Ondřej Došlý
Oscillatory properties of fourth order Sturm-Liouville differential equations

*Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, Vol. 41 (2002), No. 1, 55--65

Persistent URL: `http://dml.cz/dmlcz/120455`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **41** (2002) 55–65

# Oscillatory properties of fourth order Sturm–Liouville differential equations *

ONDŘEJ DOŠLÝ

*Mathematical Institute, Academy of Sciences of Czech Republic,*
*Žižkova 22, CZ-616 62 Brno, Czech Republic*
*e-mail: dosly@math.muni.cz*

(Received August 20, 2001)

**Abstract**

Oscillatory properties of the fourth order linear differential equation

$$y^{(IV)} - \frac{9}{16t^4} y = q(t) y \tag{$*$}$$

are investigated. Equation $(*)$ is viewed as a perturbation of the (nonoscillatory) equation

$$y^{(IV)} - \left( \frac{9}{16t^4} + \frac{5}{8t^4 \lg^2 t} \right) y = 0$$

and integral conditions on the difference $q(t) - \frac{5}{8t^4 \lg^2 t}$ are given which guarantee that (*) is (non)oscillatory. Some general aspects of this approach to the oscillation theory of higher order Sturm–Liouville equations are discussed.

**Key words:** Sturm–Liouville differential equation, oscillation and nonoscillation criteria, variational method, conditional oscillation.

**2000 Mathematics Subject Classification:** 34C10

---

*Supported partially by grant No. A1019902/199 of The Grant Agency of the Czech Academy of Sciences.

# 1 Introduction

It is a well known fact that the second order linear differential equation with the iterated logarithms

$$y''+\frac{1}{4t^2}\left(1+\frac{1}{\lg^2 t}+\ldots+\frac{1}{\lg^2 t\lg_2^2 t\ldots\lg_{n-1}^2 t}+\frac{\lambda}{\lg^2 t\lg_2^2 t\cdots\lg_n^2 t}\right)y=0, \quad (1)$$

where $\lg_2 t=\lg(\lg t)$, $\lg_n t=\lg(\lg_{n-1} t)$ and lg denotes the natural logarithm, is nonoscillatory if and only if $\lambda\le 1$. This result can be proved applying successively the transformation

$$z(t)=\sqrt{t}\,y(\lg t) \quad (2)$$

to the equation $y''=0$, then to the resulting equation $y''+\frac{1}{4t^2}y=0$, etc., see [3, 12, 16, 17, 19], where also a general background of the transformation theory of second order equations and its application in the oscillation theory of these equations can be found.

However, if we want to study the two-term fourth order equation

$$y^{(IV)}=q(t)y \quad (3)$$

using the transformation

$$z(t)=t^{\frac{3}{2}}y(\lg t) \quad (4)$$

(which is a fourth order analogue of (2)), this approach cannot be used since that transformation (4) applied to the equation $y^{(IV)}=0$ gives an equation with middle terms involving $y''$ and $y'$, so the resulting equation is no longer of the form (3), see [1] and [17] for more details concerning transformations of higher order differential equations.

For this reason, we use here a somewhat different approach, based on the factorization of disconjugate differential operators coupled with the so-called variational principle. This method enables to "detect" the first logarithmic term, namely the fact that the equation

$$y^{(IV)}-\left(\frac{9}{16t^4}+\frac{\gamma}{t^4\lg^2 t}\right)y=0 \quad (5)$$

is nonoscillatory if and only if $\gamma\le\frac{5}{8}$. A subject of the present investigation is to find what is the situation with iterated logarithmic terms, a more detailed remark concerning this problem is given in the last section. Nonoscillation of equation (5) with $\gamma=\frac{5}{8}$ enables to study the equation

$$y^{(IV)}-\frac{9}{16t^4}y=q(t)y \quad (6)$$

as a perturbation of (5) and to obtain a more refined (non)oscillation criteria than known until now. Note that this approach can be regarded as a continuation of the research of [8, 10, 11], where the equation

$$(-1)^n\left(t^\alpha y^{(n)}\right)^{(n)}=q(t)y,\quad \alpha\notin\{1,3,\ldots,2n-1\} \quad (7)$$

is regarded as a perturbation of the Euler-type equation

$$
\begin{aligned}
(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} + \tfrac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0,\\
\gamma_{n,\alpha} := (-1)^n \textstyle\prod_{j=0}^{n-1} (\lambda - j)(\lambda + \alpha - j - n) \mid_{\lambda = \frac{2n-1-\alpha}{2}} .
\end{aligned}
\tag{8}
$$

The paper is organized as follows. In the next section we recall the relationship between higher order Sturm–Liouville differential equations and linear Hamiltonian systems. We also give a statement concerning factorization of disconjugate differential operators and two variational lemmas. Section 3 contains the main results of the paper, oscillation and nonoscillation criterion for (6), where this equation is viewed as a perturbation of (5) with $\gamma = \frac{5}{8}$. The last section is devoted some remarks and comments concerning the results of the paper.

## 2 Auxiliary results

We start this section with basic oscillatory properties of higher order Sturm–Liouville differential equations

$$
L(y) := \sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)} = 0, \quad r_n(t) > 0. \tag{9}
$$

Oscillatory properties of these equations can be investigated within the scope of the oscillation theory of linear Hamiltonian systems (further LHS)

$$
x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \tag{10}
$$

where $A, B, C$ are $n \times n$ matrices with $B, C$ symmetric. Indeed, if $y$ is a solution of (9) and we set

$$
x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)} + \ldots + r_1 y' \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix},
$$

then $(x, u)$ solves (10) with $A, B, C$ given by

$$
B(t) = \operatorname{diag}\{0, \ldots, 0, r_n^{-1}(t)\}, \quad C(t) = \operatorname{diag}\{r_0(t), \ldots, r_{n-1}(t)\},
$$

$$
A = A_{i,j} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \ldots, n-1, \\ 0, & \text{elsewhere.} \end{cases}
$$

In this case we say that the solution $(x, u)$ of (10) *is generated* by the solution $y$ of (9). Moreover, if $y_1, \ldots, y_n$ are solutions of (9) and the columns of the matrix solution $(X, U)$ of (10) are generated by the solutions $y_1, \ldots, y_n$, we say that the solution $(X, U)$ *is generated by the solutions* $y_1, \ldots, y_n$.

Recall that two different points $t_1, t_2$ are said to be *conjugate* relative to system (10) if there exists a nontrivial solution $(x, u)$ of this system such that $x(t_1) = 0 = x(t_2)$. Consequently, by the above mentioned relationship between (9) and (10), these points are conjugate relative to (9) if there exists a nontrivial solution $y$ of this equation such that $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, 1, \ldots, n-1$. System (10) (and hence also equation (9)) is said to be *oscillatory* if for every $T \in \mathbb{R}$ there exists a pair of points $t_1, t_2 \in [T, \infty)$ which are conjugate relative to (10) (relative to (9)), in the opposite case (10) (or (9)) is said to be *nonoscillatory*. If $w$ is a positive function, the equation

$$L(y) = w(t)y \tag{11}$$

with the nonoscillatory operator $L$ given by (9) is said to be *conditionally oscillatory* if there exists $\lambda_0 > 0$ such that (11) with $\lambda w(t)$ instead of $w(t)$ is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The constant $\lambda_0$ is called the *oscillation constant* of (11).

A conjoined basis $(X, U)$ of (10) (i.e. a matrix solution of this system with $n \times n$ matrices $X, U$ satisfying $X^T(t)U(t) = U^T(t)X(t)$ and $\operatorname{rank}(X^T, U^T)^T = n$) is said to be the *principal solution* of (10) if $X(t)$ is nonsingular for large $t$ and for any other conjoined basis $(\bar{X}, \bar{U})$ such that the (constant) matrix $\bar{X}^T U - \bar{U}^T X$ is nonsingular $\lim_{t\to\infty} \bar{X}^{-1}(t)X(t) = 0$ holds. The last limit equals zero if and only if

$$\lim_{t\to\infty} \left( \int^t X^{-1}(s)B(s)X^{T-1}(s)\, ds \right)^{-1} = 0, \tag{12}$$

see [18]. A principal solution of (10) is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix. If $(X, U)$ is the principal solution, any conjoined basis $(\bar{X}, \bar{U})$ such that the matrix $X^T \bar{U} - U^T \bar{X}$ is nonsingular is said to be a *nonprincipal solution* of (10). Solutions $y_1, \ldots, y_n$ of (9) are said to form the *principal (nonprincipal) system of solutions* if the solution $(X, U)$ of the associated linear Hamiltonian system generated by $y_1, \ldots, y_n$ is a principal (nonprincipal) solution.

Using the relation between (9), (10) and the so-called Roundabout Theorem for linear Hamiltonian systems (see e.g. [18]), one can easily prove the following variational lemma which plays a crucial role in our investigation of oscillatory properties of (9).

**Lemma 1** ([13]) *Equation (9) is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that*

$$\mathcal{F}(y; T, \infty) := \int_T^\infty \left[ \sum_{k=0}^n r_k(t)(y^{(k)}(t))^2 \right] dt > 0$$

*for any nontrivial $y \in W^{n,2}(T, \infty)$ with compact support in $(T, \infty)$.*

We also use the following Wirtinger-type inequality.

**Lemma 2** ([14]) *Let $y \in W^{1,2}(T,\infty)$ have compact support in $(T,\infty)$ and let $M$ be a positive differentiable function such that $M'(t) \neq 0$ for $t \in [T,\infty)$. Then*

$$\int_T^\infty |M'(t)|y^2\,dt \le 4\int_T^\infty \frac{M^2(t)}{|M'(t)|}y'^2\,dt.$$

We finish this section with a statement concerning factorization of a class of (formally) self-adjoint operators.

**Lemma 3** ([4]) *Suppose that equation (9) possesses a system of positive solutions $y_1,\dots,y_{2n}$ satisfying $y_i = o(y_{i+1})$, $i=1,\dots,2n-1$, as $t\to\infty$. Then the operator $L$ given by (9) admits for large $t$ the factorization*

$$L(y) = \frac{1}{a_0(t)}\left(\frac{1}{a_1(t)}\left(\cdots\frac{r_n(t)}{a_n^2(t)}\left(\frac{1}{a_{n-1}(t)}\cdots\frac{1}{a_1(t)}\left(\frac{y}{a_0(t)}\right)'\cdots\right)'\cdots\right)'\right)',$$

*where*

$$a_0 = y_1,\ a_1 = \left(\frac{y_2}{y_1}\right)',\ a_i = \frac{W(y_1,\dots,y_{i+1})W(y_1,\dots,y_{i-1})}{W^2(y_1,\dots,y_i)},\ i=1,\dots,n-1$$

*and $a_n = (a_0\cdots a_{n-1})^{-1}$, $W(\cdot)$ being the Wronskian of the functions in brackets.*

**Lemma 4** *For any $y$ sufficiently smooth*

$$y^{(IV)} - \frac{9}{16t^4}y = \frac{1}{t^{\frac{3}{2}}}\left\{t^{1+\frac{\sqrt{10}}{2}}\left[t^{1-\sqrt{10}}\left(t^{1+\frac{\sqrt{10}}{2}}\left(\frac{y}{t^{\frac{3}{2}}}\right)'\right)'\right]'\right\}' \tag{13}$$

*and for any $y \in W_0^{2,2}(T,\infty)$, $T\in\mathbb{R}$,*

$$\int_T^\infty \left[y''^2 - \frac{9}{16t^4}y^2\right]dt = \int_T^\infty t^{1-\sqrt{10}}\left\{\left[t^{1+\frac{\sqrt{10}}{2}}\left(\frac{y}{t^{\frac{3}{2}}}\right)'\right]'\right\}^2 dt. \tag{14}$$

**Proof** The formula (13) follows immediately from Lemma 3 since the fundamental system of solutions of

$$y^{(IV)} - \frac{9}{16t^4}y = 0 \tag{15}$$

is

$$y_1 = t^{\frac{3-\sqrt{10}}{2}},\quad y_2 = t^{\frac{3}{2}},\quad y_3 = t^{\frac{3}{2}}\lg t,\quad y_4 = t^{\frac{3+\sqrt{10}}{2}}$$

and satisfies the assumptions of this lemma. Formula (14) we prove using the integration by parts

$$\int_T^\infty \left[y''^2 - \frac{9}{16t^4}y^2\right]dt = \int_T^\infty y\left(y^{(IV)} - \frac{9}{16t^4}y\right)dt$$

$$= \int_T^\infty y \left( \frac{1}{t^{\frac{3}{2}}} \left\{ t^{1+\frac{\sqrt{10}}{2}} \left[ t^{1-\sqrt{10}} \left( t^{1+\frac{\sqrt{10}}{2}} \left( \frac{y}{t^{\frac{3}{2}}} \right)' \right)' \right]' \right\}' \right) dt$$

$$= \int_T^\infty t^{1-\sqrt{10}} \left\{ \left[ t^{1+\frac{\sqrt{10}}{2}} \left( \frac{y}{t^{\frac{3}{2}}} \right)' \right]' \right\}^2 dt. \qquad \square$$

## 3 Main results

Our main result reeds as follows.

**Theorem 1** *(i) Suppose that $q(t) \geq 0$ for large $t$ and*

$$\int^\infty \left( q(t) - \frac{5}{8t^4 \lg^2 t} \right) t^3 \lg t \, dt = \infty. \tag{16}$$

*Then equation (6) is oscillatory.*

*(ii) If the second order linear differential equation*

$$(tu')' + \frac{2}{5} t^3 q(t) u = 0 \tag{17}$$

*is nonoscillatory, then equation (6) is also nonoscillatory.*

**Proof** (i) Let $T \in \mathbb{R}$ be arbitrary. According to Lemma 1, we need to construct a function $0 \not\equiv y \in W_0^{2,2}(T, \infty)$ such that

$$\mathcal{F}(y; t, \infty) := \int_T^\infty \left[ y''^2 - \left( \frac{9}{16t^4} + q(t) \right) y^2 \right] dt \leq 0. \tag{18}$$

This function we construct as follows. Let $h(t) = t^{\frac{3}{2}} \sqrt{\lg t}$ and $T < t_0 < t_1 < t_2 < t_3$ (these quantities will be specified later). Further, let $f \in C^2[t_0, t_1]$ be any function satisfying $f(t_0) = 0 = f'(t_0)$, $f(t_1) = h(t_1)$, $f'(t_1) = h'(t_1)$ and let $g$ be the solution of the (15) satisfying

$$g(t_2) = h(t_2), \; g'(t_2) = h'(t_2), \; g(t_3) = 0 = g'(t_3). \tag{19}$$

Now define the function $y \in W_0^{2,2}(T, \infty)$ as follows

$$y = \begin{cases} 0, & t \leq t_0, \\ f(t), & t_0 \leq t \leq t_1, \\ h(t), & t_1 \leq t \leq t_2, \\ g(t), & t_2 \leq t \leq t_3, \\ 0, & t \geq t_3. \end{cases}$$

Denote

$$K := \int_{t_0}^{t_1} \left[ f''^2 - \left( \frac{9}{16t^4} + q(t) \right) f^2 \right] dt.$$

We have

$$h''^2 = t^{-1}\left\{\frac{9}{16}\lg t + \frac{3}{2} + \frac{5}{8\lg t} - \frac{1}{2\lg^2 t} + \frac{1}{16\lg^3 t}\right\}$$

and hence

$$\int_{t_1}^{t_2}\left(h''^2 - \frac{9}{16t^4}h^2\right) dt = \frac{3}{2}\lg t_2 + \frac{5}{8}\int_{t_1}^{t_2}\frac{dt}{t\lg t} + L + O\left(\lg^{-1} t\right),$$

where $L$ is a real constant. Using the fact that $q(t) \geq 0$ for large $t$, if $t_2, t_3$ are sufficiently large, then

$$\int_{t_2}^{t_3}\left[g''^2 - \left(\frac{9}{16t^4} + q(t)\right)g^2\right] dt \leq \int_{t_2}^{t_3}\left[g''^2 - \frac{9}{16t^4}g^2\right] dt.$$

To estimate the last integral we use the relationship between (9) and (10). Denote $x = \binom{g}{g'}$, $u = \binom{-g'''}{g''}$, $\tilde{h} = \binom{h}{h'}$ and let $A, B, C$ be $2\times 2$ matrices of LHS (10) associated with (15). Then

$$\begin{aligned}\int_{t_2}^{t_3}\left[g''^2 - \frac{9}{16t^4}g^2\right] dt &= \int_{t_2}^{t_3}\left[u^T B(t)u + x^T C(t)x\right] dt\\ &= \int_{t_2}^{t_3}\left[u^T(x' - Ax) + x^T C(t)x\right] dt\\ &= u^T x\Big|_{t_2}^{t_3} + \int_{t_2}^{t_3} x^T\left[-u' - A^T u + C(t)x\right] dt\\ &= -u^T(t_2)x(t_2).\end{aligned}$$

Let $(X, U)$ be the principal solution of the LHS associated with (15), i.e., this solution is generated by $y_1 = t^{\frac{3-\sqrt{10}}{2}}$, $y_2 = t^{\frac{3}{2}}$. Using the fact that

$$\begin{aligned}\bar{X}(t) &= X(t)\int_t^{t_3} X^{-1}BX^{T-1}\, ds,\\ \bar{U}(t) &= U(t)\int_t^{t_3} X^{-1}BX^{T-1}\, ds - X^{T-1}(t)\end{aligned}$$

is also a conjoined basis of LHS associated with (15), according to boundary conditions (19)

$$\begin{aligned}x(t) &= X(t)\int_t^{t_3} X^{-1}BX^{T-1}\, ds\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\, ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2),\\ u(t) &= \left[U(t)\int_t^{t_3} X^{-1}BX^{T-1}\, ds - X^{T-1}(t)\right]\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\, ds\right)^{-1}\\ &\quad\times X^{-1}(t_2)\tilde{h}(t_2)\end{aligned}$$

and hence

$$-u^T(t_2)x(t_2) = \tilde h^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde h(t_2)$$
$$- \tilde h^T(t_2)U(t_2)X^{T-1}(t_2)\tilde h(t_2).$$

Since $(X,U)$ is the principal solution, the first term on the right-hand-side of the last expression tends to zero as $t_3 \to \infty$ ($t_2$ being fixed). Concerning the second term, since

$$X(t) = \begin{pmatrix} t^{\frac{3-\sqrt{10}}{2}} & t^{\frac{3}{2}} \\ \frac{3-\sqrt{10}}{2}t^{\frac{1-\sqrt{10}}{2}} & \frac{3}{2}t^{\frac{1}{2}} \end{pmatrix}, \qquad U(t) = \begin{pmatrix} \frac{27-9\sqrt{10}}{8}t^{-\frac{3+\sqrt{10}}{2}} & \frac{3}{8}t^{-\frac{3}{2}} \\ \frac{13-4\sqrt{10}}{4}t^{-\frac{1+\sqrt{10}}{2}} & \frac{3}{4}t^{-\frac{1}{2}} \end{pmatrix},$$
$$\tilde h = \begin{pmatrix} t^{\frac{3}{2}}\sqrt{\lg t} \\ \frac{t^{\frac{1}{2}}}{2}\left[\sqrt{\lg t} + \frac{1}{2\sqrt{\lg t}}\right] \end{pmatrix},$$

we have by a direct computation

$$\tilde h^T UX^{-1}\tilde h(t) = \frac{2}{\sqrt{10}}\left\{\frac{30\sqrt{10}-90}{16}\lg t + \frac{30-9\sqrt{10}}{8}(3\lg t + 1)\right.$$
$$\left. + \frac{4\sqrt{10}-10}{4}\left(\frac{9}{4}\lg t + \frac{3}{2} + \frac{1}{4\lg t}\right)\right\} = \frac{3}{2}\lg t + \frac{3}{4} + \frac{4-\sqrt{10}}{16\lg t}.$$

Summarizing the above computations

$$\mathcal{F}(y;t_0,t_3) \le K + \frac{3}{2}\lg t_2 + \frac{5}{8}\int_{t_1}^{t_2}\frac{dt}{t\lg t} + L + O(\lg^{-1} t_2) - \int_{t_1}^{t_2} q(t)t^3\lg t\,dt$$
$$+ \tilde h^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde h(t_2)$$
$$- \left[\frac{3}{2}\lg t_2 + \frac{3}{4} + O(\lg^{-1} t_2)\right].$$

Now, let $t_2 > t_1$ be such that the following conditions are satisfied:

(a) $\int_{t_1}^{t_2}\left(q(t) - \frac{5}{8t^4\lg^2 t}\right)t^3\lg t\,dt > K + L + 1$,

(b) The sum of all terms $O(\lg^{-1} t_2)$ is less than 1,

and let $t_3 > t_2$ be such that

$$\tilde h(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde h(t_2) < \frac{3}{4}.$$

For $t_2, t_3$ chosen in this way we have

$$\mathcal{F}(y;T,\infty) \le K - (K+L+1) + L - \frac{3}{4} + \frac{3}{4} + 1 \le 0.$$

Therefore, by Lemma 1 equation (6) is oscillatory.

(ii) Let $T \in \mathbb{R}$ and $y \in W_0^{2,2}(T,\infty)$ be arbitrary. According to Lemma 4 and the Wirtinger inequality (Lemma 2)

$$\int_T^\infty \left[y''^2 - \tfrac{9}{16t^4}y^2\right] dt = \int_T^\infty t^{1-\sqrt{10}} \left\{\left[t^{1+\frac{\sqrt{10}}{2}}\left(\tfrac{y}{t^{\frac{3}{2}}}\right)'\right]'\right\}^2 dt$$

$$\geq \tfrac{10}{4}\int_T^\infty t^{-1-\sqrt{10}}\left[t^{1+\frac{\sqrt{10}}{2}}\left(\tfrac{y}{t^{\frac{3}{2}}}\right)'\right]^2 dt = \tfrac{5}{2}\int_T^\infty t\left[\left(\tfrac{y}{t^{\frac{3}{2}}}\right)'\right]^2 dt.$$

Denote $u = yt^{-\frac{3}{2}}$. Since the second order equation (17) is nonoscillatory, by Lemma 1 we have

$$\int_T^\infty \left[tu'^2 - \frac{2}{5}t^3 q(t)u^2\right] dt > 0$$

if $T$ is sufficiently large. Consequently,

$$\int_T^\infty \left[y''^2 - \left(\tfrac{9}{16t^4} + q(t)\right)y^2\right] dt \geq \tfrac{5}{2}\int_T^\infty \left\{t\left[\left(\tfrac{y}{t^{\frac{3}{2}}}\right)'\right]^2 - \tfrac{2}{5}t^3 q(t)\left(\tfrac{y}{t^{\frac{3}{2}}}\right)^2\right] dt > 0$$

and this means, again by Lemma 1, that equation (6) is nonoscillatory. □

**Corollary 1** *The equation*

$$y^{(IV)} - \left(\frac{9}{16t^4} + \frac{\gamma}{t^4 \lg^2 t}\right) y = 0 \tag{20}$$

*is nonoscillatory if and only if* $\gamma \leq \frac{5}{8}$.

**Proof** The statement follows immediately from Theorem 1. Indeed, if $\gamma > \frac{5}{8}$, condition (16) reads $\int^\infty \frac{\gamma - \frac{5}{8}}{t \lg t}\, dt = \infty$. Since the second order equation $(tu')' + \frac{\gamma}{t \lg^2 y}u = 0$ is nonoscillatory for $\gamma \leq \frac{1}{4}$, we have nonoscillation of (20) for $\gamma \leq \frac{5}{8}$. □

## 4 Remarks

(i) Recall that the equation

$$L(y) = w(t)y \tag{21}$$

with $w > 0$ and the operator $L$ given by (9) is said to be *conditionally oscillatory* if there exists $\lambda_0 > 0$ such that (21) with $\lambda w$ instead of $w$ is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. Equation (20) is a typical example of conditionally oscillatory equation with $L(y) = y^{(IV)} - \frac{9}{16t^4}y$, $w(t) = \frac{1}{t^4 \lg^2 t}$ and $\lambda_0 = \frac{5}{8}$. Conditionally oscillatory equations play an important role in the spectral theory of differential operators generated by symmetric differential expressions, see [2, 5, 6, 8].

(ii) The second part of Theorem 1 claims that (6) is nonoscillatory provided the second order differential equation (17) has this property. To examine nonoscillation of (17), one can use various nonoscillation criteria. For example, it is possible to see this equation as a perturbation of the nonoscillatory second order equation

$$(tu')' + \frac{1}{4t\lg^2 t}u = 0, \tag{22}$$

i.e., we rewrite (17) into the form

$$(tu')' + \frac{1}{4t\lg^2 t}u + \left(\frac{2}{5}t^3 q(t) - \frac{1}{4t\lg^2 t}\right)u = 0. \tag{23}$$

Principal and nonprincipal solutions of (22) are $\sqrt{\lg t}$ and $\sqrt{\lg}\lg(\lg t)$, respectively, and according to the Nehari nonoscillation criterion applied to (23) equation (17) is nonocillatory if

$$\lim_{t\to\infty} \lg(\lg t)\int_t^\infty \left(\frac{2}{5}s^3 q(s) - \frac{1}{4s\lg^2 s}\right)_+ \lg s\, ds < \frac{1}{4}, \tag{24}$$

where $(\cdot)_+$ denotes the nonnegative part ($\max\{0,\cdot\}$) of the function indicated. In particular, if

$$q(t) = \frac{\gamma}{t^4\lg^2 t}\left(1 + \frac{1}{\lg^2(\lg t)}\right)$$

with $\gamma \le 5/8$ then equation (24) is nonoscillatory (for $\gamma = 5/8$ it follows from the fact that (1) with $n = 2$ is nonoscillatory for $\lambda \le 1$). Consequently, the equation

$$y^{(IV)} - \left[\frac{9}{16t^4} + \frac{\gamma}{t^4\lg^2}\left(1 + \frac{1}{\lg^2(\lg t)}\right)\right]y = 0 \tag{25}$$

is nonoscillatory for $\gamma \le 5/8$. This statement suggests two open problems:

- Is (25) oscillatory for $\gamma > 5/8$?
- Can one continue with addition of the terms with iterated logarithmic terms to get a conditionally oscillatory equation like (1) in case of second order equations?

(iii) In our paper we investigate fourth order equations since we are able to compute explicitly solutions of (15). It is an open problem whether our approach extends to the higher order equation $(-1)^n y^{(2n)} = q(t)y$. In particular, we conjecture that there exists a positive constant $\mu_n$ such that the equation

$$(-1)^n y^{(2n)} - \left\{\frac{[(2n-1)!!]^2}{4^n t^{2n}} + \frac{\lambda}{t^{2n}\lg^2 t}\right\}y = 0$$

is oscillatory if and only if $\lambda > \mu_n$. This problem, including the computation of the exact value of the constant $\mu_n$ is a subject of the present investigation.

## References


[1] Ahlbrandt, C. D., Hinton, D. B., Lewis, R. T.: *The effect of variable change on oscillation and disconjugacy criteria with applications to spectral theory and asymptotic theory.* J. Math. Anal. Appl. **81** (1981), 234–277.

[2] Ahlbrandt, C. D., Hinton, D. B., Lewis, R. T.: *Necessary and sufficient conditions for the discreteness of the spectrum of certain singular differential operators.* Canad. J. Math. **33** (1981), 229–246.

[3] Borůvka, O.: Linear Differential Transformations of the Second Order. *Oxford Univ. Press, London,* 1971.

[4] Coppel, W. A.: *Disconjugacy.* Lectures Notes in Math., No. 220, Springer Verlag, Berlin-Heidelberg, 1971.

[5] O. Došlý, *Oscillation criteria and the discreteness of the spectrum of self-adjoint, even order, differential operators*, Proc. Roy. Soc. Edinburgh **119A** (1991), 219–232.

[6] Došlý, O.: *Conditionally oscillatory equations and spectral properties of singular differential operators.* In: Proc. Conf. Ordinary Diff. Equations, Poprad 1994, 23–31.

[7] Došlý, O.: *Nehari-type oscillation criteria for self-adjoint linear differential equations.* J. Math. Anal. Appl. **182** (1994), 69–89.

[8] Došlý, O.: *Oscillation and spectral properties of a class of singular self-adjoint differential operators.* Math. Nachr. **188** (1997), 49–68.

[9] Došlý, O., Hilscher, R.: *A class of Sturm–Liouville difference equations: (non)oscillation constants and property BD.* (to appear in Comput. Math. Appl.).

[10] Došlý, O., Osička, J.: *Kneser-type oscillation criteria for self-adjoint, two term, differential equations.* Georgian J. Math. **2** (1995), 241–258.

[11] Došlý, O., Osička, J.: *Oscillation and nonoscillation of higher order self-adjoint differential equations.* (to appear in Czech. Math. J.).

[12] Hartman, P.: Ordinary Differential Equations. *John Wiley, New York,* 1964.

[13] Glazman, I. M.: Direct Methods of Qualitative Analysis of Singular Differential Operators. *Davey, Jerusalem,* 1965.

[14] Hinton, D. B., Lewis, R. T.: *Discrete spectra criteria for singular differential operators with middle terms.* Math. Proc. Cambridge Philos. Soc. **77** (1975), 337–347.

[15] Hinton, D. B., Lewis, R. T.: *Singular differential operators with spectra discrete and bounded below.* Proc. Roy. Soc. Edinburgh **84A** (1979), 117–134.

[16] Kalas, J., Ráb, M.: Ordinary Differential Equations. *Masaryk University Press, Brno,* 1995 (in Czech).

[17] Neuman, F.: Global Properties of the Linear Ordinary Differential Equations. *Kluwer Acad. Publ., Dordrecht–Boston–London,* 1991.

[18] Reid, W. T.: Sturmian Theory for Ordinary Differential Equations. *Springer Verlag, New York–Heidelberg–Berlin,* 1980.

[19] Swanson, C. A.: Comparison and Oscillation Theory of Linear Differential Equation. *Acad. Press, New York,* 1968.