PavelŽenčák

The convex interpolation of histogram by polynomial splines: The existence theorem

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 41 (2002), No. 1, 175--182

Persistent URL: `http://dml.cz/dmlcz/120449`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **41** (2002) 175–182

The Convex Interpolation of Histogram by Polynomial Splines: The Existence Theorem

Pavel Ženčák

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: zencak@risc.upol.cz

(Received February 9, 2002)

Abstract

This paper treats the interpolation of convex histogram by strictly convex polynomial splines of arbitrary degree. First the existence theorem for some special problem is proved. Finally the general existence theorem for splines of order k on $k-1$ fold refined mesh is presented.

Key words: Strict convexity, polynomial splines, interpolation, histogram.

2000 Mathematics Subject Classification: 41A15, 65D05

1 Introduction

In various applications it is often necessary to construct a smooth function that interpolates prescribed data and preserves some shape properties of them. Though many papers were devoted to such problems in the last years only few papers treats the problems of convexity (or concavity) preserving interpolation of histogram. Some of such results based on the existence of so called *histogram in convex position* are given in [2] and [3]. But the convexity criterion based on the existence of *histogram in convex position* can eliminate some cases of strictly convex data because it is only sufficient condition of existence of strictly convex interpolation function and is not necessary. Moreover only the proof of existence of the convex rational splines is given there and there is not any result for polynomial spline. In [4] the necessary and sufficient condition of histogram convexity was stateded and using it the existence of some low order splines on certain refined meshes was shown in [5]. The aim of this contribution is to prove the existence of strictly convex interpolatory spline of order k on $k-1$ fold refined mesh.

2 Some fundamentals of B-splines

Definition 2.1 Let $(\Delta t) = \{t_i\}$ be a nondecreasing sequence (which may be finite, infinite or biinfinite). The i-th normalized B-spline of order k (i.e. degree $k-1$) for knot sequence (Δt) denoted $B_i^k(x)$ is defined by the rule

$$B_i^k(x) = (t_{i+k} - t_i)[t_i, t_{i+1}, \ldots, t_{i+k}](t-x)_+^{k+1}. \tag{1}$$

Theorem 2.2 *The B-spline $B_i^k(x)$ has the following properties:*

$$\begin{array}{rl}
B_i^k(x) = 0 & \text{for } x \notin [t_i, t_{i+k}] \\
B_i^k(x) \geq 0 & \text{for } x \in [t_i, t_{i+k}] \\
B_i^k(x) > 0 & \text{for } x \in (t_i, t_{i+k}) \\
B_i^k(t_i) = 0 & \text{if } t_i < t_{i+k-1} \\
B_i^k(t_i) = 1 & \text{if } t_i = t_{i+1} = \ldots = t_{i+k-1} < t_{i+k} \\
B_i^k(t_{i+k}) = 0 & \text{if } t_{i+1} < t_{i+k} \\
B_i^k(t_{i+k}) = 1 & \text{if } t_i < t_{i+1} = t_{i+2} = \ldots = t_{i+k}
\end{array}$$

For proof see [1].

Theorem 2.3 *Let $S(x) = \sum_i b_i B_i^k(x)$ be some spline on the mesh (Δt). Then its derivatives can be computed by following rules:*

$$S^{(j)}(x) = \sum_i b_i^j B_i^{k-j}(x) \tag{2}$$

with

$$b_i^j = \begin{cases} b_i & \text{for } j = 0 \\ (k-j)\dfrac{b_i^{j-1} - b_{i-1}^{j-1}}{t_{i+k-j} - t_i} & \text{for } 0 < j < k \end{cases} \tag{3}$$

For proof see [1].

3 Convexity of histogram

Let us have given histogram $G = \{g_i\}_{i=0}^n$ on the mesh

$$(\Delta x): \quad x_0 < x_1 < \ldots < x_n < x_{n+1}, \quad \text{with } h_i = x_{i+1} - x_i,\ i = 0(1)n.$$

The convexity of histogram is defined by following way:

Definition 3.1 We say that histogram G is convex if there is convex continuous function f interpolating histogram G (i.e. $\int_{x_i}^{x_{i+1}} f(x)\,dx = h_i g_i$ for $i = 0(1)n$) on the mesh (Δx).

Let us denote $(\Delta^\alpha x) = \{x_i\}_{i=0}^{n+1} \cup \{x_i + \alpha_i h_i\}_{i=0}^{n}$ with $0 < \alpha_i < 1$ and let $S_{11}(\Delta^\alpha x)$ be space of linear splines on the refined mesh $(\Delta^\alpha x)$. Then the previous definition of convexity is equivalent to the following condition:

Theorem 3.2 (Necessary and sufficient criterion of convexity) *Histogram G is convex if and only if there is set of numbers $\{\alpha_i\}_{i=0}^{n}$ and corresponding convex function $S_{11}(x) \in S_{11}(\Delta^\alpha x)$ which interpolates histogram G.*

For proof see [4].

4 The interpolation by splines of order k

4.1 The convexity in some special subproblem

Let y_i, $i = 1(1)k-1$ (where $k > 3$) be free parameters satisfying the following conditions:

$$0 < y_1 < y_2 < \ldots < y_{k-2} < y_{k-1} < 1. \tag{4}$$

Let us define the mesh $(\Delta t) = \{t_i;\, i = 0(1)3k-2\}$ with:

$$t_i = \begin{cases} 0 & \text{for } i \in \{0, 1, \ldots k-1\} \\ y_{i-k+1} & \text{for } i \in \{k, k+1, \ldots 2k-2\} \\ 1 & \text{for } i \in \{2k-1, 2k, \ldots 3k-2\} \end{cases}$$

Let s, m, M_L, M_P and g be any numbers. Let us have given the following interpolation problem on the interval $[0,1]$: *Find spline*

$$S(x) = \sum_{i=0}^{2k-2} b_i B_i^k(x)$$

of order $k > 3$ on the mesh (Δt) such that:

$$\int_0^1 S(x)\,dx = g, \tag{5}$$

$$S(0) = s,\ S'(0) = m,\ S''(0) = M_L,\ S^{(j)}(0) = 0,\ \forall j \in \{3, 4, \ldots k-2\} \tag{6}$$

$$S''(1) = M_P,\ S^{(j)}(1) = 0,\ \forall j \in \{0, 1, \ldots k-2\} \setminus \{2\} \tag{7}$$

Lemma 4.1 *Let be given any numbers s, m, M_L, M_P, g and any mesh (Δt) then there exists spline $S(x)$ satisfying interpolation conditions (5)–(7) and its* *coefficients fulfil the following rules:*

$$b_i = s + \frac{m}{k-1}\sum_{j=1}^{i} y_j + \frac{M_L}{(k-1)(k-2)}\sum_{j=1}^{i}\sum_{l=1}^{j-1} y_j y_l, \; i = 0(1)k-2, \tag{8}$$

$$b_i = \frac{M_P}{(k-1)(k-2)}\sum_{j=1}^{2k-2-i} (1 - y_{k-j-1}), \quad i = k(1)2k-2, \tag{9}$$

$$\begin{aligned} b_{k-1} = {} & kg - s\sum_{i=0}^{k-2} y_{i+1} - \frac{m}{k-1}\sum_{i=0}^{k-2}\sum_{j=1}^{i} y_j y_{i+1} - \frac{M_L}{(k-1)(k-2)}\sum_{i=0}^{k-2}\sum_{j=1}^{i}\sum_{l=1}^{j-1} y_l y_j y_{i+1} \\ & - \frac{M_P}{(k-1)(k-2)}\sum_{i=k}^{2k-2}\sum_{j=1}^{2k-2-i} (1 - y_{k-j-1})(1 - y_{i+1-k}) \end{aligned} \tag{10}$$

Proof 1. Using Theorem 2.2 we transform the conditions (6) to the conditions:

$$b_0 = s, \; b_1^1 = m, \; b_2^2 = M_L, \; b_i^i = 0 \; \text{ for } i = 3(1)k-2.$$

Then using Theorem 2.3 we obtain the following equations:

$$\begin{aligned} s &= b_0, \\ m &= (k-1)\frac{(b_1 - b_0)}{y_1}, \\ M_L &= (k-1)(k-2)\frac{\frac{(b_2-b_1)}{y_2} - \frac{(b_1-b_0)}{y_1}}{y_1}, \\ 0 &= \frac{\frac{(b_i-b_{i-1})}{y_i} - \frac{(b_{i-1}-b_{i-2})}{y_{i-1}}}{y_{i-1}} - \frac{\frac{(b_{i-1}-b_{i-2})}{y_{i-1}} - \frac{(b_{i-2}-b_{i-3})}{y_{i-2}}}{y_{i-2}}, \quad i = 3(1)k-2. \end{aligned}$$

Solving this system we obtain the formulas (8) for b_i, $i = 0(1)k-2$.

2. The formulas (9) for computing b_i, $i = k(1)2k-2$ we obtain in the similar way from the conditions (7).

3. Using the Theorem 2.3 we can obtain the following B-spline coefficients of the antiderivative of $S(x)$:

$$b_i^{-1} = b_{i-1}^{-1} + b_i(t_{i+k} - t_i)/k, \quad i = 0(1)2k-2$$

Substituing it to the interpolation condition (5) we obtain the equation:

$$\sum_{j=0}^{2k-2}\Big(\sum_{i=0}^{j} b_i(t_{i+k} - t_i)/k\Big)\Big(B_j^k(1) - B_j^k(0)\Big) = g.$$

Substituting other computed coefficients and solving this equation we get the formula (10) for b_{k-1}. □

Theorem 4.2 *Let be given numbers $s > 0$, $m < -s$ and $g_m = -s^2/2m$. Then there exist numbers $M_L > 0$, $M_P > 0$, $\eta > 0$ and numbers y_i, $i = 1(1)k-1$ determining mesh (Δt) such that spline $S(x)$ satisfying interpolation conditions (5)–(7) is strictly convex for every $g \in (g_m, g_m + \eta)$.*

Proof The necessary and sufficient conditions of strict convexity are

$$S''(x) = \sum_{j=2}^{2k-2} b_j^2 B_j^{k-2}(x) > 0.$$

Using the nonnegativity of B-spline we get the following sufficient conditions of convexity: $b_j^2 \geq 0$, $\forall j = 2(1)2k-2$. To satisfy the strict convexity, then for each x there must exist the index i such that $x \in \operatorname{supp} B_i^{k-2}$ and $b_i^2 > 0$. It gives the following sufficient conditions of strict convexity:

$$b_j^2 \geq 0,\ j = 2(1)2k-2,\ b_2^2 > 0,\ b_{2k-2}^2 > 0,\ b_{k-1}^2 > 0,\ b_k^2 > 0,\ b_{k+1}^2 > 0. \quad (11)$$

In the rest of this proof we will show that B-spline coefficient of spline $S(x)$ satisfy the previous sufficient conditions (11). Substituting the known formulas (8)–(10) for b_j in the rule (3) we can compute simple formulas for some coefficients b_j^2:

$$\begin{aligned} b_j^2 &= \frac{M_L}{(k-1)(k-2)}, \quad \text{for } j = 2(1)k-2, \\ b_j^2 &= 0, \quad \text{for } j = k+2(1)2k-3, \\ b_{2k-2}^2 &= \frac{M_P}{(k-1)(k-2)(1-y_{k-1})}. \end{aligned}$$

If $M_L > 0$ and $M_P > 0$ than these coefficients satisfy the sufficient conditions (11). Let us denote

$$\begin{aligned} c_{k-1} &= \left(4g - s\left(1 + \sum_{i=0}^{k-2} y_{i+1}\right) - \frac{m}{k-1}\left(\sum_{i=0}^{k-2}\sum_{j=1}^{i} y_j y_{i+1} + \sum_{j=1}^{k-2} y_j\right)\right) \Big/ y_{k-1}, \\ c_k &= -4g\left(\frac{1}{y_{k-1}} + \frac{1}{1-y_1}\right) + s\left(\frac{1}{y_{k-1}} + \left(\frac{1}{y_{k-1}} + \frac{1}{1-y_1}\right)\sum_{i=0}^{k-2} y_{i+1}\right) \\ &\quad + \frac{m}{k-1}\left(\frac{1}{y_{k-1}}\sum_{j=1}^{k-2} y_j + \left(\frac{1}{y_{k-1}} + \frac{1}{1-y_1}\right)\sum_{i=0}^{k-2}\sum_{j=1}^{i} y_j y_{i+1}\right), \\ c_{k+1} &= \left(4g - s\sum_{i=0}^{k-2} y_{i+1} - \frac{m}{k-1}\sum_{i=0}^{k-2}\sum_{j=1}^{i} y_j y_{i+1}\right) \Big/ \left(1 - y_1\right). \end{aligned}$$

Then the remaining coefficients can be written in the following form:

$$b_{k-1}^2 = c_{k-1} - \frac{M_L}{(k-1)(k-2)}\left(\sum_{i=0}^{k-2}\sum_{j=1}^{i}\sum_{l=1}^{j-1} y_l y_j y_{i+1} + \sum_{j=1}^{i}\sum_{l=1}^{j-1} y_j y_l + y_{k-1}\right)$$

$$
- \frac{M_P}{(k-1)(k-2)} \sum_{i=k}^{2k-2} \sum_{j=1}^{2k-2-i} (1-y_{k-j-1})(1-y_{i+1-k})\Big) \Big/ y_{k-1},
$$

$$
\begin{aligned}
b_k^2 &= c_k + \frac{M_L}{(k-1)(k-2)} \Bigg(\Big(\frac{1}{y_{k-1}} + \frac{1}{1-y_1} \Big) \sum_{i=0}^{k-2} \sum_{j=1}^{i} \sum_{l=1}^{j-1} y_l y_j y_{i+1} \\
&+ \frac{1}{y_{k-1}} \sum_{j=1}^{i} \sum_{l=1}^{j-1} y_j y_l \Bigg) + \frac{M_P}{(k-1)(k-2)} \Bigg(\frac{1}{1-y_1} \sum_{j=1}^{k-2} (1-y_{k-j-1}) \\
&+ \Big(\frac{1}{y_{k-1}} + \frac{1}{1-y_1} \Big) \sum_{i=k}^{2k-2} \sum_{j=1}^{2k-2-i} (1-y_{k-j-1})(1-y_{i+1-k}) \Bigg), \\
b_{k+1}^2 &= c_{k+1} - \frac{M_L}{(k-1)(k-2)} \sum_{i=0}^{k-2} \sum_{j=1}^{i} \sum_{l=1}^{j-1} y_l y_j y_{i+1} \\
&- \frac{M_P}{(k-1)(k-2)} \Bigg(\sum_{i=k}^{2k-2} \sum_{j=1}^{2k-2-i} (1-y_{k-j-1})(1-y_{i+1-k}) \\
&+ \sum_{j=1}^{k-2} (1-y_{k-j-1}) - (1-y_1) \Bigg) \Bigg/ \Big(1-y_1 \Big).
\end{aligned}
$$

Properties (4) of numbers y_i imply that conditions $b_{k-1}^2 > 0$, $b_k^2 > 0$, $b_{k+1}^2 > 0$ are equivalent to conditions $c_{k-1} > 0$, $c_k \geq 0$, $c_{k+1} > 0$, if numbers M_L, M_P are positive and sufficiently small.

Let be

$$
\alpha = \frac{-s}{m}, \qquad \delta = g_m - g, \qquad y_i = \alpha - \epsilon + \frac{2(i-1)\epsilon}{k-2}, \text{ for } i = 1(1)k-1,
$$

where $0 < \epsilon < \min\{\alpha, 1-\alpha\}$. Substituting it to the inequalities $c_{k-1} > 0$, $c_k \geq 0$, $c_{k+1} > 0$ and solving this inequalities we obtain that $\delta \in (A(\epsilon), B(\epsilon)]$, where

$$
\begin{aligned}
A(\epsilon) &= \frac{-\epsilon^2 m}{6(k-2)}, \\
B(\epsilon) &= \Big(-\epsilon \Big(\epsilon k^2 + 2\epsilon^2 k^2 - 2k\epsilon^2 + 5\epsilon k + 6k - 12 - 12\epsilon \Big) m^2 \\
&\quad + 6\,(k-2)\, s^2 + 6(k-2) sm \Big) \Big/ \Big(6mk(k-1)(k-2)(2\epsilon+1) \Big).
\end{aligned}
$$

It is simple to verify that assumptions of theorem imply

$$
A(\epsilon) > 0, \qquad B(\epsilon) - A(\epsilon) = \frac{(m+s+\epsilon m)(s-\epsilon m)}{(2\epsilon+1)(k-1)mk} > 0, \qquad \lim_{\epsilon \to 0+} A(\epsilon) = 0.
$$

Therefore the conditions $c_{k-1} > 0$, $c_k \geq 0$ and $c_{k+1} > 0$ are satisfied for all $\delta \in (0, \eta]$, where $\eta = \min\{B(\epsilon) : 0 < \epsilon < \min\{\alpha, 1-\alpha\}\}$. It implies the statement of theorem. $\square$

4.2 The general existence theorem

Let us have given convex histogram G on the mesh

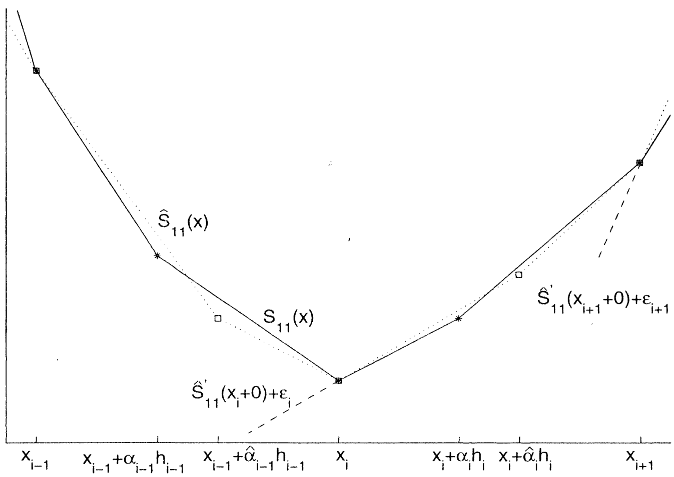
$$(\Delta x): \quad x_0 < x_1 < \ldots < x_n < x_{n+1}, \quad \text{with } h_i = x_{i+1} - x_i, \ i = 0(1)n.$$

Let us denote

$$S_{11}(x_i - 0) = \lim_{x \to x_i -} S_{11}(x),$$
$$S_{11}(x_i + 0) = \lim_{x \to x_i +} S_{11}(x).$$

Lemma 4.3 *Let G be convex histogram such that there exists convex linear spline $S_{11}(x) \in S_{11}(\Delta^\alpha x)$ interpolating G and satisfying $S'_{11}(x_i+0) < S'_{11}(x_{i+1}-0)$ for all $i = 0(1)n$ and $S'_{11}(x_{i+1} - 0) < S'_{11}(x_{i+1} + 0)$ for all $i = 0(1)n-1$. Then numbers ε_i ($\forall i = 0(1)n-1$) can be found such that there exists convex spline $\hat{S}_{11}(x) \in S_{11}(\Delta^{\hat{\alpha}} x)$ (with $\hat{\alpha}_i > \alpha_i$) satisfying $\hat{S}'_{11}(x_i+0) - \hat{S}_{11}(x_i-0) = \epsilon_i$ for every $0 \le \epsilon_i \le \varepsilon_i$.*

Proof First we will show that if $\varepsilon_{i+1} > 0$ than there exists $\varepsilon_i > 0$ and convex $\hat{S}_{11}(x)$ such that $\hat{S}'_{11}(x_i + 0) - \hat{S}_{11}(x_i - 0) = \epsilon_i$ for every $0 \le \epsilon_i \le \varepsilon_i$. Let be given any $\varepsilon_{i+1} > 0$ and let $\hat{S}_{11}(x) \in S_{11}(\Delta^\alpha x)$ satisfy the following conditions $\hat{S}_{11}(x_i) = S_{11}(x_i)$, $\hat{S}_{11}(x_{i+1}) = S_{11}(x_{i+1})$, $\hat{S}'_{11}(x_{i+1} - 0) = S'_{11}(x_{i+1} - 0) + \epsilon_{i+1}$ where $0 \le \epsilon_{i+1} \le \varepsilon_{i+1}$. Then from the interpolation condition $\int_{x_i}^{x_{i+1}} \hat{S}_{11}(x)\,dx = h_i g_i$ we obtain the condition $\hat{S}'_{11}(x_i + 0) > S'_{11}(x_i + 0) > S'_{11}(x_i - 0)$ and so $\varepsilon_i = \hat{S}'_{11}(x_i + 0) - S'_{11}(x_i - 0)$ satisfies the condition $\varepsilon_i > 0$.



Now choosing any $\varepsilon_{n+1} > 0$ and using previous technique in sequence for all $i = n(-1)0$ the statement of the lemma is proved. □

Theorem 4.4 (The existence theorem) *Let $S_{11}(x) \in S_{11}(\Delta^{\alpha}x)$ interpolating G be convex and satisfy $S'_{11}(x_i+0) < S'_{11}(x_{i+1}-0)$ for all $i = 0(1)n$ and $S'_{11}(x_{i+1}-0) < S'_{11}(x_{i+1}+0)$ for all $i = 0(1)n-1$. Then for all $k > 3$ there exists strictly convex spline $S_{k-1,1}(x) \in C^{k-2}[x_0, x_{n+1}]$ with at most $k-1$ additional knots in every subinterval of original mesh, which interpolates histogram G.*

Proof Let be $S_{k-1,1}(x_i) = S_{11}(x_i)$, $S'_{k-1,1}(x_i) = S'_{11}(x_i+0)$, $S''_{k-1,1}(x_i) = M_i$ and $S^{(j)}_{k-1,1}(x_i) = 0$ for all $j \in \{3, 4, \dots, k-1\}$, where M_i are unknown parameters. Let us gradually denote

$$\begin{aligned}
y &= \frac{x - x_i}{h_i} \\
s &= S_{k-1,1}(x_i) - S_{k-1,1}(x_{i+1}) + (x_{i+1} - x_i)S'_{k-1,1}(x_{i+1}) \\
m &= h_i \left(S'_{k-1,1}(x_i) - S'_{k-1,1}(x_{i+1})\right) \\
g &= g_i - \frac{1}{2}\left(2S_{k-1,1}(x_{i+1}) - h_i S'_{k-1,1}(x_{i+1})\right) \\
S_i(y) &= S_{k-1,1}(x) - S_{k-1,1}(x_{i+1}) + (x_{i+1} - x)S'_{k-1,1}(x_{i+1})
\end{aligned}$$

on each interval $[x_i, x_{i+1}]$. Then on each interval $[x_i, x_{i+1}]$ the spline $S_{k-1,1}(x)$ can be decomposed by the following way: $S_{k-1,1}(x) = S_i(y) + l_i(x)$, where $l(x)_i$ is line given by formula $l_i(x) = S_{k-1,1}(x_{i+1}) - (x_{i+1} - x)S'_{k-1,1}(x_{i+1})$. Since lemma 4.3 holds we can suppose without loss of generality that numbers s, m, g and spline $S_i(y)$ satisfy the assumptions of the theorem 4.2 and therefore there exist at most $k-1$ additional knots such that $S_i(y)$ is convex. Using the additivity of spline spaces we prove the convexity for the spline $S_{k-1,1}(x)$ on each interval $[x_i, x_{i+1}]$. To satisfy the smoothness (and so convexity) of spline $S_{k-1,1}(x)$ on the whole interval $[x_0, x_{n+1}]$, it is enough to put $M_i = \min\{\frac{S''_i(1)}{h_i^2}, \frac{S''_{i+1}(0)}{h_{i+1}^2}\}$. $\square$

Remark 4.1 The similar statement stand for $k = 3$ too. The proof using local spline basis is given in [4].

References

[1] de Boor, C.: A Practical Guide to Splines. *Springer Verlag, New York*, 1978

[2] Schmidt, J. W., Hess, W., Nordheim, Th.: *Shape preserving histopolation using rational quadratic splines.* Computing **44** (1990), 245–258.

[3] Schmidt, J. W., Hess, W.: *Shape preserving C^2-spline histopolation.* Journal of Approximation Theory **75**, 3 (1993), 325–345.

[4] Ženčák, P.: *Some algorithm for testing convexity of histogram.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **38** (1999), 149–163.

[5] Ženčák, P.: *Convexity of histogram and convex histopolation by polynomial splines.* In: Proceed. of SANM, Nečtiny, 1999, 327–342.