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Periodic Points for Maps in $\mathbb{R}^n$

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Abstract

A multidimensional version of the Li–Yorke theorem is stated (with some additional assumptions). Implications of 3-periodic and 5-periodic points and the existence of fixed points for multidimensional maps in $\mathbb{R}^n$ are discussed.

Key words: Li–Yorke theorem, periodic points, Sharkovskii theorem, multidimensional version.

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1 Introduction

The Sharkovskii theorem and its special case the Li–Yorke theorem are well known theorems that are included in many monographs. Both theorems are strictly one-dimensional and a full multidimensional analogy without further assumptions does not hold (for counter-examples see e.g. [Ka], [Kl]). Nevertheless many extensions and analogies (cf. [AJP], [Sch]) have appeared since these theorems have been published. Among them also some multidimensional analogies (with additional conditions or for special maps) were published (cf. [A], [Ka], [Kl], [Z]).

This paper is mainly inspired by [A], where a multidimensional version of the Li–Yorke cycle coexisting theorem is established for certain (e.g. expansive) maps by using some type of inequalities. Our aim is to state and prove some particular multidimesional version of the Sharkovskii theorem. We use new definition of the k-periodic point which differs from the one in [A], but we addopt the type of inequalities, procedures and partial steps to prove a few new theorems. We extend and improve the results from [A] for remaining inequalities representing a three-periodic point and discuss analogical situation for arbitrary k-periodic point. Because of the used type of proofs and additional assumptions (especially the conditions (3) or (15) below) it seems to be very difficult to avoid the inequalities and prove the full analogy of the Sharkovskii theorem.

Existence of fixed point is also discussed. In [Sh] the intuitive (geometric) proof of the Brouwer fixed point theorem on squares is presented; it should be assumed (otherwise the arguments in [Sh] are not correct!) that both maps $x_1 \mapsto Fix\, f(x_1, \bullet)$ and $x_2 \mapsto Fix\, f(\bullet, x_2)$ are continuous and single-valued. In our discussions we replace this condition by more general assumption of type (16).

The situation for multidimensional and multivalued maps is also mentioned.

First of all we introduce some notions and auxiliary results which will be used in the sequel.

Theorem 1.1 (Sharkovskii theorem) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a n-periodic point with $k \lhd n$ (in the Sharkovskii ordering), then it also has a k-periodic point.*

Remark 1.1 Symbol $\rhd$ denotes "*greater than*" in Sharkovskii's ordering of all positive integers, namely

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd \ldots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \ldots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \ldots$$
$$\ldots \rhd 2^n \cdot 3 \rhd 2^n \cdot 5 \rhd \ldots \rhd 2^{n+1} \cdot 3 \rhd 2^{n+1} \cdot 5 \rhd \ldots$$
$$\ldots \rhd 2^{n+1} \rhd 2^n \rhd 2^{n-1} \rhd \ldots \rhd 2^2 \rhd 2 \rhd 1.$$

Definition 1.2 An interval I *f-covers an interval* J provided $J \subset f(I)$. We write $I \to J$.

Lemma 1.3 ([Ro, p. 64]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

(a) Assume that there are two points $a \neq b$ with $f(a) > a$ and $f(b) < b$ and $[a, b]$ is contained in the domain of f ($f \in C(\mathbb{R}, \mathbb{R})$). Then there is a fixed point between a and b.

(b) If a closed interval I f-covers itself, then f has a fixed point in I.

Lemma 1.4 ([Ro, p. 65]) *Let $f \in C(\mathbb{R}, \mathbb{R})$ and assume $J_0 \to J_1 \to \cdots \to J_n = J_0$ is a loop with $J_{k+1} \subset f(J_k)$ for $k = 0, \ldots, n-1$.*

(a) Then there exists a fixed point x_0 of f^n with $f^k(x_0) \in J_k$ for $k = 1, \ldots, n$.

(b) Further assume that

(i) this loop is not a product loop formed by going p times around a shorter loop of length m where $mp = n$, and

(ii) $int(J_i) \cap int(J_k) = \emptyset$ *unless* $J_i = J_k$.

If the periodic point x_0 *of part (a) is in the interior of* J_0 *then it has least period* n.

Remark 1.2 We will use some properties of multivalued mappings (see e.g. [Go] and [HP]).

In the sequel *dim* denotes the (topological) covering dimension (for the definition see e.g. [AP]).

The definition of the Brouwer degree or an apropriate fixed-point index can be found in any standard textbook of nonlinear analysis, see e.g. [DG], [B].

2 Three-periodic point

Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$, i.e. $f = (f_1, \dots, f_n)$, $f_i = f_i(x_1, \dots, x_n)$, $\forall i = 1, \dots, n$.

Definition 2.1 We say that a point $\hat{\mathbf{x}} \in \mathbb{R}^n$ is a *k-periodic point* of function f, if conditions

$$f^k(\hat{\mathbf{x}}) = \hat{\mathbf{x}}, \quad \text{i.e. } f_i^k(\hat{\mathbf{x}}) = \hat{x}_i \;\; \forall i = 1, \dots, n \tag{1}$$

$$\begin{aligned} &\hat{x}_i \neq f_i^m(x_1, \dots, \hat{x}_i, \dots, x_n) \\ &\qquad \forall m = 1, \dots, k-1, \;\forall i = 1, \dots, n, \\ &\qquad \forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1} \end{aligned} \tag{2}$$

hold.

Remark 2.1 In the following discussion we denote

$$inf\, f_i(\dots, \hat{x}_i, \dots) := inf\, f_i(x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_n)$$

and

$$sup\, f_i(\dots, \hat{x}_i, \dots) := sup\, f_i(x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_n),$$

where infima and suprema are w.r.t. $(x_1, \dots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$.

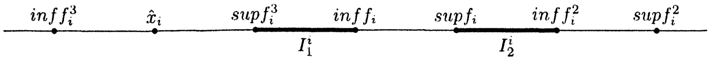
Assume that $\hat{\mathbf{x}}$ is a 3-periodic point of the function $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ such that the following conditions

$$\left.\begin{aligned} f_i(x_1, \dots, \hat{x}_i, \dots, x_n) &\neq f_i^2(y_1, \dots, \hat{x}_i, \dots, y_n) \\ f_i(x_1, \dots, \hat{x}_i, \dots, x_n) &\neq f_i^3(y_1, \dots, \hat{x}_i, \dots, y_n) \\ f_i^2(x_1, \dots, \hat{x}_i, \dots, x_n) &\neq f_i^3(y_1, \dots, \hat{x}_i, \dots, y_n) \end{aligned}\right\} \tag{3}$$

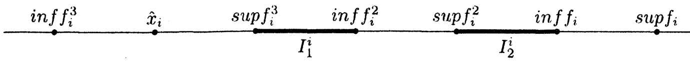
hold $\forall i = 1, \dots, n$ and $\forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) \in \mathbb{R}^{n-1}$.

Then for each $i = 1, \dots, n$ only one of the following orderings (inequalities) can appear:

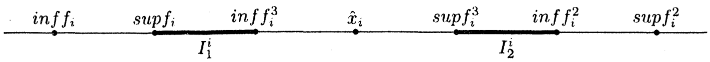
$$\left.\begin{aligned}&inf f_i^3(\dots,\hat{x}_i,\dots) \le \hat{x}_i \le sup\, f_i^3(\dots,\hat{x}_i,\dots) < inf f_i(\dots,\hat{x}_i,\dots)\\ &\le sup\, f_i(\dots,\hat{x}_i,\dots) < inf f_i^2(\dots,\hat{x}_i,\dots) \le sup\, f_i^2(\dots,\hat{x}_i,\dots)\end{aligned}\right\} \quad (4)$$



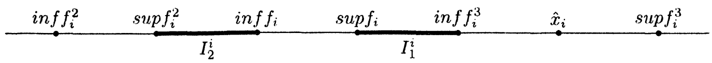
$$\left.\begin{aligned}&inf f_i^3(\dots,\hat{x}_i,\dots) \le \hat{x}_i \le sup\, f_i^3(\dots,\hat{x}_i,\dots) < inf f_i^2(\dots,\hat{x}_i,\dots)\\ &\le sup f_i^2(\dots,\hat{x}_i,\dots) < inf f_i(\dots,\hat{x}_i,\dots) \le sup\, f_i(\dots,\hat{x}_i,\dots)\end{aligned}\right\} \quad (5)$$



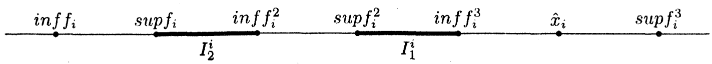
$$\left.\begin{aligned}&inf f_i(\dots,\hat{x}_i,\dots) \le sup\, f_i(\dots,\hat{x}_i,\dots) < inf f_i^3(\dots,\hat{x}_i,\dots) \le \hat{x}_i\\ &\le sup\, f_i^3(\dots,\hat{x}_i,\dots) < inf f_i^2(\dots,\hat{x}_i,\dots) \le sup\, f_i^2(\dots,\hat{x}_i,\dots)\end{aligned}\right\} \quad (6)$$



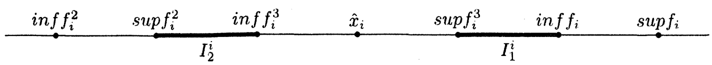
$$\left.\begin{aligned}&inf f_i^2(\dots,\hat{x}_i,\dots) \le sup\, f_i^2(\dots,\hat{x}_i,\dots) < inf f_i(\dots,\hat{x}_i,\dots)\\ &\le sup\, f_i(\dots,\hat{x}_i,\dots) < inf f_i^3(\dots,\hat{x}_i,\dots) \le \hat{x}_i \le sup\, f_i^3(\dots,\hat{x}_i,\dots)\end{aligned}\right\} \quad (7)$$



$$\left.\begin{aligned}&inf f_i(\dots,\hat{x}_i,\dots) \le sup\, f_i(\dots,\hat{x}_i,\dots) < inf f_i^2(\dots,\hat{x}_i,\dots)\\ &\le sup\, f_i^2(\dots,\hat{x}_i,\dots) < inf f_i^3(\dots,\hat{x}_i,\dots) \le \hat{x}_i \le sup\, f_i^3(\dots,\hat{x}_i,\dots)\end{aligned}\right\} \quad (8)$$



$$\left.\begin{aligned}&inf f_i(\dots,\hat{x}_i,\dots) \le sup\, f_i(\dots,\hat{x}_i,\dots) < inf f_i^3(\dots,\hat{x}_i,\dots) \le \hat{x}_i\\ &\le sup\, f_i^3(\dots,\hat{x}_i,\dots) < inf f_i^2(\dots,\hat{x}_i,\dots) \le sup\, f_i^2(\dots,\hat{x}_i,\dots)\end{aligned}\right\} \quad (9)$$



We can now state and prove the following three Theorems.

Theorem 2.2 *Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ and assume that the inequalities (4) or (7) hold for all $i = 1, \dots, n$. Assume furthermore that conditions*

$$\left.\begin{aligned}&\forall(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \in I_2^1 \times \dots \times I_2^{i-1} \times I_2^{i+1} \times \dots \times I_2^n :\\ &dim(Fix(f_i) \cap I_2^i) = 0 \;\; i = 1,\dots,n \text{ and all elements of } Fix(f_i) \cap I_2^i\\ &\text{have nontrivial fixed-point indices, defined as the local Brouwer}\\ &\text{degree of the maps } x_i - f_i\end{aligned}\right\} \quad (10)$$

and

$$\left.\begin{array}{l}\forall(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in I_1^1\times\dots\times I_1^{i-1}\times I_1^{i+1}\times\dots\times I_1^n \textit{ and}\\ \forall m>1\ :\ dim(Fix(f_i^m)\cap I_1^i)=0\ i=1,\dots,n \textit{ and all elements of}\\ Fix(f_i^m)\cap I_1^i \textit{ have nontrivial fixed-point indices, defined as the local}\\ \textit{Brouwer degree of the maps } x_i-f_i^m\end{array}\right\}\quad(11)$$

are satisfied for all $i=1,\dots,n$. *(The intervals* I_2^j *and* I_1^j, $j=1,\dots,n$ *are defined by the inequalities (4) or (7).) Then* f *has an* N*-periodic point for all* $N\in\mathbb{N}$ *(in the sense of Definition 2.1).*

Proof First assume that inequalities (4) hold for all $i=1,\dots,n$.

Then $I_1^i\longleftrightarrow I_2^i\hookleftarrow$ holds for all $i=1,\dots,n$ (i.e. I_2^i f_i-covers both I_1^i and I_2^i and I_1^i f_i-covers I_2^i, see Definition 1.2), where intervals I_1^i and I_2^i are defined by inequalities (4).

$N=1$. Applying Lemma 1.3 for $I_2^i\longrightarrow I_2^i$ we get that $(\forall(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in I_2^1\times\dots\times I_2^{i-1}\times I_2^{i+1}\times\dots\times I_2^n)$ $\forall i=1,\dots,n$ there exists a point $z_i\in I_2^i$ such that $f_i(x_1,\dots,z_i,\dots,x_n)=z_i$.

For each $i=1,\dots,n$ we can define a multivalued mapping $\varphi_i : I_2^1\times\dots\times I_2^{i-1}\times I_2^{i+1}\times\dots\times I_2^n\rightsquigarrow I_2^i$ by

$$\varphi_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)=Fix\ f_i(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\cap I_2^i.$$

This mapping is u.s.c. and under condition (10) φ_i has a (single-valued) continuous selection (see [Go, Theorem 74.7] and pp. 82–83 and 90–95 in [HP] or [A]). We denote this selection also by φ_i. The hypersurfaces defined by functions φ_i, $i=1,\dots,n$ must intersect at least in one point which is an 1-periodic (i.e. fixed) point of f.

$N\geq 2$. For each $i=1,\dots,n$ we can consider the following loop of intervals

$$I_1^i\longrightarrow\underbrace{I_2^i\longrightarrow\dots\longrightarrow I_2^i}_{(N-1)-times}\longrightarrow I_1^i.$$

By Lemma 1.4 it holds that $(\forall(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in I_1^1\times\dots\times I_1^{i-1}\times I_1^{i+1}\times\dots\times I_1^n)$ there exists $y_i\in I_1^i$ such that $f_i^N(x_1,\dots,y_i,\dots,x_n)=y_i$ and $f_i^k(x_1,\dots,y_i,\dots,x_n)\in I_2^i$ $\forall k=1,\dots,N-1$.

It remains to show, that $f_i^k(x_1,\dots,y_i,\dots,x_n)\neq y_i$ $\forall k=1,\dots,N-1$. Assume that there exists $k\in\{1,\dots,N-1\}$ such that $y_i=f_i^k(x_1,\dots,y_i,\dots,x_n)\in I_2^i$. Thus we have $y_i\in I_1^i\cap I_2^i$. Then either $I_1^i\cap I_2^i=\emptyset$ which is a contradiction, or $I_1^i\cap I_2^i=\{inf f_i(\dots,\hat{x}_i,\dots)\}=\{sup\, f_i(\dots,\hat{x}_i,\dots)\}\equiv\{f_i(\dots,\hat{x}_i,\dots)\}$.

For $N=2$ it holds that

$$\begin{aligned}y_i=f_i^2(x_1,\dots,y_i,\dots,x_n)&=f_i^2(x_1,\dots,f_i(\dots,\hat{x}_i,\dots),\dots,x_n)\\&=f_i(\dots,\hat{x}_i,\dots)>sup\, f_i^3(\dots,\hat{x}_i,\dots)\end{aligned}$$

and also

$$f_i^2(x_1,\dots,f_i(\dots,\hat{x}_i,\dots),\dots,x_n)=f_i^3(x_1,\dots,\hat{x}_i,\dots,x_n)\leq sup\, f_i^3(\dots,\hat{x}_i,\dots)$$

which is a contradiction.

For $N > 2$ it holds that

$$f_i^2(x_1, \ldots, f_i(\ldots, \hat{x}_i, \ldots), \ldots, x_n) = f_i^2(x_1, \ldots, y_i, \ldots, x_n) \in I_2^i$$

and also

$$f_i^2(x_1, \ldots, f_i(\ldots, \hat{x}_i, \ldots), \ldots, x_n) = f_i^3(x_1, \ldots, \hat{x}_i, \ldots, x_n) \notin I_2^i$$

which is a contradiction.

Consequently $f_i^k(x_1, \ldots, y_i, \ldots, x_n) \neq y_i \ \forall k = 1, \ldots, N-1, \ \forall i = 1, \ldots, n.$

For each $N > 1$ and each $i = 1, \ldots, n$ we can define multivalued mapping $\psi_{i,N} : I_1^1 \times \ldots \times I_1^{i-1} \times I_1^{i+1} \times \ldots \times I_1^n \rightsquigarrow I_1^i$ by

$$\psi_{i,N}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = Fix \ f_i^N(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n) \cap I_1^i.$$

Then $\psi_{i,N}$ is u.s.c. and under condition (11) there exists a (single-valued) continuous selection. We denote this selection also by $\psi_{i,N}$. For each $N > 1$ the hypersurfaces defined by functions $\psi_{i,N}$, $i = 1, \ldots, n$ must intersect at least in one point which is an N-periodic point of f.

If we consider inequalities (7), then the proof is the same with only one change:

For $N = 2$ it holds that

$$\begin{aligned} y_i = f_i^2(x_1, \ldots, y_i, \ldots, x_n) &= f_i^2(x_1, \ldots, f_i(\ldots, \hat{x}_i, \ldots), \ldots, x_n) \\ &= f_i(\ldots, \hat{x}_i, \ldots) < \mathit{inf} f_i^3(\ldots, \hat{x}_i, \ldots) \end{aligned}$$

and also

$$\begin{aligned} f_i^2(x_1, \ldots, y_i, \ldots, x_n) &= f_i^2(x_1, \ldots, f_i(x_1, \ldots, \hat{x}_i, \ldots, x_n), \ldots, x_n) \\ &= f_i^3(x_1, \ldots, \hat{x}_i, \ldots, x_n) \geq \mathit{inf} f_i^3(\ldots, \hat{x}_i, \ldots) \end{aligned}$$

which is a contradiction. □

Remark 2.2 If the set $Fix(f)$ is e.g. finite then $dim \ Fix(f) = 0$.

Theorem 2.3 *Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ and assume that the inequalities (5) or (8) hold for all $i = 1, \ldots, n$. Assume furthermore that conditions*

$$\left. \begin{array}{l} \forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in I_1^1 \times \ldots \times I_1^{i-1} \times I_1^{i+1} \times \ldots \times I_1^n : \\ dim(Fix(f_i) \cap I_1^i) = 0 \ i = 1, \ldots, n \text{ and all elements of } Fix(f_i) \cap I_1^i \\ \text{have nontrivial fixed-point indices, defined as the local Brouwer} \\ \text{degree of the maps } x_i - f_i \end{array} \right\} \quad (12)$$

and

$$\left. \begin{array}{l} \forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in I_2^1 \times \ldots \times I_2^{i-1} \times I_2^{i+1} \times \ldots \times I_2^n \text{ and} \\ \forall m > 1 \ : \ dim(Fix(f_i^m) \cap I_2^i) = 0 \ i = 1, \ldots, n \text{ and all elements of} \\ Fix(f_i^m) \cap I_2^i \text{ have nontrivial fixed-point indices, defined as the local} \\ \text{Brouwer degree of the maps } x_i - f_i^m \end{array} \right\} \quad (13)$$

are satisfied for all $i = 1, \dots, n$. *(The intervals* I_2^j *and* I_1^j, $j = 1, \dots, n$ *are defined by the inequalities (5) or (8).) Then* f *has an* N*-periodic point for all* $N \in \mathbb{N}$ *(in the sense of Definition 2.1).*

Proof First assume that inequalities (5) hold for all $i = 1, \dots, n$.

Then $\hookrightarrow I_1^i \longleftrightarrow I_2^i$ holds $\forall i = 1, \dots, n$.

$N = 1$. Applying Lemma 1.3 for $I_1^i \longrightarrow I_1^i \ \forall i = 1, \dots, n$ we get that $(\forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in I_1^1 \times \dots \times I_1^{i-1} \times I_1^{i+1} \times \dots \times I_1^n)$ there exists a point $z_i \in I_1^i$ such that $f_i(x_1, \dots, z_i, \dots, x_n) = z_i$.

For each $i = 1, \dots, n$ we can define a multivalued mapping $\varphi_i : I_1^1 \times \dots \times I_1^{i-1} \times I_1^{i+1} \times \dots \times I_1^n \rightsquigarrow I_1^i$ by

$$\varphi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = Fix\ f_i(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n) \cap I_1^i.$$

This mapping is u.s.c. and under condition (12) φ_i has a (single-valued) continuous selection. We denote this selection also by φ_i. The hypersurfaces defined by functions φ_i, $i = 1, \dots, n$ must intersect at least in one point which is an 1-periodic (i.e. fixed) point of f.

$N \geq 2$. For each $i = 1, \dots, n$ we can consider following loop of intervals

$$I_2^i \longrightarrow \underbrace{I_1^i \longrightarrow \dots \longrightarrow I_1^i}_{(N-1)-times} \longrightarrow I_2^i.$$

By Lemma 1.4 it holds that $(\forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in I_2^1 \times \dots \times I_2^{i-1} \times I_2^{i+1} \times \dots \times I_2^n)$ there exists $y_i \in I_2^i$ such that $f_i^N(x_1, \dots, y_i, \dots, x_n) = y_i$ and $f_i^k(x_1, \dots, y_i, \dots, x_n) \in I_1^i \ \forall k = 1, \dots, N-1$.

It remains to show, that $f_i^k(x_1, \dots, y_i, \dots, x_n) \neq y_i \ \forall k = 1, \dots, N-1$. Assume that there exists $k \in \{1, \dots, N-1\}$ such that $y_i = f_i^k(x_1, \dots, y_i, \dots, x_n) \in I_1^i$. Thus we have $y_i \in I_1^i \cap I_2^i$. Then either $I_1^i \cap I_2^i = \emptyset$ which is a contradiction, or $I_1^i \cap I_2^i = \{inf f_i^2(\dots, \hat{x}_i, \dots)\} = \{sup f_i^2(\dots, \hat{x}_i, \dots)\} \equiv \{f_i^2(\dots, \hat{x}_i, \dots)\}$.

For $N = 2$ it holds that

$$\begin{aligned} y_i = f_i(x_1, \dots, y_i, \dots, x_n) &= f_i(x_1, \dots, f_i^2(\dots, \hat{x}_i, \dots), \dots, x_n) \\ &= f_i^2(\dots, \hat{x}_i, \dots) > sup f_i^3(\dots, \hat{x}_i, \dots) \end{aligned}$$

and also

$$\begin{aligned} f_i(x_1, \dots, y_i, \dots, x_n) &= f_i(x_1, \dots, f_i^2(\dots, \hat{x}_i, \dots), \dots, x_n) \\ &= f_i^3(x_1, \dots, \hat{x}_i, \dots, x_n) \leq sup f_i^3(\dots, \hat{x}_i, \dots) \end{aligned}$$

which is a contradiction.

For $N > 2$ it holds that

$$f_i^2(x_1, \dots, f_i^2(\dots, \hat{x}_i, \dots), \dots, x_n) = f_i^2(x_1, \dots, y_i, \dots, x_n) \in I_1^i$$

and also

$$f_i^2(x_1, \dots, f_i^2(\dots, \hat{x}_i, \dots), \dots, x_n) = f_i^4(x_1, \dots, \hat{x}_i, \dots, x_n)$$

$$= f_i(x_1, \ldots, \hat{x}_i, \ldots, x_n) \notin I_1^i$$

which is contradiction.

For each $N > 1$ and each $i = 1, \ldots, n$ we can define multivalued mapping $\psi_{i,N} : I_2^1 \times \ldots \times I_2^{i-1} \times I_2^{i+1} \times \ldots \times I_2^n \rightsquigarrow I_2^i$ by

$$\psi_{i,N}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = Fix\ f_i^N(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n) \cap I_2^i.$$

Then $\psi_{i,N}$ is u.s.c. and under condition (13) there exists a (single-valued) continuous selection. We denote this selection also by $\psi_{i,N}$. For each $N > 1$ the hypersurfaces defined by functions $\psi_{i,N}$, $i = 1, \ldots, n$ must intersect at least in one point which is an N-periodic point of f.

If we consider inequalities (8), then the proof is the same with only one change:

For $N = 2$ holds

$$y_i = f_i(x_1, \ldots, y_i, \ldots, x_n) = f_i(x_1, \ldots, f_i^2(\ldots, \hat{x}_i, \ldots), \ldots, x_n)$$
$$= f_i^2(\ldots, \hat{x}_i, \ldots) < inf\, f_i^3(\ldots, \hat{x}_i, \ldots)$$

and also

$$f_i(x_1, \ldots, y_i, \ldots, x_n) = f_i(x_1, \ldots, f_i^2(\ldots, \hat{x}_i, \ldots), \ldots, x_n)$$
$$= f_i^3(x_1, \ldots, \hat{x}_i, \ldots, x_n) \geq inf\, f_i^3(\ldots, \hat{x}_i, \ldots)$$

which is a contradiction. □

Theorem 2.4 *Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ and assume that the inequalities (6) or (9) hold for all $i = 1, \ldots, n$. Assume furthermore that condition*

$$\left.\begin{array}{l} \forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in I_2^1 \times \ldots \times I_2^{i-1} \times I_2^{i+1} \times \ldots \times I_2^n \textit{ and} \\ \forall m \geq 1\ :\ dim(Fix(f_i^m) \cap I_2^i) = 0\ i = 1, \ldots, n \textit{ and all elements of} \\ Fix(f_i^m) \cap I_2^i \textit{ have nontrivial fixed-point indices, defined as the local} \\ \textit{Brouwer degree of the maps } x_i - f_i^m \end{array}\right\} \quad (14)$$

is satisfied for all $i = 1, \ldots, n$. (The intervals I_2^j and I_1^j, $j = 1, \ldots, n$ are defined by the inequalities (6) or (9).) Then f has an N-periodic point for all $N \in \mathbb{N}$ (in the sense of Definition 2.1).

Proof First assume that inequalities (6) hold for all $i = 1, \ldots, n$.

Then $\hookrightarrow I_1^i \longleftrightarrow I_2^i \hookleftarrow$ holds $\forall i = 1, \ldots, n$.

$N = 1$. Applying Lemma 1.3 for $I_2^i \longrightarrow I_2^i$ we get that $(\forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in I_2^1 \times \ldots \times I_2^{i-1} \times I_2^{i+1} \times \ldots \times I_2^n)$ $\forall i = 1, \ldots, n$ there exists a point $z_i \in I_2^i$ such that $f_i(x_1, \ldots, z_i, \ldots, x_n) = z_i$.

For each $i = 1 \ldots, n$ we can define a multivalued mapping $\varphi_i : I_2^1 \times \ldots \times I_2^{i-1} \times I_2^{i+1} \times \ldots \times I_2^n \rightsquigarrow I_2^i$ by

$$\varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = Fix\ f_i(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n) \cap I_2^i.$$

This mapping is u.s.c. and under condition (14) φ_i has a (single-valued) continuous selection. We denote this selection also by φ_i. The hypersurfaces defined by functions φ_i, $i = 1, \dots, n$ must intersect at least in one point which is an 1-periodic (i.e. fixed) point of f.

$N \geq 2$. For each $i = 1, \dots, n$ we can consider following loop of intervals

$$I_2^i \longrightarrow \underbrace{I_1^i \longrightarrow \dots \longrightarrow I_1^i}_{(N-1)-times} \longrightarrow I_2^i.$$

By Lemma 1.4 it holds that $(\forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in I_2^1 \times \dots \times I_2^{i-1} \times I_2^{i+1} \times \dots \times I_2^n)$ there exists $y_i \in I_2^i$ such that $f_i^N(x_1, \dots, y_i, \dots, x_n) = y_i$ and $f_i^k(x_1, \dots, y_i, \dots, x_n) \in I_1^i \ \forall k = 1, \dots, N-1$.

It remains to show, that $f_i^k(x_1, \dots, y_i, \dots, x_n) \neq y_i \ \forall k = 1, \dots, N-1$. Assume that there exists $k \in \{1, \dots, N-1\}$ such that $y_i = f_i^k(x_1, \dots, y_i, \dots, x_n) \in I_1^i$. Thus we have $y_i \in I_1^i \cap I_2^i$. Then either $I_1^i \cap I_2^i = \emptyset$ which is a contradiction, or $I_1^i \cap I_2^i = \{\mathit{inf} f_i^3(\dots, \hat{x}_i, \dots)\} = \{\mathit{sup} f_i^3(\dots, \hat{x}_i, \dots)\} \equiv \{f_i^3(\dots, \hat{x}_i, \dots)\} = \{\hat{x}_i\}$.

For $N = 2$ it holds that

$$y_i = f_i^2(x_1, \dots, y_i, \dots, x_n) = f_i^2(x_1, \dots, \hat{x}_i, \dots, x_n) > \mathit{sup} f_i^3(\dots, \hat{x}_i, \dots)$$

and also

$$\begin{aligned} f_i^2(x_1, \dots, y_i, \dots, x_n) &= f_i^2(x_1, \dots, f_i(x_1, \dots, y_i, \dots, x_n), \dots, x_n) \\ = f_i^3(x_1, \dots, y_i, \dots, x_n) &= f_i^3(x_1, \dots, \hat{x}_i, \dots, x_n) \leq \mathit{sup} f_i^3(\dots, \hat{x}_i, \dots) \end{aligned}$$

which is a contradiction.

For $N > 2$ it holds that

$$f_i^2(x_1, \dots, \hat{x}_i, \dots, x_n) = f_i^2(x_1, \dots, y_i, \dots, x_n) \in I_1^i$$

and also

$$f_i^2(x_1, \dots, \hat{x}_i, \dots, x_n) \notin I_1^i$$

which is contradiction.

For each $N > 1$ and each $i = 1, \dots, n$ we can define multivalued mapping $\psi_{i,N} : I_2^1 \times \dots \times I_2^{i-1} \times I_2^{i+1} \times \dots \times I_2^n \rightsquigarrow I_2^i$ by

$$\psi_{i,N}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \mathit{Fix}\ f_i^N(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n) \cap I_2^i.$$

Then $\psi_{i,N}$ is u.s.c. and under condition (14) there exists a (single-valued) continuous selection. We denote this selection also by $\psi_{i,N}$. For each $N > 1$ the hypersurfaces defined by functions $\psi_{i,N}$, $i = 1, \dots, n$ must intersect at least in one point which is an N-periodic point of f.

If we consider the inequalities (9), then the proof is the same with only one change:

For $N = 2$ holds

$$y_i = f_i^2(x_1, \dots, y_i, \dots, x_n) = f_i^2(x_1, \dots, \hat{x}_i, \dots, x_n) < \mathit{inf} f_i^3(\dots, \hat{x}_i, \dots)$$

and also

$$f_i^2(x_1,\dots,y_i,\dots,x_n) = f_i^2(x_1,\dots,f_i(x_1,\dots,y_i,\dots,x_n),\dots,x_n)$$

$$= f_i^3(x_1,\dots,y_i,\dots,x_n) = f_i^3(x_1,\dots,\hat{x}_i,\dots,x_n) \geq \mathit{inf} f_i^3(\dots,\hat{x}_i,\dots)$$

which is a contradiction. □

Remark 2.3 If we add to Definition 2.1 of *k-periodic point* the condition

$$\left.\begin{array}{l} f_i^m(x_1,\dots,\hat{x}_i,\dots,x_n) \neq f_i^l(y_1,\dots,\hat{x}_i,\dots,y_n) \\ \forall m,l = 1,\dots,k,\ l \neq m,\ \forall i = 1,\dots,n, \\ \forall(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n),(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n) \in \mathbb{R}^{n-1} \end{array}\right\} \tag{15}$$

(which is only generalization of condition (3) for $k \in \mathbb{N}$) it seems to be cumbersome to verify this additional condition.

In previous Theorems 2.2, 2.3 and 2.4 we considered the same ordering (i.e. one of the inequalities (4)–(9)) in each direction (i.e. $\forall i = 1,\dots,n$). It is not difficult to prove analogous theorems if we admit different (but considering again one of inequalities (4)–(9)) ordering in each direction. The only problem is with the location of conditions of type (10) and (11), because they are dependent on the given inequalities—see the following example.

Example 2.1 We consider a function $g \in C(\mathbb{R}^2,\mathbb{R}^2)$. Assume that $\hat{\mathbf{x}} = (\hat{x}_1,\hat{x}_2)$ is a 3-periodic point of g such that inequalities (4) hold for $i = 1$ and inequalities (5) hold for $i = 2$. In this case we have to state the conditions of type (10) and (11) for each $i = 1, 2$ separately:

(a) $\forall x_2 \in I_1^2 :\ \mathit{dim}\,(\mathit{Fix}(g_1) \cap I_2^1) = 0$ and all elements of $\mathit{Fix}(g_1) \cap I_2^1$ have nontrivial fixed-points indices

(b) $\forall x_1 \in I_2^1 :\ \mathit{dim}\,(\mathit{Fix}(g_2) \cap I_1^2) = 0$ and all elements of $\mathit{Fix}(g_2) \cap I_1^2$ have nontrivial fixed-points indices

(c) $\forall x_2 \in I_2^2$ and $\forall m > 1 :\ \mathit{dim}\,(\mathit{Fix}(g_1^m) \cap I_1^1) = 0$ and all elements of $\mathit{Fix}(g_1^m) \cap I_1^1$ have nontrivial fixed-points indices

(d) $\forall x_1 \in I_1^1$ and $\forall m > 1 :\ \mathit{dim}\,(\mathit{Fix}(g_2) \cap I_2^2) = 0$ and all elements of $\mathit{Fix}(g_2) \cap I_2^2$ have nontrivial fixed-points indices

Then function g has an N-periodic point for all $N \in \mathbb{N}$.

Proof: The proof is analogous to the proofs of Theorem 2.2 (for $i = 1$ with ordering (4)) and Theorem 2.3 (for $i = 2$ with ordering (5)).

For $i = 1$ it holds that $I_1^1 \longleftrightarrow I_2^1 \hookleftarrow$.

$N = 1$. Applying Lemma 1.3 for $I_2^1 \longleftrightarrow I_2^1$ we get that $(\forall x_2 \in I_1^2)$ there exists a point $z_1 \in I_2^1$ such that $g_1(z_1,x_2) = z_1$.

We can define a multivalued mapping $\varphi_1 : I_1^2 \rightsquigarrow I_2^1$ by

$$\varphi_1(x_2) = \mathit{Fix}\ g_1(\cdot,x_2) \cap I_2^1.$$

This mapping is u.s.c. and thanks condition (a) φ_1 has a (single-valued) continuous selection. We denote this selection also by $\varphi_1 : I_1^2 \to I_2^1$.

$N > 1$. We can consider following loop of intervals

$$I_1^1 \longrightarrow \underbrace{I_2^1 \longrightarrow \cdots \longrightarrow I_2^1}_{(N-1)-times} \longrightarrow I_1^1.$$

By Lemma 1.4 it holds that $(\forall x_2 \in I_2^2)$ there exists $y_1 \in I_1^1$ such that $g_1^N(y_1, x_2) = y_1$ and $g_1^k(y_1, x_2) \in I_2^1$ $\forall k = 1, \dots, N-1$.

It remains to show that $g_1^k(y_1, x_2) \neq y_1$ $\forall k = 1, \dots, N-1$. Assume that there exists $k \in \{1, \dots, N-1\}$ such that $y_1 = g_1^k(y_1, x_2) \in I_2^1$. Thus we have $y_1 \in I_1^1 \cap I_2^1$. Then either $I_1^1 \cap I_2^1 = \emptyset$ which is a contradiction, or $I_1^1 \cap I_2^1 = \{inf_{x_2}\, g_1(\hat{x}_1, x_2)\} = \{sup_{x_2}\, g_1(\hat{x}_1, x_2)\} \equiv \{g_1(\hat{x}_1, \cdot)\}$

For $N = 2$ it holds that

$$y_1 = g_1^2(y_1, x_2) = g_1^2(g_1(\hat{x}_1, \cdot), x_2) = g_1(\hat{x}_1, \cdot) > sup_{x_2}\, g_1^3(\hat{x}_1, x_2)$$

and also

$$g_1^2(g_1(\hat{x}_1, \cdot), x_2) = g_1^3(\hat{x}_1, x_2) \leq sup_{x_2}\, g_1^3(\hat{x}_1, x_2)$$

which is a contradiction.

For $N > 2$ it holds that

$$g_1^2(g_1(\hat{x}_1, \cdot), x_2) = g_1(y_1, x_2) \in I_2^1$$

and also

$$g_1^2(g_1(\hat{x}_1, \cdot), x_2) = g_1^3(\hat{x}_1, x_2) \notin I_2^1$$

which is a contradiction.

For each $N > 1$ we can define the multivalued mapping $\psi_{1,N} : I_2^2 \rightsquigarrow I_1^1$ by

$$\psi_{1,N}(x_2) = Fix\ g_1^N(\cdot, x_2) \cap I_1^1.$$

Then $\psi_{1,N}$ is u.s.c. and thanks condition (c) there exists a (single-valued) continuous selection. We denote this selection also by $\psi_{1,N} : I_2^2 \to I_1^1$.

For $i = 2$ it holds that $\hookrightarrow I_1^2 \longleftrightarrow I_2^2$.

$N = 1$. Applying Lemma 1.3 for $I_1^2 \longleftrightarrow I_1^2$ we get that $(\forall x_1 \in I_2^1)$ there exists a point $z_2 \in I_1^2$ such that $g_2(x_1, z_2) = z_2$.

We can define a multivalued mapping $\varphi_2 : I_2^1 \rightsquigarrow I_1^2$ by

$$\varphi_2(x_1) = Fix\ g_2(x_1, \cdot) \cap I_1^2.$$

This mapping is u.s.c. and thanks condition (b) φ_2 has a (single-valued) continuous selection. We denote this selection also by $\varphi_2 : I_2^1 \to I_1^2$.

$N > 1$. We can consider following loop of intervals

$$I_2^2 \longrightarrow \underbrace{I_1^2 \longrightarrow \cdots \longrightarrow I_1^2}_{(N-1)-times} \longrightarrow I_2^2.$$

By Lemma 1.4 it holds that $(\forall x_1 \in I_1^1)$ there exists $y_2 \in I_2^2$ such that $g_2^N(x_1, y_2) = y_2$ and $g_2^k(x_1, y_2) \in I_1^2 \ \forall k = 1, \dots, N-1$.

It remains to show that $g_2^k(x_1, y_2) \neq y_2 \ \forall k = 1, \dots, N-1$. Assume that there exists $k \in \{1, \dots, N-1\}$ such that $y_2 = g_2^k(x_1, y_2) \in I_1^2$. Thus we have $y_2 \in I_1^2 \cap I_2^2$. Then either $I_1^2 \cap I_2^2 = \emptyset$ which is a contradiction, or $I_1^2 \cap I_2^2 = \{inf_{x_1} g_2^2(x_1, \hat{x}_2)\} = \{sup_{x_1} g_2^2(x_1, \hat{x}_2)\} \equiv \{g_2^2(\cdot, \hat{x}_2)\}$

For $N = 2$ it holds that

$$y_2 = g_2(x_1, y_2) = g_2(x_1, g_2^2(\cdot, \hat{x}_2)) = g_2^2(\cdot, \hat{x}_2) > sup_{x_1} g_2^3(x_1, \hat{x}_2)$$

and also

$$g_2(x_1, y_2) = g_2(x_1, g_2^2(\cdot, \hat{x}_2)) = g_2^3(x_1, \hat{x}_2) \leq sup_{x_1} g_2^3(x_1, \hat{x}_2)$$

which is a contradiction.

For $N > 2$ it holds that

$$g_2^2(x_1, g_2^2(\cdot, \hat{x}_2)) = g_2^2(x_1, y_2) \in I_1^2$$

and also

$$g_2^2(x_1, g_2^2(\cdot, \hat{x}_2)) = g_2^4(x_1, \hat{x}_2) = g_2(x_1, \hat{x}_2) \notin I_1^2$$

which is a contradiction.

For each $N > 1$ we can define multivalued mapping $\psi_{2,N} : I_1^1 \rightsquigarrow I_2^2$ by

$$\psi_{2,N}(x_1) = Fix \ g_2^N(x_1, \cdot) \cap I_2^2.$$

Then $\psi_{2,N}$ is u.s.c. and thanks condition (d) there exists a (single-valued) continuous selection. We denote this selection also by $\psi_{2,N} : I_1^1 \to I_2^2$.

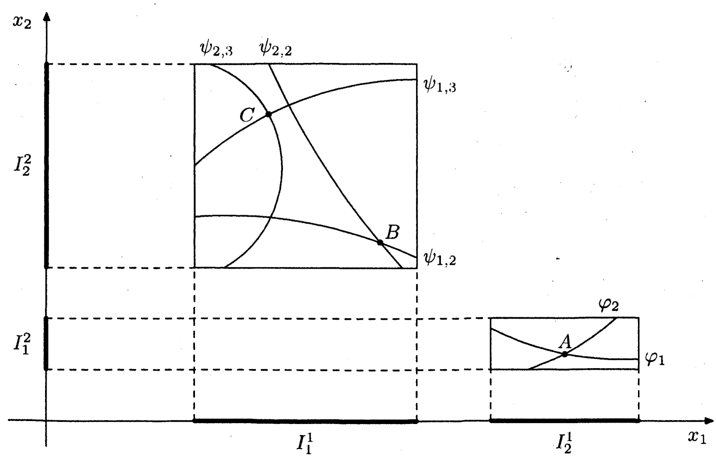


Fig. 1: Possible locations of the first three periodic points of g from Example 2.1, where A is a 1-periodic point, B is a 2-periodic point and C is a 3-periodic point.

Conclusion:

(1) The functions φ_1 and φ_2 must intersect at least in one point which is a 1-periodic (i.e. fixed) point of g.

(2) For all $N > 1$ the functions $\psi_{1,N}$ and $\psi_{2,N}$ must also intersect at least in one point which is a N-periodic point of g. (See Fig. 1.) ○

We can avoid the complication related to the location of conditions of type (10) and (11) using a stronger condition

$$\left.\begin{array}{l}\forall(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in\mathbb{R}^{n-1} \text{ and } \forall m\geq 1\ :\\ dim\,(Fix(f_i^m)) = 0\ \forall i=1,\dots,n \text{ and all elements of } Fix(f_i^m) \text{ have}\\ \text{nontrivial fixed-point indices, defined as the local Brouwer degree}\\ \text{of the maps } x_i - f_i^m\end{array}\right\}\quad(16)$$

Now we are able to state the following theorem.

Theorem 2.5 *Assume that for* $f \in C(\mathbb{R}^n,\mathbb{R}^n)$ *there exists a 3-periodic point* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *such that for each* $i = 1,\dots,n$ *one of inequalities (4)–(9) hold. If condition (16) holds for all* $i = 1,\dots,n$, *then the function* f *has an* N*-periodic point for all* $N \in \mathbb{N}$.

Proof Proof of this theorem follows for each $i = 1,\dots,n$ the corresponding (in the sense of used inequalities) proof of one of the preceding Theorems 2.2–2.4. (See also Example 2.1.) □

3 Five-periodic point

In the previous section we have discussed the situation for a 3-periodic point. Analogically the case of 5-periodic point can be described.

First of all we use the following notation. If $\hat{\mathbf{x}} \in \mathbb{R}^n$ is a k-periodic point of the function $f \in C(\mathbb{R}^n,\mathbb{R}^n)$, then for fixed $i = 1,\dots,n$ and for all $j = 1,\dots,k-1$ we denote the interval $[inf\,f_i^j(\dots,\hat{x}_i,\dots), sup\,f_i^j(\dots,\hat{x}_i,\dots)]$ by the number $j+1$ and the interval $[inf\,f_i^k(\dots,\hat{x}_i,\dots), sup\,f_i^k(\dots,\hat{x}_i,\dots)]$ by the number 1.

Assume a function $f \in C(\mathbb{R}^n,\mathbb{R}^n)$ has a 3-periodic point $\hat{\mathbf{x}} \in \mathbb{R}^n$ satisfying condition (15), the discussion then splits into six cases (i.e. inequalities (4)–(9)). These six ($6 = 3!$) cases arise from the permutations of the numbers 1, 2 and 3, which represent the intervals

$$[inf\,f_i^3(\dots,\hat{x}_i,\dots), sup\,f_i^3(\dots,\hat{x}_i,\dots)],\quad [inf\,f_i(\dots,\hat{x}_i,\dots), sup\,f_i(\dots,\hat{x}_i,\dots)]$$

and

$$[inf\,f_i^2(\dots,\hat{x}_i,\dots), sup\,f_i^2(\dots,\hat{x}_i,\dots)].$$

The following table shows the correspondence between the permutations and the inequalities:

combination	inequalities
1 2 3	(4)
1 3 2	(5)
2 1 3	(6)
3 2 1	(7)
2 3 1	(8)
3 1 2	(9)

If we now consider a 5-periodic point $\hat{\mathbf{x}} \in \mathbb{R}^n$ satisfying condition (15), the discussion splits into 120 ($120 = 5!$) cases. This number is exactly the number of permutations of numbers 1, 2, 3, 4 and 5, which represent intervals

$$[\mathit{inf}\, f_i^5(\ldots,\hat{x}_i,\ldots), \mathit{sup}\, f_i^5(\ldots,\hat{x}_i,\ldots)], \quad [\mathit{inf}\, f_i(\ldots,\hat{x}_i,\ldots), \mathit{sup}\, f_i(\ldots,\hat{x}_i,\ldots)],$$

$$[\mathit{inf}\, f_i^2(\ldots,\hat{x}_i,\ldots), \mathit{sup}\, f_i^2(\ldots,\hat{x}_i,\ldots)], \quad [\mathit{inf}\, f_i^3(\ldots,\hat{x}_i,\ldots), \mathit{sup}\, f_i^3(\ldots,\hat{x}_i,\ldots)]$$

and

$$[\mathit{inf}\, f_i^4(\ldots,\hat{x}_i,\ldots), \mathit{sup}\, f_i^4(\ldots,\hat{x}_i,\ldots)].$$

Each permutation corresponds to some inequalities. For example permutation (2 1 4 3 5) represents the inequalities

$$\begin{aligned} \mathit{inf}\, f_i(\ldots,\hat{x}_i,\ldots) &\le \mathit{sup}\, f_i(\ldots,\hat{x}_i,\ldots) < \mathit{inf}\, f_i^5(\ldots,\hat{x}_i,\ldots) \le \hat{x}_i \\ &\le \mathit{sup}\, f_i^5(\ldots,\hat{x}_i,\ldots) < \mathit{inf}\, f_i^3(\ldots,\hat{x}_i,\ldots) \le \mathit{sup}\, f_i^3(\ldots,\hat{x}_i,\ldots) \\ &< \mathit{inf}\, f_i^2(\ldots,\hat{x}_i,\ldots) \le \mathit{sup}\, f_i^2(\ldots,\hat{x}_i,\ldots) \\ &< \mathit{inf}\, f_i^4(\ldots,\hat{x}_i,\ldots) \le \mathit{sup}\, f_i^4(\ldots,\hat{x}_i,\ldots). \end{aligned}$$

If we denote the set of all permutations of numbers 1, 2, 3, 4 and 5 by the symbol $\mathcal{M}$, then this set splits into two disjoint sets

$$\begin{aligned} \mathcal{M}_0 = \{&(5\ 3\ 2\ 4\ 1), (1\ 4\ 2\ 3\ 5), (2\ 5\ 3\ 4\ 1), (1\ 4\ 3\ 5\ 2), (3\ 1\ 4\ 5\ 2), \\ &(2\ 5\ 4\ 1\ 3), (4\ 2\ 5\ 1\ 3), (3\ 1\ 5\ 2\ 4), (5\ 3\ 1\ 2\ 4), (4\ 2\ 1\ 3\ 5)\} \end{aligned}$$

and

$$\mathcal{M} - \mathcal{M}_0.$$

Now we are able to state the following two theorems:

Theorem 3.1 *Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ and let $\hat{\mathbf{x}} \in \mathbb{R}^n$ be its 5-periodic point. If for all $i = 1, \ldots, n$ one of the inequalities corresponding to one of the permutations from set $\mathcal{M} - \mathcal{M}_0$ hold and if we add appropriate conditions of type (16), then function f has an N-periodic point for all $N \in \mathbb{N}$.*

Remark 3.1 Note that f has a 3-periodic point.

Theorem 3.2 *Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ and let $\hat{\mathbf{x}} \in \mathbb{R}^n$ be its 5-periodic point. If for all $i = 1, \ldots, n$ one of the inequalities corresponding to one of the permutations from set $\mathcal{M}_0$ hold and if we add appropriate conditions of type (16), then function f has an N-periodic point for all $N \in \mathbb{N}$, $5 \rhd N$.*

Remark 3.2 Proofs of Theorems 3.1 and 3.2 are similar to the proofs from the preceding section, they are only a bit more technically demanding.

Analogously as for 3-periodic point we can consider for 5-periodic point for each $i = 1, \ldots, n$ different ordering (i.e. inequalities) and state similar theorems.

4 Fixed point

Theorem 4.1 *Let* $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ *and let* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *be its k-periodic point,* $k \in \mathbb{N}$, $k \neq 1$, *satisfying condition (15). Assume that the condition*

$$\left.\begin{array}{l} \forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1} : dim(Fix(f_i)) = 0 \ , i = 1, \ldots, n \\ \textit{and all elements of } Fix(f_i) \textit{ have nontrivial fixed-point indices,} \\ \textit{defined as the local Brouwer degree of the maps } x_i - f_i \end{array}\right\} \quad (17)$$

holds. Then f *has a fixed point.*

Remark 4.1 In this section we denote

$$f(\ldots, \hat{x}_i, \ldots) \equiv f(x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)$$

to simplify the notation.

Proof of Theorem 4.1 Let $i \in \{1, \ldots, n\}$ and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ be arbitrary but fixed. Let $(m = 1, \ldots, k)$

$$a^i = \max\{f_i^m(\ldots, \hat{x}_i, \ldots); f_i^{m+1}(\ldots, \hat{x}_i, \ldots) > f_i^m(\ldots, \hat{x}_i, \ldots)\}$$

and

$$b^i = \min\{f_i^m(\ldots, \hat{x}_i, \ldots); f_i^{m+1}(\ldots, \hat{x}_i, \ldots) < f_i^m(\ldots, \hat{x}_i, \ldots)\}.$$

It now holds that $f_i(\ldots, a^i, \ldots) > a^i$ and $f_i(\ldots, b^i, \ldots) < b^i$. Then by the Lemma 1.3 there exists a point c^i between a^i and b^i such that $f_i(\ldots, c^i, \ldots) = c^i$.

For each $i = 1, \ldots, n$ we define a multivalued mapping $\varphi_i : \mathbb{R}^n \rightsquigarrow \mathbb{R}$ by

$$\varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = Fix\,(f_i(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n)).$$

Then every φ_i is u.s.c. and under assumption (17) there exists a (single-valued) continuous selection of φ_i (we denote this selection also by φ_i).

The hypersurfaces defined by the continuous mappings φ_i $i = 1, \ldots n$ must intersect at least in one point which is a fixed point of function f. □

Theorem 4.2 *Let* $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ *and let* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *be its k-periodic point,* $k \in \mathbb{N}$, $k \neq 1$, *such that conditions*

$$-\infty < inf f_i^m(\ldots, \hat{x}_i, \ldots) \quad \textit{and} \quad sup\, f_i^m(\ldots, \hat{x}_i, \ldots) < \infty$$

(where infima and suprema are w.r.t. $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$*) are satisfied for all* $i = 1, \ldots, n$ *and all* $m = 1, \ldots k$. *Assume furthermore that condition (17) holds. Then* f *has a fixed point.*

Proof For each $i = 1, \ldots, n$ and $m = 1, \ldots, k$ we denote

$$A_i^m = inf f_i^m(\ldots, \hat{x}_i, \ldots) \quad \text{and} \quad B_i^m = sup f_i^m(\ldots, \hat{x}_i, \ldots).$$

Let $i \in \{1, \ldots, n\}$ and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ be arbitrary but fixed. We denote

$$a^i = \min_{m=1,\ldots k} A_i^m > -\infty, \qquad b^i = \max_{m=1,\ldots k} B_i^m < \infty$$

and

$$I^i = [a^i, b^i].$$

Then $I^i \subset f_i(I^i)$ and by the Lemma 1.3 there exists a point c^i between a^i and b^i such that $f_i(\ldots, c^i, \ldots) = c^i$.

For each $i = 1, \ldots n$ we define a multivalued mapping $\varphi_i : \mathbb{R}^n \rightsquigarrow \mathbb{R}$ by

$$\varphi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \ldots, x_n) = Fix\,(f_i(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n)).$$

Then every φ_i is u.s.c. and under assumption (17) there exists a (single-valued) continuous selection of φ_i (we denote this selection also by φ_i).

The hypersurfaces defined by the continuous mappings φ_i $i = 1, \ldots n$ must intersect at least in one point which is a fixed point of function f. □

5 Concluding remarks

The situation for other periodic points can be discussed and described similarly as for 3-periodic and 5-periodic points. For example for 4-periodic point we (thanks condition (15)) obtain $4! = 24$ different inequalities. Eight of them imply only fixed and 2-periodic point, the rest ensures the existence of k-periodic points for all $k \in \mathbb{N}$. Exploring the 6-periodic point and its $6! = 720$ inequalities would take a long time and so it is no longer profitable to use this procedure for k-periodic points with $k > 5$.

Although the full analogy of the Sharkovskii theorem for multidimensional maps seems to hold (in the spirit of the above investigations), we are not able to prove it technically without stated assumptions (especially condition (15)).

5.1 Multidimensional and multivalued analogy

This remark is inspired by [AJP], where the multivalued versions of the Sharkovskii theorem for M-maps are established.

Definition 5.1 A multivalued function $\varphi : \mathbb{R} \rightsquigarrow \mathbb{R}$ is called an *M-map* if φ is upper-semi-continuous (u.s.c) map whose nonempty sets of values are either single points or compact intervals.

Definition 5.2 Let φ be an M-map. By the *k-orbit* $\mathcal{O}$ of φ we mean a set $\mathcal{O} = \{x_j\}_{j=1}^k$ of points $x_j \in \mathbb{R}$ such that

(i) $x_{j+1} \in \varphi(x_j)$ for all $j = 1, \dots, k$,

(ii) $x_1 = x_{n+1}$,

(iii) $\mathcal{O}$ cannot be formed by going p-times around a shorter m-orbit, where $mp = k$.

If the points x_j, $j = 1, \dots, k$, forming the k-orbit $\mathcal{O}$ are mutually different, then we speak about a *primary k-orbit*.

Theorem 5.3 ([AJP, Theorem 4]) *Let an M-map* $\varphi : \mathbb{R} \rightsquigarrow \mathbb{R}$ *have an n-orbit,* $n \in \mathbb{N}$. *Then* φ *has also a k-orbit, for every* $k \triangleleft n$, *with the exception of at most three orbits.*

Theorem 5.4 ([AJP, Corollary 7]) *Let* $\varphi : \mathbb{R} \rightsquigarrow \mathbb{R}$ *be an M-map and let* φ *have an n-orbit, where* $n = 2^m q$, $m \in \mathbb{N}_0$ *and q is odd, and let n be maximal in the Sharkovskii ordering.*

1. *If* $q > 3$, *the* φ *has a k-orbit, for every* $k \triangleleft n$, *except* $k = 2^{m+2}$.
2. *If* $q = 3$, *the* φ *has a k-orbit, for every* $k \triangleleft n$, *except* $k = 2^{m+1}3, 2^{m+2}, 2^{m+1}$.
3. *If* $q = 1$, *the* φ *has a k-orbit, for every* $k \triangleleft n$.

Proof For the proofs of Theorems 5.3 and 5.4 see [AJP]. □

Now we can consider the function $f : \mathbb{R}^n \to (2^{\mathbb{R}} - \{\emptyset\})^n$ such that for all $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ the functions

$$f_i(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n) : \mathbb{R} \rightsquigarrow \mathbb{R}, \quad i = 1, \dots, n$$

are M-maps.

Definition 5.5 We say that a set

$$\{\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(k)}\},$$

where $\mathbf{X}^{(j)} = \{x_1^{(j)}, \dots, x_n^{(j)}\} \in \mathbb{R}^n$, $j = 1, \dots, k$ is a *k-orbit* of f, if $\forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ and $\forall i = 1, \dots, n$ a set $\mathcal{O}_i = \{x_i^{(1)}, \dots, x_i^{(k)}\}$ is a k-orbit of an M-map $f_i(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)$.

Now we discuss the following situation: let $\mathcal{O} = \{\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(k)}\}$ be a k-orbit of the function f. Let $i \in \{1, \dots, n\}$ and $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ be arbitrary but fixed. We can define a mapping

$$\left.\begin{array}{l} F_i : \mathbb{R} \rightsquigarrow \mathbb{R} \\ F_i(y) = f_i(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \text{ for all } y \in \mathbb{R}. \end{array}\right\} \tag{18}$$

Then this mapping is an M-map and a sequence $\mathcal{O}_i = \{x_i^{(1)}, \dots, x_i^{(k)}\}$ is its k-orbit.

By Theorem 5.3 or 5.4 applied to an M-map F_i there exist m-orbits for every $m \triangleleft k$, with the exception of at most three orbits.

For each $i \in \{1, \dots, n\}$ and for all $m \triangleleft k$ we can define following multivalued mapping on $\mathbb{R}^{n-1}$ by

$$\varphi_{F_i,m}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \mathbf{O}_i^m := \{\mathcal{O}_i^m; \mathcal{O}_i^m \text{ is } m\text{-orbit of } F_i\}.$$

Since each orbit is represented by the set we obtain a set $\mathbf{O}_i^m$ of sets. The sets which represent the orbits are not ordered and therefore we are not able to state further properties of $\varphi_{F_i,m}$.

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