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Periodic Solutions of Certain Third Order Differential Systems with Nonlinear Dissipation *

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Abstract

We present n-dimensional analogues to some results obtained by Ezeilo and Omari [8], by studying the existence of T-periodic solutions for certain third-order nonlinear differential systems of the form $X''' + AX'' + G(t, X') + CX = P(t)$, where the dissipative term G and forcing term P are vector-valued functions, and A and C are nonsingular constant matrices. We shall demonstrate in this study that the transition from the scalar to the vector field is by no means trivial.

Key words: Nonlinear dissipation, sharp and nonuniform nonresonance, Leray–Schauder alternative/continuation method, Mawhin's coincidence degree.

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1 Introduction

In the paper by Ezeilo and Omari [8], the problem of nonresonant oscillations of the solutions of the third order scalar differential equation

$$x''' + ax'' + g(x') + cx = p(t) \tag{1.1}$$

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and when $g = g(t, x')$ depends also on the t-variable, its generalisation

$$x''' + ax'' + g(t, x') + cx = p(t) \tag{1.2}$$

where a and c are nonzero constants, and g and p are scalar-valued functions, has been extensively investigated subject to the 2π-periodic boundary conditions

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = x''(0) - x''(2\pi) = 0 \tag{1.3}$$

Arising from the analysis of an appropriately posed eigenvalue problem, their main results establish the existence of 2π-periodic solutions employing first the sharp nonresonance conditions

$$(g_1) \qquad k^2 + \alpha^-(|x'|) < \frac{g(x')}{x'} < (k+1)^2 - \alpha^+(|x'|), \quad k \in \mathbb{N},$$

where $\alpha^\pm : (0, +\infty) \to \mathbb{R}$ are two nonincreasing functions such that

$$\lim_{|x'| \to +\infty} |x'|\alpha^\pm(|x'|) = +\infty,$$

and then the nonuniform assumptions

$$(g_2) \qquad k^2 \le \gamma^-(t) \le \liminf_{|x'| \to \infty} \frac{g(t, x')}{x'} \le \limsup_{|x'| \to \infty} \frac{g(t, x')}{x'} \le \gamma^+(t) \le (k+1)^2$$

uniformly in $x' \in \mathbb{R}$ for a.e. $t \in [0, 2\pi]$, where $\gamma^\pm \in L^1(0, 2\pi)$ such that strict inequalities hold on subsets of $[0, 2\pi]$ of positive measure, according as g is autonomous or nonautonomous. Some uniqueness results are also given by appropriate modifications of the above conditions.

Since then several other articles have appeared in the literature dealing with similar equations in the scalar case. Notable among these is the work of Andres and Vlček [5] who dealt with the problem of existence of periodic solutions for certain parametric differential equations involving large nonlinearities of the form (1.1) with the coefficient a, however, t-variable, and c nonlinear. More general equations than (1.1) and [5] involving nonlinear coefficients have also been studied in Aftabizadeh, Xu and Gupta [1] and Rachůnková [13]. The survey paper by Andres [4] which gives a comprehensive bibliographical review of some third order equations since 1969, also includes existence results that are consequences of nonlinear pertubations of linear problems at resonance as well as at nonresonance up to [5] (see [1–3], [7–9], [11] and [14]).

Alternative sharp hypotheses have recently been proposed by Minhós [11] for the problem (1.1)–(1.3), by weakening the condition on the oscillation of g, with the conditions (g_1) replaced by the two conditions

$$(g_3) \qquad k^2 \le \liminf_{|x'| \to \pm\infty} \frac{g(x')}{x'} \le \limsup_{|x'| \to \pm\infty} \frac{g(x')}{x'} \le (k+1)^2$$

and

$$(\mathcal{G}) \qquad k^2 < \limsup_{x' \to +\infty} \frac{2\mathcal{G}(x')}{x'^2}, \qquad \liminf_{x' \to +\infty} \frac{2\mathcal{G}(x')}{x'^2} < (k+1)^2,$$

where $\mathcal{G}$ denotes the primitive of the nonlinear function g, that is,

$$\mathcal{G}(y) = \int_0^y g(\tau)\, d\tau.$$

Our interest in this paper is to study the vector versions of the above problems, and seek to evolve similar or even equivalent hypotheses of the sharp nonresonance type for their solvability.

Specifically, we shall investigate nonlinear differential systems of the form

$$X''' + AX'' + G(t, X') + CX = P(t) \tag{1.4}$$

subject to the T-periodic boundary conditions

$$X(0) - X(T) = X'(0) - X'(T) = X''(0) - X''(T) = 0 \tag{1.5}$$

on $[0, T]$ with $T > 0$, using the vector analogue of condition (g_1) in the first instance here. Investigations using the remaining three conditions are being considered separately and will appear shortly.

Accordingly, $X \in \mathbb{R}^n$, A and C are constant real $n \times n$ nonsingular matrices, and $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $P : [0, T] \to \mathbb{R}^n$ are n-vectors, which are T-periodic in t. We shall assume further that G satisfies the Carathéodory conditions, that is, $G(\cdot, X')$ is measurable for every $X' \in \mathbb{R}^n$; $G(t, \cdot)$ is continuous for a.e. $t \in [0, T]$, and for each $r > 0$, there exists an integrable function $\gamma_r \in L^1([0, T], \mathbb{R})$ such that $\|G(t, X')\| \leq \gamma_r(t)$, for $\|X'\| \leq r$ and a.e. $t \in [0, T]$.

As in the cited paper above, our main results are built around the unbounded nonlinear perturbations of an associated linear differential operator. However, the transition from the scalar to the vector field has invariably introduced aspects of linear algebra, multi-variable calculus and analysis which cannot be ignored. These new additions inevitably neccessitate the need to redefine, modify and re-present some of the results obtained for the scalar case. Moreover, an abstract framework suitable for the application of Mawhin's coincidence degree [10] is provided in line with the approach given in Afuwape, Omari and Zanolin [2], to guarantee the solvability of our stated problem in an appropriate functional setting.

Let X be a point of the Euclidean space $\mathbb{R}^n$ equipped with the usual norm $\|X\|$. For any pair $X, Y \in \mathbb{R}^n$, we shall write $\langle X, Y \rangle$ for the usual scalar product of X and Y so that in particular, $\langle X, X \rangle = \|X\|^2$.

It is standard result that if D is a real $n \times n$ symmetric matrix, then for any $X \in \mathbb{R}^n$,

$$\delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2, \tag{1.6}$$

where δ_d and Δ_d are respectively the least and greatest eigenvalues of D. In general, $\lambda_i(D)$ shall denote the eigenvalues of any matrix D, and $\|D\|_2$ its spectral norm.

The following Banach spaces will also be frequently refered to:

(i) the classical spaces of k times continuously differentiable functions $C^k([0, T], \mathbb{R}^n)$, $k \geq 0$ an integer, where $C^0 = C$ and $C^\infty = \cap_{k \geq 0} C^k$ with norms $\|X\|_{C^k}$ and $\|X\|_\infty$ respectively;

(ii) the space of T-periodic functions $C_T^k([0,T],\mathbb{R}^n)$ defined by

$$C_T^k = \{X : [0,T] \to \mathbb{R}^n : X \in C^k \text{ and } X \text{ is } T\text{-periodic}\}$$

with the norm on C^k;

(iii) $L^p([0,T],\mathbb{R}^n)$, $1 \le p < +\infty$, the usual Lebesgue spaces with the norms $\|X\|_{L^p}$ and $\|X\|_\infty$ for $p = +\infty$;

(iv) the Sobolev space $W_T^{k,1}([0,T],\mathbb{R}^n)$ defined by

$$\begin{aligned} W_T^{k,1} = \{X : J \to \mathbb{R}^n : & \ X, X', \dots, X^{(k-1)} \text{ are absolutely} \\ & \text{continuous on } [0,T],\ X^{(k)} \in C_T(0,T) \text{ and} \\ & X^{(i)}(0) - X^{(i)}(T) = 0,\ i = 0,1,2,\dots,k-1,\ k \in \mathbb{N}\} \end{aligned}$$

with corresponding norm $\|X\|_{W_T^{k,1}}$.

2 Preliminary analysis and the abstract setting

We shall define the linear differential operator $\mathcal{L} : dom\,\mathcal{L} \subset L^\infty \to L^1$ by

$$\mathcal{L}X := -X''' - AX'' - BX' - CX \tag{2.1}$$

where

$$\begin{aligned} dom\,\mathcal{L} = \{\, X \in L^\infty : X \in C^2, & \text{ with } X'' \text{ absolutely continuous on } [0,T] \\ & \text{and satisfying (1.5)}\} \end{aligned}$$

In the Hilbert space L^2, we shall fix the orthonormal basis $\{\phi_{k,i}, \psi_{k,i}\}$ with

$$\phi_{0,i}(t) = \left(\frac{2}{T}\right)^{\frac{1}{2}}, \qquad \psi_{0,i}(t) = 0$$

$$\phi_{k,i}(t) = \left(\frac{2}{T}\right)^{\frac{1}{2}} \cos(k\omega t), \qquad \psi_{k,i}(t) = \left(\frac{2}{T}\right)^{\frac{1}{2}} \sin(k\omega t),$$

for $i = 1, \dots, n$, where $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, $t \in \mathbb{R}$.

Thus, if $X \in dom\,\mathcal{L} \subset L^2$, its Fourier series expansion is given by

$$X \sim \sum_{i=1}^{n} \sum_{k=0}^{\infty} (a_{k,i}\phi_{k,i} + b_{k,i}\psi_{k,i})$$

with $b_{0,i} = 0$, $\omega = \frac{2\pi}{T}$, $k \in \mathbb{N}$, $i = 1, \dots, n$.

It follows that $\mathcal{L}X$ would have the expansion

$$\begin{aligned} \mathcal{L}X(t) \sim\ & I\sqrt{\frac{2}{T}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} -(k\omega)^3 \left[a_{k,i}\sin(k\omega t) + b_{k,i}\cos(k\omega t)\right] \\ & + A\sqrt{\frac{2}{T}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} (k\omega)^2 \left[a_{k,i}\cos(k\omega t) + b_{k,i}\sin(k\omega t)\right] \end{aligned}$$

$$
\begin{aligned}
&+ B\sqrt{\frac{2}{T}}\sum_{i=1}^{n}\sum_{k=1}^{\infty} k\omega\left[a_{k,i}\sin(k\omega t) - b_{k,i}\cos(k\omega t)\right] \\
&- C\sqrt{\frac{2}{T}}\sum_{i=1}^{n}\sum_{k=0}^{\infty}\left[a_{k,i}\cos(k\omega t) + b_{k,i}\sin(k\omega t)\right] \\
= &-\sqrt{\frac{2}{T}}\sum_{i=1}^{n}\sum_{k=0}^{\infty}\left[(C - k^2\omega^2 A)a_{k,i} + k\omega(B - k^2\omega^2 I)b_{k,i}\right]\cos(k\omega t) \\
&+ \sqrt{\frac{2}{T}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\left[k\omega(B - k^2\omega^2 I)a_{k,i} - (C - k^2\omega^2 A)b_{k,i}\right]\sin(k\omega t) \\
= &\sum_{i=1}^{n}\sum_{k=0}^{\infty}\left[-(\lambda_{k,i}a_{k,i} + \mu_{k,i}b_{k,i})\phi_{k,i} + (\mu_{k,i}a_{k,i} - \lambda_{k,i}b_{k,i})\psi_{k,i}\right] \qquad (2.2)
\end{aligned}
$$

where

$$\lambda_{0,i} = 1, \quad \mu_{0,i} = 0, \quad \lambda_{k,i} = C - k^2\omega^2 A, \quad \mu_{k,i} = k\omega(B - k^2\omega^2 I),$$

for each $i = 1, \ldots, n$ and $k \in \mathbb{N}$. Therefore, $X \in ker\mathcal{L}$ if and only if for each $i = 1, \ldots, n$ and $k \in \mathbb{N}$

$$\lambda_{k,i}a_{k,i} + \mu_{k,i}b_{k,i} = 0 = \mu_{k,i}a_{k,i} - \lambda_{k,i}b_{k,i}.$$

This occurs if and only if for each $i = 1, \ldots, n$ and $k \in \mathbb{N}$, either

$$a_{k,i} = 0 = b_{k,i}, \tag{2.3}$$

or

$$\lambda_{k,i} = C - k^2\omega^2 A = 0 = k\omega(B - k^2\omega^2 I) = \mu_{k,i} \tag{2.4}$$

Let

$$K := \{k \in \mathbb{N} : k^2\omega^2 A - C = 0 = k\omega(k^2\omega^2 I - B)\}$$

Then, K is finite since A, B, C are $n \times n$ (finite) symmetric matrices. Thus

$$ker\mathcal{L} = \{X \in dom\mathcal{L} : X = \sum_{i=1}^{n}\sum_{k\in K}(a_{k,i}\phi_{k,i} + b_{k,i}\psi_{k,i})\}$$

It follows that $ker\mathcal{L}$ fits the unique continuation property.

Moreover, since $ker\mathcal{L} = ker\mathcal{L}^\star$, where $\mathcal{L}^\star$ is the adjoint of $\mathcal{L}$,

$$Im\mathcal{L} = \{Z \in L^1 : \int z_i\phi_{k,i} = 0 = \int z_i\psi_{k,i}\},$$

for each $k \in K$, $i = 1, \ldots, n$, where $\phi_{k,i}, \psi_{k,i} \in \mathcal{L}^\star$, so that $ker\mathcal{L}$ and $Im\mathcal{L}$ are orthogonal.

We choose the projection $Q : L^1 \to L^1$ given by

$$QZ = \sum_{i=1}^{n}\sum_{k\in K}\left[\phi_{k,i}\int z_i\phi_{k,i} + \psi_{k,i}\int z_i\psi_{k,i}\right]$$

and let $\mathcal{P} = \mathcal{Q}|_{L^\infty} : L^\infty \to L^\infty$. Setting, for each $k \in K$, $i = 1, \ldots, n$,

$$a_{k,i} = \int x_i \phi_{k,i}, \qquad b_{k,i} = \int x_i \psi_{k,i}, \qquad X \in L^\infty,$$

we observe that $Im\mathcal{P} = ker\,\mathcal{L}$ and $ker\,\mathcal{Q} = Im\,\mathcal{L}$, so that

$$L^\infty = ker\mathcal{P} \oplus ker\,\mathcal{L} \quad \text{and} \quad L^2 = Im\,\mathcal{L} \oplus Im\,\mathcal{Q}$$

as topological direct sums.

Note that for every $X \in E = L^2$, we write $X = \overline{X} + \widetilde{X}$, where $\overline{X} = \mathcal{P}X = \frac{1}{T}\int_0^T X(t)\,dt$ and $\widetilde{X} = X - \overline{X}$. Observe that $\mathcal{P}\widetilde{X} = 0$.

Thus, $E = \overline{E} + \widetilde{E}$, where $\overline{E} = \mathcal{P}(E) = Im\mathcal{P}$ and $\widetilde{E} = ker\mathcal{P} = \{X \in E : \mathcal{P}X = 0\}$.

Hence, $ker\mathcal{L}$ is finite dimensional, $Im\mathcal{L}$ is closed in L^1 and $\dim ker\mathcal{L} = \operatorname{codim} Im\mathcal{L} = 2n$, so that $\mathcal{L}$ is a Fredholm mapping of index zero.

The right inverse of $\mathcal{L}$, denoted by $\mathcal{K}$, and defined by $\mathcal{K} : dom\mathcal{K} \subset L^1 \to dom\,\mathcal{L} \cap ker\mathcal{P} \subset L^\infty$, such that $dom\mathcal{K} = Im\mathcal{L}$ and $Im\mathcal{K} = ker\mathcal{P}$, is associated to the pair $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{L}\mathcal{K}(I - \mathcal{Q})Z = Z$, for each $Z \in L^1$, and $\mathcal{K}\mathcal{L}X = X$, for each $X \in dom\,\mathcal{L} \cap ker\mathcal{P}$, so that $\mathcal{K}$ is a compact linear operator.

Thus, by virtue of the Carathéodory assumptions on G, the Nemytskii operator defined by $G : L^\infty \to L^1$. $GX'(\cdot) = G(\cdot X'(\cdot))$ is $\mathcal{L}$-compact on every bounded subset of L^∞, since $\mathcal{K}$ is compact.

Finally, we take $E \equiv I$ (the identity map) and $F \equiv -\mathcal{P}(\cdot) \in L^1$.

Under the above conditions, the T-periodic solutions of (1.4)–(1.5) are the solutions $X \in dom\,\mathcal{L}$ of the operator equation

$$\mathcal{L}X = EGX + F, \qquad F \in Im\,\mathcal{L} \tag{2.5}$$

The reader who is interested in the solvability of the abstract equation (2.5) may refer to Afuwape et al [2] and Omari and Zanolin [12] for further details. Our approach will look more closely at the conditions (2.3) and (2.4) and examine some of the several options opened up as a result of them. These two conditions represent the resonance and nonresonance situations respectively.

3 Solvability of $X''' + AX'' + G(t, X') + CX = P(t)$

Sequel to condition (2.4) of section 2, we recall that the linear homogeneous system

$$X''' + AX'' + BX' + CX = 0 \tag{3.1}$$

has no nontrivial T-periodic solution if and only if either

$$\lambda_i(A^{-1}C) \neq k_i^2\omega^2, \quad \text{with } k_i = 0, 1, 2, \ldots, \tag{3.2}$$

or

$$C \neq 0, \quad \lambda_i(B) \neq k^2\omega^2, \quad \text{with } k = 1, 2, \ldots, \tag{3.3}$$

Consequently, by the Fredholm alternative, the PBVP

$$X''' + AX'' + BX' + CX = P(t) \tag{3.4}$$

together with (1.5) has exactly one solution (in the Carathéodory sense), for every $P \in L^1$ subject to either (3.2) or (3.3).

Condition (3.2) has been used exclusively in a preceding article [3] to evolve various nonresonance results for systems of the type

$$X''' + AX'' + BX' + sH(t, X) = P(t) \tag{3.5}$$

subject to (1.5).

The problem of obtaining analogous existence results for systems such as (1.4) subject to nonresonant conditions based on (3.3) is our main focus in this paper. Indeed, (3.3) implies that for the associated eigenvalue problem

$$X''' + AX'' + CX = -\lambda X' \tag{3.6}$$

together with (1.5), we easily deduce that

(i) any $\lambda \neq k^2\omega^2$, for each $k = 1, 2, \ldots$, is not an eigenvalue; and

(ii) $\lambda = k^2\omega^2$, for some $k = 1, 2, \ldots$, is an eigenvalue if and only if $C = k^2\omega^2 A$, A nonsingular.

We observe that (i) implies in particular that any $\lambda < \omega^2$ is not an eigenvalue, and also by (ii), the first possible eigenvalue is $\lambda = \omega^2$.

Each of the statements (i) or (ii) has an important bearing on the solvability of the PBVP for the non-autonomous system

$$X''' + AX'' + \lambda X' + CX = P(t) \tag{3.7}$$

with $P \in L^1$.

It is clear for instance, from (i) and the Fredholm alternative, that a solution for (1.4) can be expected if the ratio $\langle G(t, X'), X'\rangle / \|X'\|^2$ is such that

$$k^2\omega^2 < \frac{\langle G(t, X'), X'\rangle}{\|X'\|^2} < (k+1)^2\omega^2,$$

for $\|X'\|$ sufficiently large, and a.e. $t \in [0, T]$, provided that some control is put on the closeness of the ratio to $k^2\omega^2$ and $(k+1)^2\omega^2$.

The main role of statement (ii) is to provide an adequate background against which the sharpness of our conditions on G can be tested.

Our existence result is based on the following proposition for the Leray–Schauder alternative:

Proposition 3.1 *Let B be a suitable nonsingular constant matrix such that the homogeneous linear system*

$$X''' + AX'' + BX' + CX = 0$$

has no nontrivial T-periodic solution.

If all the T-periodic solutions of the λ-dependent family of equations

$$X''' + AX'' + (1-\lambda)BX' + CX + \lambda G(\cdot, X') = \lambda P(t)$$

are uniformly a-priori bounded independently of $\lambda \in (0,1)$, that is, there exists an open bounded set $\Omega \in W_T^{3,1}$ such that $0 \in \Omega$, and for any $\lambda \in (0,1)$ each solution $X_\lambda \in W_T^{3,1}$ of the λ-dependent system ultimately satisfies $X_\lambda \notin \partial\Omega$.

Then the equation (1.4)-(1.5) has at least one T-periodic solution.

Proof Let us define the following operators

$$\begin{array}{ll} L : dom\, L = W_T^{3,1} \subset L^\infty \to L^1, & X \longmapsto LX := X''' + AX'' + CX \\ N : \overline{\Omega} \subset W_T^{3,1} \to L^1, & X \longmapsto NX := P(t) - G(\cdot, X') \\ A : dom\, A \subset W_T^{3,1} \to L^1, & X \longmapsto AX := -BX \end{array}$$

The above homotopy therefore translates into the equivalent functional equation

$$LX - (1-\lambda)AX - \lambda NX = 0 \quad \text{where} \quad (X, \lambda) \in (dom\, L \cap \overline{\Omega}) \times (0,1)$$

with

$$LX - (1-\lambda)AX - \lambda NX \neq 0 \quad \text{for every} \quad (X, \lambda) \in (dom\, L \cap \partial\Omega) \times (0,1)$$

and $ker(L - A) = \{0\}$.

Clearly L is a linear Fredholm mapping of index zero. Moreover, N and A are L-completely continuous and thus N is L-compact on $\overline{\Omega}$ (see Rachůnková [13]). The assertion of the Proposition now follows from Mawhin [10] (Theorem IV.5). □

Let ν and β be constants defined by

$$\nu = \frac{1}{2}(k^2\omega^2 + (k+1)^2\omega^2), \qquad \beta = \frac{1}{2}((k+1)^2\omega^2 - k^2\omega^2).$$

The following preliminary result will be required in the construction of the *a-priori* estimate Ω of Proposition 3.1:

Lemma 3.2 *Let $X = X(t)$ be any twice continuously differentiable function of t and A be any constant matrix. Then, there exists a constant $\delta > 0$ such that*

$$\beta^2 \int_0^T \|X'\|^2\, dt \le \int_0^T \|X''' + \nu X'\|^2\, dt$$

$$\delta \left(\int_0^T \|X''\|^2\, dt \right)^{\frac{1}{2}} \le \int_0^T \|X''' + AX''\|\, dt$$

Proof From the Fourier expansion of $X(t)$ given componentwise as

$$x_i(t) \sim a_{0,i} + \sum_{k=1}^{\infty}(a_{k,i}\cos(k\omega t) + b_{k,i}\sin(k\omega t)), \tag{3.8}$$

$i = 1, \dots, n$, $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, we have

$$x_i'(t) \sim \sum_{k=1}^{\infty} k\omega(-a_{k,i}\sin(k\omega t) + b_{k,i}\cos(k\omega t)) \tag{3.9}$$

$$x_i'''(t) \sim \sum_{k=1}^{\infty} k^3\omega^3(a_{k,i}\sin(k\omega t) - b_{k,i}\cos(k\omega t)) \tag{3.10}$$

Thus,

$$\begin{aligned}
\int_0^T \|X''' + \nu X'\|^2\, dt &= \frac{T}{2}\sum_{i=1}^{n}\sum_{k=1}^{\infty} k^2\omega^2(k^2\omega^2 - \nu)^2({a_{k,i}}^2 + {b_{k,i}}^2) \\
&\geq \beta^2\,\frac{T}{2}\sum_{i=1}^{n}\sum_{k_i=1}^{\infty} k^2\omega^2({a_{k,i}}^2 + {b_{k,i}}^2) \\
&= \beta^2 \int_0^T \|X'\|^2\, dt
\end{aligned} \tag{3.11}$$

by the definition of β given above, and the first inequality follows. Furthermore, setting $A = \mu I$, we have

$$\begin{aligned}
\int_0^T \|X''' + \mu X''\|\, dt &= \int_0^T \sum_{i=1}^{n}\sum_{k=1}^{\infty} k^2\omega^2\big[\big(k\omega\, a_{k,i} - \mu\, b_{k,i}\big)\sin(k\omega t) \\
&\quad - \big(k\omega\, b_{k,i} + \mu\, a_{k,i}\big)\cos(k\omega t)\big]\, dt \\
&\geq \int_0^T \sum_{i=1}^{n}\sum_{k=1}^{\infty} k^{-2}\omega^{-2}\big(k\omega + \mu\big)^{-1}\big(a_{k,i} + b_{k,i}\big)^{-1}\, dt \\
&\geq T\,\Big[\sum_{i=1}^{n}\sum_{k=1}^{\infty}\big(k^2\omega^2 + \mu^2\big)^{-1} \\
&\quad \times \sum_{i=1}^{n}\sum_{k=1}^{\infty} k^{-4}\omega^{-4}\big(a_{k,i}^2 + b_{k,i}^2\big)^{-1}\Big]^{-\frac{1}{2}} \\
&\geq \Big(\frac{1}{2T}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\big(k^2\omega^2 + \mu^2\big)^{-1}\Big)^{-\frac{1}{2}} \\
&\quad \times \Big(\frac{T}{2}\sum_{i=1}^{n}\sum_{k=1}^{\infty} k^4\omega^4\big(a_{k,i}^2 + b_{k,i}^2\big)\Big)^{\frac{1}{2}}
\end{aligned} \tag{3.12}$$

by the Hölder inequality, so that the second inequality of the lemma also holds, with $\delta = n\sqrt{2T}(\sum_{k=1}^{\infty}(k^2\omega^2 + \mu^2)^{-1})^{-\frac{1}{2}}$. □

We shall now prove an existence result for (1.4)–(1.5) which represents vector analogue of Theorem 1 of Ezeilo and Omari [8].

Theorem 3.3 *Let C be a nonsingular matrix and assume that G satisfies*

$$(\mathcal{G}_1) \qquad k^2\omega^2 + \alpha^-(\|X'\|) \le \frac{\langle G(t,X'),X'\rangle}{\|X'\|^2} \le (k+1)^2\omega^2 - \alpha^+(\|X'\|),$$

uniformly in $X'' \in \mathbb{R}^n$ with $\|X'\| \ge r > 0$, and a.e. $[0,T]$, where $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, and $\alpha^\pm : \mathbb{R}^n_+ \to \mathbb{R}$ are two functions which are such that

$$(\mathcal{G}_2) \qquad \lim_{\|X'\|\to+\infty} \|X'\|\alpha^\pm(\|X'\|) = +\infty$$

Then system (1.4)–(1.5) has at least one solution, for every $P \in L^1([0,T],\mathbb{R}^n)$ and all arbitrary matrix A.

Proof We shall consider (1.4) in the equivalent form

$$X''' + AX'' + \nu X' + CX = \nu X' - G(t,X') + P(t) \tag{3.13}$$

Then for each $P \in L^1([0,T],\mathbb{R}^n)$, there exists exactly one function $W = \mathcal{K}P \in W^{3,1}_T([0,T],\mathbb{R}^n)$, satisfying (1.5) and

$$W''' + AW'' + \nu W' + CW = P(\cdot), \tag{3.14}$$

by the Fredholm alternative, where $\mathcal{K} : dom\mathcal{K} \subset L^1 \to W^{3,1}_T$

Making the change of variable $Z = X - W$, (3.13) becomes

$$\begin{aligned} Z''' + AZ'' + \nu Z' + CZ &= \nu Z' + \nu W' - G(Z' + W') \\ &= \nu Z' - \gamma(\cdot, Z') \end{aligned} \tag{3.15}$$

where we set $\gamma(t,X') = G(t,X'+W'(t)) - \nu W'(t)$, $X' \in \mathbb{R}^n$, $t \in [0,T]$. Then, γ is continuous and moreover by hypothesis $(\mathcal{G}_1)$ and $(\mathcal{G}_2)$,

$$\begin{aligned} \|\gamma(t,X') - \nu X'\| &= \|G(t,X'+W'(t)) - \nu(X'+W'(t))\| \\ &\le (\|X'\| + \|W'\|_\infty)\left(\frac{\langle G(t,X'),X'\rangle}{\|X'\|^2} - \nu\right) \\ &\le (\|X'\| + \|W'\|_\infty)(\beta - \alpha^+(\|X'\|)) \\ &\le \beta\|X'\| - k_1, \end{aligned} \tag{3.16}$$

uniformly in $X'' \in \mathbb{R}^n$ and a.e. $[0,T]$ with $\|X'\| \ge r_1$, for every $k_1 > 0$ and $r_1 > 0$ depending on k_1 and $\|W'\|_\infty$.

This implies that for a suitable constant $k_2 > 0$,

$$\|\gamma(t,X') - \nu X'\|^2 \le \beta^2\|X'\|^2 - 2\beta k_1\|X'\| + k_2, \tag{3.17}$$

for all $X' \in \mathbb{R}^n$ and $t \in [0,T]$.

Next, we define the Nemytskii operator

$$N : C^1([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}^n) \quad \text{by} \quad NZ = \nu Z' - \gamma(\cdot, Z').$$

Then, the Hammerstein operator $\mathcal{K}N : C^1([0,T],\mathbb{R}^n) \to C^1([0,T],\mathbb{R}^n)$ is completely continuous and its (possible) fixed points in C^1 are the solutions in C^3 of (3.15) which, by the transformation $Z = X - W$, determine the solutions in $W^{3,1}$ of (1.4).

For solving the fixed point problem

$$Z = \mathcal{K}NZ,$$

in C^1, we use the Leray-Schauder continuation method given by Proposition 3.1. Accordingly, we consider the problems depending on a parameter $\lambda \in [0,1]$,

$$Z = \lambda \mathcal{K}NZ. \tag{3.18}$$

It is sufficient to find a constant $R > 0$ such that $\|Z\| < R$ for every Z satisfying (3.18) for $\lambda \in (0,1)$, or equivalently,

$$Z''' + AZ'' + \gamma_\lambda(t, Z') + CZ = 0, \tag{3.19}$$

where we set $\gamma_\lambda(t, X') = (1-\lambda)\nu X' + \lambda\gamma(t, X')$, $X' \in \mathbb{R}^n$.

Therefore, let $Z \in C^3$ be a solution to (3.19) for some $\lambda \in (0,1)$. Multiplying (3.19) scalarly by $Z'''(t)+\nu Z'(t)$ and integrating over $[0,T]$ using (1.5), observing that

$$\int_0^T \langle AZ'' + CZ,\ Z''' + \nu Z'\rangle\, dt = 0,$$

we obtain

$$\int_0^T \langle Z''' + \gamma_\lambda(t, Z'),\ Z''' + \nu Z'\rangle\, dt = 0 \tag{3.20}$$

That is

$$\int_0^T \left(\langle Z''', Z'''\rangle + \langle Z''', \nu Z'\rangle + \langle \gamma_\lambda(t, Z'), Z'''\rangle + \langle \gamma_\lambda(t, Z'), \nu Z'\rangle\right) dt = 0 \tag{3.21}$$

Noting that $\langle Z, Z\rangle = \|Z\|^2$, it is easily verified that (3.21) can be written as

$$\int_0^T \|Z''' + \nu Z'\|^2\, dt + \int_0^T \|Z''' + \gamma_\lambda(t, Z')\|^2\, dt - \int_0^T \|\gamma_\lambda(t, Z') - \nu Z'\|^2\, dt = 0 \tag{3.22}$$

so that, on dropping the second integral,

$$\int_0^T \|Z''' + \nu Z'\|^2\, dt \le \int_0^T \|\gamma_\lambda(t, Z') - \nu Z'\|^2\, dt = \lambda^2 \int_0^T \|\gamma(t, Z') - \nu Z'\|^2\, dt \tag{3.23}$$

which, by (3.17), yields

$$\int_0^T \|Z''' + \nu Z'\|^2 \, dt \le \beta^2 \int_0^T \|Z'\|^2 \, dt - 2\beta k_1 \int_0^T \|Z'\| \, dt + k_2 T \tag{3.24}$$

On the other hand, by Lemma 3.2, we know that for every $Z \in C^3$ satisfying (1.5), we have

$$\beta^2 \int_0^T \|Z'\|^2 \, dt \le \int_0^T \|Z''' + \nu Z'\|^2 \, dt,$$

so that from (3.24), we derive

$$\|Z'\|_{L^2} = \int_0^T \|Z'\| \, dt \le (2\beta k_1)^{-1} k_2 T := c_1 \tag{3.25}$$

Next, we observe that by (3.16), γ satisfies, for some constants k_3, $k_4 > 0$, the condition

$$\|\gamma(t, X')\| \le \|\gamma(t, X') - \nu X'\| + \nu \|X'\| \le k_3 \|X'\| + k_4$$

Hence, integrating (3.19) over $[0, T]$, we obtain

$$\begin{aligned} \| \int_0^T Z(t) \, dt \| \; &\le \; \int_0^T \|C^{-1} \gamma_\lambda(t, Z')\| \, dt \le \|C^{-1}\|_2 \int_0^T \|\gamma(t, Z')\| \, dt \\ &\le \; \|C^{-1}\|_2 (k_3 \|Z'\|_{L^1} + k_4 T) \le \delta_c^{-1} (k_3 c_1 + k_4 T) \\ &:= c_2 \end{aligned} \tag{3.26}$$

Thus, combining (3.25) and (3.26) yields

$$\begin{aligned} \|Z\|_\infty \le T^{-1} \| \int_0^T Z(t) \, dt\| + \|Z'\|_{L^1} \; &\le \; T^{-1} c_2 + c_1 \\ &:= c_3 \end{aligned} \tag{3.27}$$

Also, observing from (3.19) that

$$\begin{aligned} \|Z''' + AZ''\|_{L^1} \; &\le \; (1-\lambda)\nu \int_0^T \|Z'\| \, dt + \lambda \int_0^T \|\gamma(t, Z')\| \, dt + T \|C\|_2 \|Z\|_\infty \\ &\le \; \nu c_1 + (k_3 c_1 + k_4 T) + T c_3 \Delta_c := c_4, \end{aligned} \tag{3.28}$$

we conclude by the second inequality of Lemma 3.2 that

$$\|Z''\|_{L^2} \le \delta^{-2} c_4 := c_5 \tag{3.29}$$

and then

$$\|Z'\|_\infty \le \sqrt{T} \|Z''\|_{L^2} = c_5 \sqrt{T} := c_6 \tag{3.30}$$

It follows from (3.27) and (3.30) that

$$\|Z\|_{C^1} = \|Z\|_\infty + \|Z'\|_\infty \le c_3 + c_6 := c_7 \tag{3.31}$$

for every solution Z of (3.19), for arbitary $\lambda \in (0, 1)$. Thus $\|Z\|_{C^1} \le \Omega$ follows, for some $\Omega > c_7 > 0$.

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