yields

$$\overline{P}_{ijk} - \frac{1}{n+1}(\overline{P}_k\overline{h}_{ij} + \overline{P}_i\overline{h}_{jk} + \overline{P}_j\overline{h}_{kl}) =$$

$$\frac{1}{n+1}\Big(t\lambda(x,y) + \frac{r_{00}}{2L}\Big)\big\{(n+1)C_{ijk} - (C_k h_{ij} + C_i h_{jk} C_j h_{ki})\big\}. \qquad (9)$$

(7) leads to

Theorem 2 *Let F^n *P-Finsler space and $\overline{F}^n$ an arbitrary Finsler space. If there exists a projective Randers change $\overline{L}(x,y) = L(x,y) + \rho(x,y)$, then we have a (9) for tensors $\overline{P}_{ijk}$ and C_{ijk}.*

By virtue of Theorem 2 the above yields two corollaries:

- If F^n is a C-reducible space, then $\overline{F}^n$ is a P-reducible space.
- If $\overline{F}^n$ is a P reducible space, then F^n is a C-reducible space.

Next we are concerned with an assumption $\overline{F}^n$ is a *P-Finsler space, that is $\overline{C}_{ijk|o} = \lambda(x,y)\overline{C}_{ijk}$. Consequently (1) gives

$$\lambda(x,y)\overline{C}_{ijk} = tP_{ijk} + \frac{r_{00}}{2L}C_{ijk} + \frac{1}{2L}(h_{ij}q_k + h_{jk}q_i + h_{ki}q_j) \qquad (10)$$

Since $\overline{L}\,\overline{h}^{ij} = Lh^{ij}$ holds

$$q_k = \frac{2}{n+1}\Big(\lambda(x,y)\overline{LC}_k - \overline{L}P_k - \frac{r_{00}}{2}C_k\Big)$$

Substitution in (10) leads to

$$\lambda(x,y)\overline{C}_{ijk} = \frac{t}{n+1}\big\{(n+1)P_{ijk} - (P_k h_{ij} + P_i h_{jk} + P_j h_{ki})\big\}$$

$$-\frac{r_{00}}{2(n+1)L}\big\{(n+1)C_{ijk} - (C_k h_{ij} + C_i h_{jk} + C_j h_{ki})\big\}$$

$$+\frac{1}{(n+1)L}\big\{\lambda(x,y)\overline{L}(\overline{C}_k h_{ij} + \overline{C}_i h_{jk} + \overline{C}_j h_{ki})\big\}. \qquad (11)$$

Therefore $h_{ij} = L\frac{\overline{h}_{ij}}{\overline{L}}$, then (11) is written in the form

$$\frac{\lambda(x,y)}{n+1}\{(n+1)\overline{C}_{ijk} - (\overline{C}_k\overline{h}_{ij} + \overline{C}_i\overline{h}_{jk} + \overline{C}_j\overline{h}_{ki})\} =$$

$$= \frac{t}{n+1}\{(n+1)P_{ijk} - (P_k h_{ij} + P_i h_{jk} + P_j h_{ki})\}$$

$$= \frac{r_{00}}{2(n+1)L}\{(n+1)C_{ijk} - (C_k h_{ij} + C_i h_{jk} + C_j h_{ki})\} \qquad (12)$$

From (12) we obtain following

Proposition 1 *Let $\overline{F}^n$ be a *P-Finsler space and F^n an arbitrary Finsler space. If there exists a projective Randers change $\overline{L}(x,y) = L(x,y) + \rho(x,y)$, then we get the relation (12) for tensors $\overline{C}_{ijk}$, P_{ijk} and C_{ijk}.*

From this Proposition 1 follows that F^n is C-reducible, then $\overline{F}^n$ is C-reducible, too.

4 Example

It is well-known, that a Finsler space induced by a Funk metric is a *P-Finsler space, where: $P_{ijk} = -KLC_{ijk}$ $(K \in \mathbb{R}^+)$ [7]. If exists a projective Randers change between a *P-Finsler space induced by Funk metric, and an arbitrary Finsler space, then this space necessarily is a P-reducible Finsler space.

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