Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 41 (2002), No. 1, 161--173

Persistent URL: `http://dml.cz/dmlcz/120448`

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Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **41** (2002) 161–173

The Problem $\nabla \cdot \mathbf{v} = f$ and Singular Integrals on Orlicz Spaces*

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(Received February 28, 2001)

Abstract

The main goal of this paper is to prove that singular integrals satisfying Calderón–Zygmund conditions are well-defined on certain set of Orlicz spaces. This fact enables us to decide upon the solvability of the problem $\nabla\cdot\mathbf{v} = f$ on such spaces. This paper was inspired by similar results known for the Riesz transform (see [4]).

Key words: Singular integrals, Bogovsky operator, Orlicz spaces.

2000 Mathematics Subject Classification: 31A10, 35A05, 35F15

1 Introduction

The theory of singular integrals plays an important role in many parts of mathematics. We apply this theory to construct a solution to the problem

$$\nabla \cdot \mathbf{v} = f \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{a}, \tag{1.2}$$

where Ω is a bounded or unbounded domain. The problem (1.1), (1.2) was studied for a function f such that $f \in L^p(\Omega)$, with $p \in (1, \infty)$. Moreover, the function f was assumed to satisfy a compatibility condition in the form

*Supported by the grant of Palacky University No. 3110 3007.

$\int_\Omega f\,dx = \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dx$ when Ω is a bounded domain. We refer the reader to [2, pp. 117–139] for more details about this case. Another application of the singular integrals stems from the fact that they form fundamental solutions to the Laplace equation or the Stokes equations, and one can use them to obtain appropriate interior estimates or estimates near the boundary. First results about the singular integrals, published in [1], contain information about their behaviour on L^p-spaces. In [4, p. 97 ff.] these results are generalized to the Riesz transform on Orlicz spaces.

In this paper we prove that the singular integrals are well-defined on appropriate Orlicz spaces as well, obtaining thereby a generalization of the results from [1]. We apply our results to singular integrals, which generate a solution to the problem (1.1)–(1.2). This problem is of a remarkable importance, since, for example, it provides a construction of appropriate test functions in the theory of compressible Navier–Stokes equations.

2 Preliminaries

For $p \in R^m$, $p = (p_1, \dots, p_m)$, $m \in N$, we denote by $|p|$ the Euclidean norm

$$|p| = \sqrt{\sum_{i=1}^{m} p_i^2}.$$

Using the definition of Young function Φ and its complementary function Ψ (see [5, p. 134]) we adopt the usual notation, namely $\widetilde{L}_\Phi(\Omega)$ for the Orlicz class, i.e.

$$u \in \widetilde{L}_\Phi(\Omega) \quad \text{if } \rho(u;\Phi) := \int_\Omega \Phi(u(x))\,dx < \infty\,,$$

$L_\Phi(\Omega)$ for the Orlicz space, which is the set of all measurable functions u with $\|u\|_\Phi < \infty$, where

$$\|u\|_\Phi = \sup_v \int_\Omega |uv|\,dx,$$

with $v \in \widetilde{L}_\Psi(\Omega)$ and $\rho(v;\Psi) \le 1$, and $E_\Phi(\Omega)$ for the closure of $B(\Omega)$ (the set of bounded measurable functions) with respect to the Orlicz norm $\|\cdot\|_\Phi$. There exists an equivalent (Luxemburg) norm, given by

$$|||u|||_\Phi = \inf\{\lambda > 0;\ \int_\Omega \Phi\Big(\frac{u}{\lambda}\Big)\,dx \le 1\}\,. \tag{2.1}$$

We further denote by $W^k L_\Phi(\Omega)$ the set of all functions u defined almost everywhere on Ω such that all distributional derivatives $D^\alpha u$, with $|\alpha| \le k$, belong to $L_\Phi(\Omega)$, and by $W_0^k L_\Phi(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{k,\Phi}$, where

$$\|u\|_{k,\Phi} = \Big(\sum_{|\alpha|\le k} |||D^\alpha u|||_\Phi^2\Big)^{\frac{1}{2}}\,.$$

$D_0^1 L_\Phi(\Omega)$ means the closure of $C_0^\infty(\Omega)$ in the seminorm $|\cdot|_{1,\Phi}$,

$$|u|_{1,\Phi} = \Big(\sum_{|\alpha|=1} |||D^\alpha u|||_\Phi^2 \Big)^{\frac{1}{2}},$$

and $H_\Phi^{0,\Phi}(\partial\Omega)$ stands for the closure of $B(\partial\Omega)$ in the norm

$$||u||_{H_\Phi^{0,\Phi}} = ||u||_\Phi + [u]_\Phi ,$$

where

$$[u]_\Phi = \inf\Big\{ s > 0; \int_{\partial\Omega}\int_{\partial\Omega} \Phi\Big(\frac{1}{s}\frac{|u(x)-u(y)|}{|x-y|}\Big)\frac{1}{|x-y|^{N-1}}\,dx\,dy \le 1 \Big\}.$$

Here, $\partial\Omega$ denotes the boundary of Ω.

In a similar way, we can define vector-valued Orlicz spaces for G-function G, where

$$||\mathbf{u}||_G = \inf\{\lambda > 0; \int_\Omega G\Big(\frac{\mathbf{u}(x)}{\lambda}\Big)\,dx \le 1\}. \tag{2.2}$$

Here the G-function G is expressed in one of the following forms:

1. $G_1(t) = \Phi(|t|)$,
2. $G_2(t) = \sum_{i=1}^m \Phi(t_i)$,

where Φ is a Young function and $t = (t_1, \dots, t_m)$. $H_{0,\Phi}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the norm $||\cdot||_{H_\Phi}$, where

$$||\mathbf{u}||_{H_\Phi} = ||\mathbf{u}||_{G_2} + ||\nabla \cdot \mathbf{u}||_\Phi.$$

It will cause no confusion if we denote by $||\cdot||_{1,\Phi}$ the norm of vector-valued functions

$$||\mathbf{u}||_{1,\Phi} = \Big(\sum_{|\alpha|\le 1} ||D^\alpha \mathbf{u}||_{G_2}^2 \Big)^{\frac{1}{2}}$$

for $\mathbf{u} \in W_0^1 L_\Phi(\Omega)$, because the Young function Φ will be assumed to satisfy the global Δ_2-condition (see below for the definition) and hence the G-functions G_i are equivalent. For more details about the Orlicz spaces we refer the reader to [3] and [5].

Now let us state basic definitions and assertions which will be useful in the sequel.

Definition 2.1 A Young function Φ satisfies the global Δ_2-condition if

$$\Phi(2t) \le k\Phi(t) \quad \text{for all } t > 0$$

and some constant $k > 0$.

Proposition 2.2 [4, p. 17] *Let Φ be a Young function satisfying the global Δ_2-condition. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq \frac{b\Phi(t_1)}{t_1^p}, \quad 0 < t_1 \leq t_2 \,. \tag{2.3}$$

Definition 2.3 The function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \leq \Phi(t) \leq c\omega(ct), \quad t \in [0, \infty).$$

The following proposition is generalization of the similar assertion for $u \in W_0^{1,p}(\Omega)$ (see [2, p. 48]).

Proposition 2.4 *Let Ω be a bounded domain, Φ a Young function and $u \in W_0^1 L_\Phi(\Omega)$. Then*

1. $\rho(u, \Phi) \leq c\rho(c\nabla u, G_1)$
2. $\rho(u, \Phi) \leq c\rho(c\nabla u, G_2)$,

where the G-functions G_i are defined as above.

Proof Since Ω is bounded, we have $\Omega \subset L_d = \{x \in R^N; 0 \leq |x_i| < d/2, i = 1, \ldots, N\}$, and we can derive the estimate

$$|u(x)| \leq \frac{1}{2} \int_{-d/2}^{d/2} |\nabla u| \, dx_N$$

for $u \in C_0^\infty(\Omega)$. Applying the function Φ to this estimate and integrating over Ω we obtain the estimate 1. In the second case it is enough to realize that the following estimate holds:

$$|u(x)| \leq \frac{1}{2} \sum_{i=1}^{N} \int_{-d/2}^{d/2} \left| \frac{\partial u}{\partial x_i} \right| dx_i \,. \qquad \square$$

3 Singular integrals on Orlicz spaces

We will investigate singular integrals

$$Tf(x) := \int_{R^N} K(x, x-y) f(y) \, dy \,,$$

with the kernel

$$K(x, y) := \frac{k(x, y)}{|y|^N} \,,$$

where the function $k(x, y)$ satisfies the Calderón–Zygmund conditions, i.e.

1. for any $x \in \Omega$, $y \in R^N \setminus \{0\}$ and every $\beta > 0$

$$k(x, y) = k(x, \beta y),$$

2. for every $x \in \Omega$, $k(x, y)$ is integrable on the sphere $|y| = 1$ and

$$\int_{|y|=1} k(x, y)\, dy = 0,$$

3. for some $q > 1$, there exists $C > 0$ such that

$$\int_{|y|=1} |k(x, y)|^q\, dy \leq C, \quad \text{uniformly in } x,$$

4. there exists a constant $B > 0$ such that

$$|k(x, y)| \leq B \quad \text{for every } x \in \Omega.$$

Let us denote

$$\lambda_{\alpha,f} := |\Omega_{\alpha,f}|,$$

where $\Omega_{\alpha,f} := \{x \in R^N;\ |f(x)| > \alpha\}$ and $|\Omega_{\alpha,f}|$ means the Lebesgue's measure of $\Omega_{\alpha,f}$. Set

$$K_\epsilon(x, z) := \begin{cases} K(x, z) & |z| \geq \epsilon, \\ 0 & |z| < \epsilon \end{cases}$$

and

$$T_\epsilon f(x) := \int_{R^N} K_\epsilon(x, x - y) f(y)\, dy.$$

Definition 3.1 An operator T is said to be of weak type (p, q) if

$$\lambda(\alpha, Tf) \leq \left(\frac{A\|f\|_p}{\alpha}\right)^q, \quad f \in L^p(R^N),\ \alpha > 0,$$

with A independent of f. An operator T is said to be of weak type (Φ, Φ) if

$$\Phi(\alpha)\lambda(\alpha, Tf) \leq A \int_{R^N} \Phi(f(x))\, dx, \quad f \in \widetilde{L}_\Phi(R^N),\ \alpha > 0,$$

with A independent of f.

Theorem 3.2 *Let Φ be a Young function satisfying the global Δ_2-condition. Then there exists $c > 0$ such that*

$$\Phi(\alpha)\lambda(\alpha, T_\epsilon f) \leq c \int_{R^N} \Phi(f(x))\, dx$$

for $f \in L_\Phi(R^N)$ and for all $\alpha \in [0, \infty)$, with c independent of ϵ.

Proof We can derive the following estimate

$$\Phi(\alpha)\lambda(\alpha, T_\epsilon f) \le \Phi(\alpha)\lambda\Big(\frac{\alpha}{2}, T_\epsilon f_\alpha\Big) + \Phi(\alpha)\lambda\Big(\frac{\alpha}{2}, T_\epsilon f^\alpha\Big),$$

with

$$f_\alpha := \begin{cases} f(x) & \text{if } |f(x)| \le \alpha, \\ 0 & \text{if } |f(x)| > \alpha \end{cases}$$

and

$$f^\alpha := \begin{cases} f(x) & \text{if } |f(x)| > \alpha, \\ 0 & \text{if } |f(x)| \le \alpha. \end{cases}$$

Since Φ is a convex function, we have

$$\frac{\Phi(x_1)}{x_1} \le \frac{a\Phi(ax_2)}{x_2}, \quad a > 1$$

for $0 < x_1 \le x_2$. As $T_\epsilon f$ is of weak type $(1,1)$, we obtain

$$\Phi(\alpha)\lambda\Big(\frac{\alpha}{2}, T_\epsilon f^\alpha\Big) \le \frac{c\Phi(\alpha)}{\alpha}\int_{R^N} |f^\alpha(x)|\,dx \le c_1\int_{R^N}\Phi(f(x))\,dx\,.$$

For the remaining part of the proof we use Proposition 2.2 and the fact that $T_\epsilon f$ is of weak type (p,p) with $p > 1$. Thus,

$$\Phi(\alpha)\lambda\Big(\frac{\alpha}{2}, T_\epsilon f_\alpha\Big) \le \frac{c\Phi(\alpha)}{\alpha^p}\int_{R^N} |f_\alpha(x)|^p\,dx \le c\int_{R^N}|f(x)|^p\frac{\Phi(f(x))}{|f(x)|^p}\,dx\,. \qquad \square$$

Theorem 3.3 *Let Φ be a Young function satisfying the global Δ_2-condition and let Φ^γ be quasiconvex for some $\gamma \in (0,1)$. Then*

$$\int_{R^N}\Phi(T_\epsilon f(x))\,dx \le c\int_{R^N}\Phi(f(x))\,dx \tag{3.1}$$

for each $\epsilon \in (0,1)$, with c independent of ϵ. Moreover, $\lim_{\epsilon\to 0} T_\epsilon f = Tf$ exists in $L_\Phi(R^N)$, and

$$\int_{R^N}\Phi(Tf(x))\,dx \le c\int_{R^N}\Phi(f(x))\,dx\,. \tag{3.2}$$

Proof The key fact is T_ϵ being of weak type (p,p), $p \ge 1$.

First, we obtain

$$\int_{R^N}\Phi(T_\epsilon f(x))\,dx = \int_0^\infty \lambda(\alpha, T_\epsilon f)\,d\Phi(\alpha)$$

$$\le \int_0^\infty \lambda\Big(\frac{\alpha}{2}, T_\epsilon f^\alpha\Big)d\Phi(\alpha) + \int_0^\infty \lambda\Big(\frac{\alpha}{2}, T_\epsilon f_\alpha\Big)d\Phi(\alpha) = I_1 + I_2\,.$$

Now, Φ^γ is quasiconvex for some $\gamma \in (0,1)$, hence

$$\int_0^t \frac{d\Phi(u)}{u} \le \frac{c\Phi(ct)}{t} \quad \text{for } t > 0,\ c > 0$$

(see [4, p. 6]). We conclude that

$$I_1 \le \int_0^\infty \frac{c}{\alpha}\Big(\int_{|f(x)|>\alpha} |f(x)|\,dx\Big)\,d\Phi(\alpha)$$

$$= c\int_{R^N} |f(x)|\Big(\int_0^{|f(x)|} \frac{d\Phi(\alpha)}{\alpha}\Big)dx \le c_1\int_{R^N} |f(x)|\frac{\Phi(f(x))}{|f(x)|}\,dx\,.$$

Let p be as in Proposition 2.2. Then we can verify that, for $p_1 > p$, $\lim_{t\to\infty}\frac{\Phi(t)}{t^{p_1}} = 0$, and

$$\int_t^\infty \frac{d\Phi(u)}{u^{p_1}} = \frac{\Phi(u)}{u^{p_1}}\Big|_t^\infty + p_1\int_t^\infty \frac{\Phi(u)}{u^{p_1+1}}\,du.$$

Applying the above identity we obtain

$$I_2 \le c\int_0^\infty \frac{1}{\alpha^{p_1}}\Big(\int_{|f(x)|<\alpha} |f(x)|^{p_1}\,dx\Big)\,d\Phi(\alpha)$$

$$= c\int_{R^N} |f(x)|^{p_1}\Big(\int_{|f(x)|}^\infty \frac{d\Phi(\alpha)}{\alpha^{p_1}}\Big)dx \le c\int_{R^N} \Phi(f(x))\,dx\,.$$

As $f \in L_\Phi(R^N)$ and Φ satisfies the global Δ_2-condition, we have $f = f_1 + f_2$ for some $f_1 \in C_0^\infty(R^N)$ and f_2 with a sufficiently small norm $\|\cdot\|_\Phi$. Hence the estimate

$$\|T_{\frac{1}{n}}f - T_{\frac{1}{m}}f\|_\Phi \le c\max_{R^N}|\nabla f_1|\delta\Big(\frac{1}{n}\Big) + c\|f_2\|_\Phi$$

holds with $\frac{1}{n} \ge \frac{1}{m}$, where $\delta\Big(\frac{1}{n}\Big) \to 0$ for $n \to +\infty$. Therefore, for every $f \in L_\Phi(R^N)$, we have the fundamental and hence convergent sequence $T_{\frac{1}{n}}f$ in $L_\Phi(R^N)$. Thus,

$$\|Tf\|_\Phi \le c\|f\|_\Phi \tag{3.3}$$

for $f \in L_\Phi(R^N)$, and at the same time we obtain

$$\rho(Tf,\Phi) \le c\rho(f,\Phi)\,. \tag{3.4}$$

□

Theorem 3.4 *Let us define the operator $\overline{T}$ by*

$$\overline{T}f(x) := \int_{R^N} K(x-y)f(y)\,dy\,,$$

where $K \in L^1(R^N)$. Then

$$\|\overline{T}f\|_\Phi \le c\|f\|_\Phi \tag{3.5}$$

and

$$\rho(\overline{T}f,\Phi) \le c\rho(f,\Phi) \tag{3.6}$$

for $f \in L_\Phi(R^N)$, where Φ is a Young function satisfying the global Δ_2-condition and Φ^γ is quasiconvex for some $\gamma \in (0,1)$.

Proof By the Young theorem, we get

$$\|\overline{T}f\|_p \le c\|f\|_p, f \in L^p(R^N), \quad p \in [1,\infty).$$

So, $\overline{T}$ is of type (p,p) for any $p \in [1,\infty)$ and the more so it is of weak type (p,p). The rest of the proof is the same as that of Theorem 3.3. □

4 The problem $\nabla \cdot \mathbf{v} = f$ on Orlicz spaces I.

Let Ω be a bounded domain in R^N, $N \ge 2$. We can formulate the following problem: *Given* $f \in L_\Phi(\Omega)$ *with*

$$\int_\Omega f\,dx = 0, \tag{4.1}$$

find a vector field $\mathbf{v} : \Omega \to R$ *such that*

$$\nabla \cdot \mathbf{v} = f \quad in\ \Omega, \tag{4.2}$$

$$\mathbf{v} \in W_0^1 L_\Phi(\Omega), \tag{4.3}$$

and

$$\|\mathbf{v}\|_{1,\Phi} \le c\|f\|_\Phi. \tag{4.4}$$

Theorem 4.1 *Let* Ω *be a bounded domain with Lipschitzian boundary. Let* Φ *be a Young function satisfying the global* Δ_2*-condition and such that* Φ^γ *is quasiconvex for some* $\gamma \in (0,1)$. *Then, for any* $f \in L_\Phi(\Omega)$ *satisfying* (4.1), *the problem* (4.2)–(4.4) *has at least one solution* $\mathbf{v} \in W_0^1 L_\Phi(\Omega)$. *If* $f \in C_0^\infty(\Omega)$, *then* $\mathbf{v} \in C_0^\infty(\Omega)$.

Proof Let $f \in C_0^\infty(\Omega)$ and $\int_\Omega f dx = 0$. Using the fact that $\Omega = \cup_{i=1}^{m+\nu}\Omega_i$, where Ω_i is a star-shaped domain with respect to an open ball B_i, and the decomposition $f = \sum_{i=1}^{m+\nu} f_i$ with $f_i \in C_0^\infty(\Omega_i)$ and $\int_{\Omega_i} f_i\,dx = 0$ (for details see [2, p. 127]), we reduce the problem to the case when Ω is a star-shaped domain with respect to an open ball $B_R(x_0)$. In such case we obtain the solution to the problem (4.2)–(4.4) expressed as a sum of functions $\mathbf{v}_i$, $i = 1,\dots,m+\nu$, where $\mathbf{v}_i$ is a solution to the problem (4.2)–(4.4) with $f = f_i$ in Ω_i. So, let us suppose that Ω is a star-shaped domain with respect to the open ball $B_R(x_0)$ and $f \in C_0^\infty(\Omega)$. The solution to the problem (4.1)–(4.4) which we are interested in has the form

$$\mathbf{v}(x) = \int_{\Omega'} \widetilde{f}(y)\Big[\frac{x-y}{|x-y|^N}\int_{|x-y|}^\infty \omega\Big(y + \xi\frac{x-y}{|x-y|}\Big)\xi^{N-1}\,d\xi\Big]dy,$$

where $\widetilde{f} \in C_0^\infty(\Omega')$, $\widetilde{f}$ and Ω' respectively denote f and Ω after the change of variables

$$x \to x' = \frac{x-x_0}{R}.$$

Here $\omega \in C_0^\infty(R^N)$ is such that

- $supp(\omega) \subset B(= B_1(0))$
- $\int_B \omega \, dx = 1$.

The formulae for derivatives of $\mathbf{v}$ can be derived in the same way as in [2, p. 119]; so we can compute

$$\frac{\partial}{\partial x_j} v_i(x) = \int_\Omega K_{ij}(x, x-y) f(y)\, dy + \int_\Omega G_{ij}(x,y) f(y)\, dy$$

$$+ f(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \omega(y)\, dy, \quad i,j = 1, \ldots, N,$$

where $|G_{ij}(x,y)| \le \frac{c}{|x-y|^{N-1}}$, $x, y \in \Omega$, and $K_{ij} = \frac{k_{ij}(x, x-y)}{|x-y|^N}$ with k_{ij} satisfying Calderón–Zygmund conditions. Then, using convexity, the global Δ_2-condition, Theorem 3.3, Theorem 3.4 and Proposition 2.4 (returning to the original variables by $x' \to x$) we get

$$\rho(\mathbf{v}, G_2) + \rho(\nabla \mathbf{v}, \widetilde{G}_2) \le c\rho(f, \Phi),$$

where $G_2(t) = \sum_{i=1}^N \Phi(t_i)$, $t = (t_1, \ldots, t_N)$ and $\widetilde{G}_2(s) = \sum_{i,j=1}^N \Phi(s_{ij})$, $s = (s_{ij})_{i,j=1}^N$. This inequality yields

$$\|\mathbf{v}\|_{1,\Phi} \le c\|f\|_\Phi .$$

Since Φ satisfies the global Δ_2-condition, we can associate to each $f \in L_\Phi(\Omega)$ a sequence $\{f_m\}_{m=1}^\infty$, $f_m \in C_0^\infty(\Omega)$, such that $\int_\Omega f_m \, dx = 0$ and $f_m \to f$ in $L_\Phi(\Omega)$. Denoting $\mathbf{v}_m$ the solution of (4.2)–(4.4) with $f = f_m$, we can verify that $\{\mathbf{v}_m\}_{m=1}^\infty$ converges to $\mathbf{v}$ strongly in the space $W^1 L_\Phi(\Omega)$. This follows from Theorems 3.3 and 3.4 and from the linearity of singular and weakly singular integrals which represent the solution $\mathbf{v}_m$ and its derivatives. □

Let Ω be a bounded domain in R^N, $N \ge 2$. We can formulate the following problem: *Given* $\mathbf{a} \in H_\Phi^{0,\Phi}(\partial\Omega)$ *and* $f \in L_\Phi(\Omega)$ *with*

$$\int_\Omega f\, dx = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\, dS, \tag{4.5}$$

find a vector field $\mathbf{v} : \Omega \to R$ *such that*

$$\nabla \cdot \mathbf{v} = f \quad \text{in } \Omega, \tag{4.6}$$

$$\mathbf{v} \in W^1 L_\Phi(\Omega), \quad \mathbf{v}|_{\partial\Omega} = \mathbf{a}, \tag{4.7}$$

and

$$\|\mathbf{v}\|_{1,\Phi} \le c(\|f\|_\Phi + \|\mathbf{a}\|_{H_\Phi^{0,\Phi}}). \tag{4.8}$$

Theorem 4.2 *Let Ω be a bounded domain with Lipschitzian boundary. Let Φ be a Young function satisfying the global Δ_2-condition and such that Φ^γ is quasiconvex for some $\gamma \in (0,1)$. Then, for any $f \in L_\Phi(\Omega)$ satisfying (4.5), the problem (4.6)–(4.8) has at least one solution $\mathbf{v} \in W^1 L_\Phi(\Omega)$.*

Proof Let $\mathbf{a} \in H_\Phi^{0,\Phi}(\partial\Omega)$. Using the existence of a continuous linear operator from $H_\Phi^{0,\Phi}(\partial\Omega)$ onto $W^1 L_\Phi(\Omega)$ (see [6]), there exists $\mathbf{U} \in W^1 L_\Phi(\Omega)$ such that $\mathbf{U}|_{\partial\Omega} = \mathbf{a}$ in the sense of traces. We will seek the solution to our problem in the form

$$\mathbf{v} = \mathbf{u} + \mathbf{U}\,.$$

The problem then reduces to the question of finding some $\mathbf{u}$ such that

$$\nabla \cdot \mathbf{u} = f - \nabla \cdot \mathbf{U} = F, \quad \mathbf{u}|_{\partial\Omega} = 0$$

with $\int_\Omega F\,dx = 0$. Theorem 4.1 then guarantees the existence of a solution $\mathbf{u}$ satisfying the estimate

$$||\mathbf{u}||_{1,\Phi} \le c||F||_\Phi\,.$$

Finally, we have

$$||\mathbf{v}||_{1,\Phi} \le c(||f||_\Phi + ||\mathbf{U}||_{1,\Phi}) \le c(||f||_\Phi + ||\mathbf{a}||_{H_\Phi^{0,\Phi}})\,. \qquad \square$$

5 The problem $\nabla \cdot \mathbf{v} = f$ on Orlicz spaces II.

Assume that Φ has the following properties:

1. Φ is a Young function
2. $\Phi(t) = \sum_{p=1}^{\infty} \frac{1}{p!}\Phi^{(p)}(0)t^p, \quad \forall t \in [0,\infty)$.

Theorem 5.1 *Let Ω be a bounded domain with Lipschitzian boundary. Let $f \in \widetilde{L}_\Phi(\Omega)$ is such that (4.1) holds. Suppose that, given $p_0 \ge 1$, there exists p such that $p \ge p_0$ and $\Phi^{(p)}(0) \ne 0$. Assume that there exists a function $M(t)$ such that*

$$M(t) \le c\sum_{p=1}^{\infty} \frac{\Phi^{(p)}(0)}{(2p)!}t^p, \quad t \in R_0^+\,.$$

Then there exists a solution $\mathbf{v} \in W_0^1 L_{\Phi_2}(\Omega)$ to the problem (4.2)–(4.4) such that

$$||\mathbf{v}||_{1,\Phi_2} \le c||f||_\Phi\,,$$

where Φ_2 is a Young function which satisfies the inequality $\Phi_2(t) \le M(t)$ for all $t \in [0,\infty)$.

Proof Since $f \in \widetilde{L}_\Phi(\Omega)$, there exists a solution of the problem (4.2)–(4.4) such that $\mathbf{v} \in W_0^{1,p}(\Omega)$ for all $p \in [2, \infty)$ and

$$||\mathbf{v}||_{1,p} \leq cp||f||_p \,.$$

By Fatou's lemma and Lebesgue's theorem we get, for $G(t) = \sum_{i=1}^N \Phi_2(t_i)$, $t = (t_1, \dots, t_N)$ and $\overline{G}(s) = \sum_{i,j=1}^N \Phi_2(s_{ij})$, $s = (s_{ij})_{i,j=1}^N$,

$$\rho(\mathbf{v}, G) + \rho(\nabla \mathbf{v}, \overline{G}) \leq c\rho(cf, \Phi) \,.$$

Using (2.1), we find

$$||\mathbf{v}||_{1,\Phi_2} \leq c||f||_\Phi \,. \qquad \square$$

6 The problem $\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{g}$ on Orlicz spaces

Given $\mathbf{g} \in H_{0,\Phi}(\Omega)$ with

$$\int_\Omega \nabla \cdot \mathbf{g} \, dx = 0, \tag{6.1}$$

find a vector field $\mathbf{w} : \Omega \to R$ such that

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{g} \quad in \ \Omega, \tag{6.2}$$

$$\mathbf{w} \in W_0^1 L_\Phi(\Omega), \tag{6.3}$$

$$||\mathbf{w}||_{1,\Phi} \leq c||\nabla \cdot \mathbf{g}||_\Phi, \tag{6.4}$$

and

$$||\mathbf{w}||_{G_i} \leq c||\mathbf{g}||_{G_i}. \tag{6.5}$$

Theorem 6.1 *Let Ω be a bounded domain with Lipschitzian boundary. Let Φ be a Young function and $g \in H_{0,\Phi}(\Omega)$, where Φ is a Young function satisfying the global Δ_2-condition such that Φ^γ is quasiconvex for some $\gamma \in (0, 1)$. Then there exists at least one solution $\mathbf{w} \in W_0^1 L_{\Phi_1}(\Omega)$ of the problem (6.2)–(6.4). In particular, if $\nabla \cdot \mathbf{g} \in C_0^\infty(\Omega)$, then $\mathbf{w} \in C_0^\infty(\Omega)$.*

Proof Using a decomposition of Ω analogous to the one in the proof of Theorem 4.1 we show that it suffices to consider problem (4.2)–(4.3) and (6.3)–(6.4) with $f = \nabla \cdot \mathbf{g}$. However, the existence of a solution to this problem is guaranteed by Theorem 4.1. The estimates (6.3)–(6.4) follow from properties of singular and weakly singular integrals. $\square$

Remark 6.2 Using the technique from the proof of Theorem 5.1, we can prove analogous results as in Theorem 5.1 for $f = \nabla \cdot \mathbf{g}$.

7 The problem $\nabla \cdot \mathbf{v} = f$ on exterior domains

Let Ω be an exterior domain in R^N, $N \geq 2$. We can formulate the following problem: *Given $f \in L_\Phi(\Omega)$, find a vector field $\mathbf{v} : \Omega \to R$ such that*

$$\nabla \cdot \mathbf{v} = f \quad \textit{in } \Omega, \tag{7.6}$$

$$\mathbf{v} \in D_0^1 L_\Phi(\Omega), \tag{7.7}$$

and

$$|\mathbf{v}|_{1,\Phi} \leq c||f||_\Phi. \tag{7.8}$$

Theorem 7.1 *Let Ω be a bounded domain with Lipschitzian boundary. Let Φ be a Young function satisfying the global Δ_2-condition such that Φ^γ is quasiconvex for some $\gamma \in (0,1)$. Let $f \in L_\Phi(\Omega)$. Then there exists a solution to the problem (7.6)-(7.8).*

Proof Let $\{f_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ be a sequence which converges to f in $L_\Phi(\Omega)$. Set

$$\mathbf{v}_m = \nabla \psi_m + \mathbf{w}_m, \quad m \in N,$$

where

$$\Delta \psi_m = f_m \quad \text{in } R^N$$

and, for some $R > 2\delta(\Omega^c)$ (here $\delta(\Omega) := \sup_{x,y\in\Omega} |x-y|$),

$$\nabla \cdot \mathbf{w}_m = 0 \quad \text{in } \Omega_R, \text{ with } \Omega_R := \Omega \cap B_R(0),$$

$$\mathbf{w}_m = -\nabla \psi_m \quad \text{on } \partial\Omega,$$

and

$$\mathbf{w}_m = 0 \quad \text{on } \partial B_R(0).$$

By the representation $\psi_m = \mathcal{E} * f_m$, with $\mathcal{E}$ the fundamental solution of Laplace equation, and by Theorems 3.3 and 3.4, we obtain

$$|\psi_m|_{2,\Phi} \leq c||f_m||_\Phi.$$

Moreover, since

$$\int_{\partial\Omega} \nabla\psi_m \cdot \mathbf{n}\, dS = 0 \quad \text{for all } m \in N,$$

we deduce the existence of a solenoidal field $\mathbf{w}_m \in W^1 L_{\Phi_1}(\Omega_R)$ from Theorem 4.2 such that

$$||\mathbf{w}_m||_{1,\Phi} \leq c||\nabla \cdot (\phi \nabla \psi_m)||_\Phi$$

extending $\mathbf{w}_m$ by zero outside of Ω_R and for $\phi \in C^1(R^N)$ such that $\phi = 1$ if $|x| < \frac{R}{2}$, $\phi = 0$ if $|x| \geq R$. Using the representation of ψ_m, we obtain

$$||\mathbf{w}_m||_{1,\Phi} \leq c||f_m||_\Phi.$$

So we get that $\mathbf{v}_m$ is a solution of our problem for $f = f_m$. The rest of proof is analogous to that of Theorem 4.1. □

Remark 7.2 To prove the existence of a solution of the corresponding problem with nonhomogeneous boundary condition $\mathbf{v} = \mathbf{a}$ on $\partial\Omega$ we can use the same technique as in the proof of Theorem 4.2.

Acknowledgment I am thankful to I. Straškraba for valuable remarks.

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