Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **41** (2002) 131–145

Topological Essentiality for Multivalued Weighted Mappings

Robert SKIBA

Faculty of Mathematics and Informatics,
Nicholas Copernicus University,
Chopina 12/18, 87-100 Toruń, Poland
e-mail: robo@mat.uni.torun.pl

(Received November 12, 2001)

Abstract

In this paper we introduce a concept of topological essentiality for multivalued weighted mappings. Topological essentiality can be defined on a large class of mappings than topological degree. Topological essentiality was systemmaticaly studied in [5], [8], [6], [15].

Key words: Topological essentiality, fixed point, weighted mappings.

2000 Mathematics Subject Classification: 47H04, 47H10, 54C60, 55M20

0 Introduction

First we recall some well known notions and introduce necessary notations. All the considered topological spaces are assumed to be metric. A space X is called an absolute retract ($X \in AR$) if for each space Y and for any homeomorphism $h : X \to Y$ such that $h(X)$ is a closed subset of Y, the set $h(X)$ is a retract of Y; that is, there exists a continuous map $r : Y \to h(X)$ such that $r(y) = y$ for every $y \in h(X)$.

Let X and Y be two spaces and assume that for every point $x \in X$ a nonempty finite subset $\varphi(x)$ of Y is given; in this case we say that $\varphi : X \multimap Y$ is a multivalued map. The symbol $f : X \to Y$ is reserved for singlevalued mappings. A multivalued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) or lower semicontinuous (l.s.c.) provided for any open $V \subset Y$ the set

$$\varphi^{-1}(V) = \{x \in X \mid \varphi(x) \subset V\}$$

or the set

$$\varphi_+^{-1}(V) = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\},$$

respectively, is open. We say that φ is continuous when it is both l.s.c. and u.s.c.. See [7] for more details concerning multivalued mappings.

We shall say that two mappings $\varphi, \psi : X \multimap Y$ have a coincidence if there exists a point $x \in X$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$.

This paper is organized as follows. Section 1 is devoted to weighted mappings. In section 2 we give the definition and the most important properties of topological essentiality for w-maps. Section 3 is concerned with applications of the topological essentiality.

1 Weighted mappings

Definition 1.1 A weighted mapping from X to Y with coefficients in a commutative ring with unity Ω (or simply a w-map) is a pair $\varphi = (\sigma_\varphi, w_\varphi)$ satisfying the following conditions:

- $\sigma_\varphi : X \multimap Y$ is a multivalued upper semicontinuous mapping;
- $w : X \times Y \to \Omega$ is a function with the following properties:
  - $w_\varphi(x, y) = 0$ for every $y \notin \sigma_\varphi(x)$;
  - if U is an open subset of Y and $x \in X$ is such that $\sigma_\varphi(x) \cap \partial U = \emptyset$, then there exists an open neighbourhood V of the point x such that
$$\sum_{y \in U} w_\varphi(x, y) = \sum_{y \in U} w_\varphi(z, y),$$
    for every $z \in V$.

Note 1.1 For our comfort a multivalued weighted mapping from X to Y, i.e. $\varphi = (\sigma_\varphi, w_\varphi)$, we shall denote by $\varphi : X \multimap Y$. So, by $\varphi(x)$ we shall mean $\sigma_\varphi(x)$ for every $x \in X$. The mapping σ_φ from the above definition will be called a support of φ. By a weight of φ we shall understand a function w_φ, i.e. $w_\varphi : X \times Y \to \Omega$. Moreover, each continuous map $f : X \to Y$ can be considered as a weighted one by assigning the coefficient 1 to each $f(x)$.

Now we shall give some examples of weighted maps.

Example 1.1 Let $\varphi : X \multimap Y$ be a continuous map such that for all $x \in X$ $\varphi(x)$ consist of 1 or exactly n points (with n fixed). A weight $w_\varphi : X \times Y \to \mathbb{Z}$ we define by the formula:

$$w_\varphi(x, y) = \begin{cases} 0 & \text{if } y \notin \varphi(x) \\ n & \text{if } \{y\} = \varphi(x) \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that a pair $\varphi = (\sigma_\varphi = \varphi, w_\varphi)$ is a w-map.

Example 1.2 let $f : X \to SP^nY$ be a singlevalued map and let $\Pi : SP^nY \multimap Y$ be a multivalued map which is defined by

$$\Pi(x_1^{k_1} \cdots x_s^{k_s}) = \{x_1, \dots, x_s\},$$

where SP^nY denotes the n-th symmetric product of Y and $x_1^{k_1} \cdots x_s^{k_s}$ denotes an equivalence class in SP^nY (see [11]). Then f induces a w-map $\varphi = (\sigma_\varphi, w_\varphi)$, where

$$\sigma_\varphi : X \multimap Y \quad \text{and} \quad w_\varphi : X \times Y \to \mathbb{Z}$$

are defined as follows:

$$\sigma_\varphi(x) = \Pi \circ f(x)$$

and

$$w_\varphi(x, y) = \begin{cases} k_i & \text{if } y \in \sigma_\varphi(x) \\ 0 & \text{if } y \notin \sigma_\varphi(x). \end{cases}$$

Now we recall the notion of the w-homotopy.

Definition 1.2 Let $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ be w-maps. We say that φ is w-homotopic to ψ ($\varphi \sim_w \psi$) if there exists a w-map $\mathbb{H} : X \times [0,1] \multimap Y$ such that:

$$w_{\mathbb{H}}((x, 0), y) = w_\varphi(x, y)$$

and

$$w_{\mathbb{H}}((x, 1), y) = w_\psi(x, y).$$

Let us underline that we do not demand in order to

$$\sigma_{\mathbb{H}}(x, 0) = \sigma_\varphi(x) \quad \text{and} \quad \sigma_{\mathbb{H}}(x, 1) = \sigma_\psi(x).$$

Below we shall list important and well known properties of w-maps.

Proposition 1.1 *If $\psi, \varphi : X \multimap Y$ are w-maps, then $\psi \cup \varphi = (\sigma_{\psi\cup\varphi}, w_{\psi\cup\varphi})$ is also one, where*

$$\sigma_{\psi\cup\varphi} : X \multimap Y \quad \text{and} \quad w_{\psi\cup\varphi} : X \times Y \to \Omega$$

are defined by the formulas:

$$\sigma_{\psi\cup\varphi}(x) = \sigma_\psi(x) \cup \sigma_\varphi(x)$$

and

$$w_{\psi\cup\varphi}(x, y) = w_\psi(x, y) + w_\varphi(x, y),$$

for every $x \in X$ and $y \in Y$.

Proposition 1.2 *If $\varphi : X \multimap Y$ is a w-map and $\lambda \in \Omega$, then $\lambda\varphi = (\sigma_{\lambda\varphi}, w_{\lambda\varphi})$ is also one, where*

$$\sigma_{\lambda\varphi}(x) = \sigma_\varphi(x) \quad \text{and} \quad w_{\lambda\varphi}(x,y) = \lambda \cdot w_\varphi(x,y),$$

for every $x \in X$ and $y \in Y$.

Proposition 1.3 *If $\psi : X \multimap Y$ and $\varphi : Y \multimap Z$ are w-maps, then $\varphi \circ \psi : X \multimap Z$ is a w-map, where its support $\sigma_{\varphi\circ\psi}$ is the composition of σ_φ and σ_ψ and a weight $w_{\varphi\circ\psi} : X \times Z \to \Omega$ is defined by the formula:*

$$w_{\varphi\circ\psi}(x,z) = \sum_{y\in Y} w_\psi(x,y) \cdot w_\varphi(y,z),$$

for every $x \in X$ and $z \in Z$.

Proposition 1.4 *If $\varphi : X_1 \multimap Y_1$ and $\psi : X_2 \multimap Y_2$ are w-maps, then $\varphi \times \psi = (\sigma_{\varphi\times\psi}, w_{\varphi\times\psi})$ is also one, where*

$$\sigma_{\varphi\times\psi} : X_1 \times X_2 \multimap Y_1 \times Y_2$$

and

$$w_{\varphi\times\psi} : (X_1 \times X_2) \times (Y_1 \times Y_2) \to \Omega$$

are defined as follows:

$$\sigma_{\varphi\times\psi}(x_1, x_2) = \sigma_\varphi(x_1) \times \sigma_\psi(x_2)$$

and

$$w_{\varphi\times\psi}((x_1,x_2),(y_1,y_2)) = w_\varphi(x_1,y_1) \cdot w_\psi(x_2,y_2),$$

for every $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in Y_1$, $y_2 \in Y_2$.

Now we shall consider some algebraic properties of w-maps. They will play a crucial role in topological essentiality.

Definition 1.3 Let E be a normed space and let $\psi, \varphi : X \multimap E$ be two w-maps. By $\psi + \varphi : X \multimap E$ we shall understand a pair $\psi + \varphi = (\sigma_{\psi+\varphi}, w_{\psi+\varphi})$, where

$$\sigma_{\psi+\varphi} : X \multimap E \quad \text{and} \quad w_{\psi+\varphi} : X \times E \to \Omega$$

are defined as follows:

$$\sigma_{\psi+\varphi}(x) = \{u + v \mid u \in \psi(x) \text{ and } v \in \varphi(x)\};$$

$$w_{\psi+\varphi}(x,u) = \sum_{e\in E} w_\psi(x, u-e) \cdot w_\varphi(x,e).$$

Due to our definition we obtain the following:

Proposition 1.5 *The above pair $\psi + \varphi = (\sigma_{\psi+\varphi}, w_{\psi+\varphi})$ is a w-map.*

Proof It is sufficient to show that $\psi + \varphi$ can be represented as the composition of some w-maps. Let

$$\triangle : X \to X \times X, \qquad f : E \times E \to E$$

be w-maps, where

$$\triangle(x) = (x, x), \qquad f(u, v) = u + v,$$

for every $x \in X$, $u, v \in E$. From Proposition 1.3 and 1.4 it follows that

$$f \circ (\psi \times \varphi) \circ \triangle$$

is a w-map. We show that $\psi + \varphi = f \circ (\psi \times \varphi) \circ \triangle$. Indeed, it is not difficult to see that

$$\sigma_{\psi+\varphi}(x) = \sigma_{f\circ(\psi\times\varphi)\circ\triangle}(x).$$

Now, it remains to show that for every $x \in X$, $u \in E$

$$w_{\psi+\varphi}(x, u) = w_{(f\circ(\psi\times\varphi))\circ\triangle}(x, u).$$

So, let $x \in X$ and $u \in E$, then

$$\begin{aligned}
&w_{(f\circ(\psi\times\varphi))\circ\triangle}(x, u) = \\
&= \sum_{(x_1,x_2)\in X\times X} w_\triangle(x, (x_1, x_2)) \cdot w_{f\circ(\psi\times\varphi)}((x_1, x_2), u) \\
&= w_\triangle(x, (x, x)) \cdot w_{f\circ(\psi\times\varphi)}((x, x), u) \\
&= 1 \cdot \Big[\sum_{(e_1,e_2)\in E\times E} w_f((e_1, e_2), u) \cdot w_{\psi\times\varphi}((x, x), (e_1, e_2)) \Big] \\
&= \sum_{(e_1,e_2)\in f^{-1}(u)} w_\psi(x, e_1) \cdot w_\varphi(x, e_2),
\end{aligned}$$

because $(e_1, e_2) \in f^{-1}(u)$ if and only if $e_1 + e_2 = u$. Consequently

$$= \sum_{e\in E} w_\psi(x, u - e) \cdot w_\varphi(x, e).$$

This completes the proof. □

Definition 1.4 Let E be a normed space and let $\psi, \varphi : X \multimap E$ be two w-maps. By $\psi - \varphi : X \multimap E$ we shall understand a pair $\psi - \varphi = (\sigma_{\psi-\varphi}, w_{\psi-\varphi})$, where

$$\sigma_{\psi-\varphi} : X \multimap E \quad \text{and} \quad w_{\psi-\varphi} : X \times E \to \Omega$$

are defined as follows:

$$\sigma_{\psi-\varphi}(x) = \{u - v \mid u \in \psi(x) \text{ and } v \in \varphi(x)\};$$

$$w_{\psi-\varphi}(x, u) = \sum_{e\in E} w_\psi(x, u + e) \cdot w_\varphi(x, e).$$

Now, reasoning exactly as in the proof of Proposition 1.5 (of course, a function $f : E \times E \to E$ we have to define by the formula: $f(u,v) = u - v$) we obtain:

Proposition 1.6 *A pair $\psi - \varphi = (\sigma_{\psi-\varphi}, w_{\psi-\varphi})$ is a w-map.*

Definition 1.5 Let $\mathbb{E}$ be a normed space and $\varphi : X \multimap \mathbb{E}$ and let $s : X \to \mathbb{R}$ be a continous function. By $s\varphi : X \multimap \mathbb{E}$ we shall understand a pair $s\varphi = (\sigma_{s\varphi}, w_{s\varphi})$, where

$$\sigma_{s\varphi} : X \multimap \mathbb{E} \quad \text{and} \quad w_{s\varphi} : X \times \mathbb{E} \to \Omega$$

are defined as follows:

$$\sigma_{s\varphi}(x) = \{s(x)u \mid u \in \varphi(x)\}$$

and

$$w_{s\varphi}(x,u) = \begin{cases} w_\varphi(x, \frac{u}{s(x)}) & \text{if } s(x) \neq 0 \\ \sum_{e \in \mathbb{E}} w_\varphi(x,e) & \text{if } s(x) = 0,\ u = 0 \\ 0 & \text{if } s(x) = 0,\ u \neq 0. \end{cases}$$

Now we shall prove the following:

Proposition 1.7 *A pair $s\varphi = (\sigma_{s\varphi}, w_{s\varphi})$ defined above is a w-map.*

Proof We show that

$$s\varphi = f \circ (s \times \varphi) \circ \triangle,$$

where $f : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ is defined by the formula:

$$f(\alpha, e) = \alpha \cdot e,$$

for every $\alpha \in \mathbb{R}$, $e \in \mathbb{E}$. We can easily verify that

$$\sigma_{s\varphi}(x) = \sigma_{f\circ(s\times\varphi)\circ\triangle}(x).$$

To complete the proof of the proposition, it suffices now to prove that for any $x \in X$, $u \in \mathbb{E}$

$$w_{f\circ(s\times\varphi)\circ\triangle}(x,u) = w_{s\varphi}(x,u).$$

Indeed, let $x \in X$ and $u \in \mathbb{E}$, we obtain

$$\begin{aligned}
& w_{f\circ(s\times\varphi)\circ\triangle}(x,u) = \\
& = \sum_{(x_1,x_2)\in X\times X} w_\triangle(x,(x_1,x_2)) w_{f\circ(s\times\varphi)}((x_1,x_2),u) \\
& = w_\triangle(x,(x,x)) w_{f\circ(s\times\varphi)}((x,x),u) \\
& = \sum_{(\alpha,e)\in\mathbb{R}\times\mathbb{E}} w_{s\times\varphi}((x,x),(\alpha,e)) w_f((\alpha,e),u) \\
& = \sum_{(\alpha,e)\in\mathbb{R}\times\mathbb{E}} w_s(x,\alpha) w_\varphi(x,e) w_f((\alpha,e),u) \\
(\star)\ & = \sum_{e\in\mathbb{E}} w_s(x,s(x)) w_\varphi(x,e) w_f((s(x),e),u) =
\end{aligned}$$

Now we distinguish three cases:

Case 1 ($s(x) \neq 0$) Then we get

$$(\star) = w_s(x, s(x)) w_\varphi \left(x, \frac{u}{s(x)}\right) w_f \left(\left(s(x), \frac{u}{s(x)}\right), u\right) = w_\varphi \left(x, \frac{u}{s(x)}\right),$$

because $f((s(x), e)) = u$ if and only if $e = \frac{u}{s(x)}$.

Case 2 ($s(x) = 0, u = 0$) Let us observe that $s(x) \cdot e = u$ holds for all $e \in \mathbb{E}$. Therefore

$$(\star) = \sum_{e \in \mathbb{E}} w_\varphi(x, e) w_f((s(x), e), u) = \sum_{e \in \mathbb{E}} w_\varphi(x, e).$$

Case 3 ($s(x) = 0, u \neq 0$) Since $s(x)e = u$ does not hold for any $e \in \mathbb{E}$, we get that

$$(\star) = \sum_{e \in \mathbb{E}} w_\varphi(x, e) \cdot 0 = 0.$$

This ends the proof. □

Remark 1.1 Let $\varphi, \psi : X \multimap \mathbb{E}$ be two w-maps. It is easy to see that the following equation:

$$w_{\varphi+\psi}(x, u) = w_\varphi(x, u) + w_\psi(x, u)$$

is not true in general.

Let us recall now the notion of the index of the w-map.

Definition 1.6 Let $\varphi : X \multimap Y$ be a w-map and let a space X be connected. Then the sum

$$\sum_{y \in Y} w_\varphi(x, y)$$

is called the index of the weighted mapping, where $x \in X$. We shall denote it by $I_w(\varphi)$.

Remark 1.2 The above definition is correct because the sum

$$\sum_{y} w_\varphi(x, y)$$

does not depend on $x \in X$ if the space X is connected (see lemma 2.3 in [9]).

Proposition 1.8 *The above index has the following properties (see [10]).*

(a) If $\varphi, \psi : X \multimap Y$ are w-homotopic, then

$$I_w(\varphi) = I_w(\psi).$$

(b) If $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ are w-maps, then

$$I_w(\psi \circ \varphi) = I_w(\psi) \cdot I_w(\varphi).$$

(c) If $\varphi : X_1 \multimap Y_1$ and $\psi : X_2 \multimap Y_2$ are w-maps, then

$$I_w(\varphi \times \psi) = I_w(\varphi) \cdot I_w(\psi).$$

(d) If $f : X \to Y$ is a continuous map, then $I_w(f) = 1$ (see Note 1.1).

In view of the above Propositions we obtain:

Proposition 1.9 *Let $\varphi, \psi : X \multimap E$ be two w-maps and let $s : X \to \mathbb{R}$ be continuous map. Then:*

(a) $I_w(\varphi + \psi) = I_w(\varphi) \cdot I_w(\psi)$;

(b) $I_w(\varphi - \psi) = I_w(\varphi) \cdot I_w(\psi)$;

(c) $I_w(s \cdot \varphi) = I_w(\varphi)$.

Finally this section, we recall the Schauder fixed point theorem for w-maps.

Theorem 1.1 *If $X \in AR$, then any w-map $\varphi : X \multimap X$ with $I_w(\varphi) \neq 0$ has a fixed point.*

2 Topological essentiality

Now we are in a position to define a notion of topological essentiality for weighted mappings. In what follows $\mathbb{E}, \mathbb{F}$ are two real normed spaces and $\mathbb{U}$ is an open connected bounded subset of $\mathbb{E}$. By $\overline{\mathbb{U}}$ we shall denote the closure of $\mathbb{U}$ in $\mathbb{E}$. From now on we consider only w-maps φ with $I_w(\varphi) \neq 0$. Moreover, we assume that Ω is the field. We let:

$$\mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F}) = \{\varphi : \overline{\mathbb{U}} \multimap \mathbb{F} \mid \varphi \text{ is a } w\text{-map and } 0 \notin \varphi(\partial\mathbb{U})\};$$

$$\mathbb{W}_{\mathbb{C}}(\mathbb{U}, \mathbb{F}) = \{\varphi : \overline{\mathbb{U}} \multimap \mathbb{F} \mid \varphi \text{ is a } w\text{-map and compact}\};$$

$$\mathbb{W}_0(\mathbb{U}, \mathbb{F}) = \{\varphi : \overline{\mathbb{U}} \multimap \mathbb{F} \mid \varphi \in \mathbb{W}_{\mathbb{C}}(\mathbb{U}, \mathbb{F}) \text{ and } \varphi(x) = \{0\} \text{ for every } x \in \partial\mathbb{U}\}$$

Definition 2.1 A w-map $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ is called essential (with respect to $\mathbb{W}_0(\mathbb{U}, \mathbb{F})$) provided for any $\psi \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$ there exists a point $x \in \mathbb{U}$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$.

Let us observe that if $\mathbb{E} = \mathbb{F}$ then the notion of essentiality can be reintepreted as $\mathbb{Z}_2$ topological degree. We give now some examples of essential w-maps.

Example 2.1 Let $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{B},\mathbb{R})$, where $\mathbb{B}$ is an open ball at $0 \in \mathbb{E}$ with radius $r > 0$. If there exist $x_0, x_1 \in \partial\mathbb{B}$ such that

$$u < 0 \quad \text{for every } u \in \varphi(x_0)$$

and

$$v > 0 \quad \text{for every } v \in \varphi(x_1),$$

then φ is an essential w-map.

To see this, we shall need the following lemma:

Lemma 2.1 ([1]) *Let $\varphi : [a,b] \multimap \mathbb{R}$ be a w-map from the interval $[a,b]$ to $\mathbb{R}$ with $I_w(\varphi) \neq 0$. Suppose that $\varphi(a) \subset \mathbb{R}^-$ and $\varphi(b) \subset \mathbb{R}^+$. Then $0 \in \varphi(x)$ for some $x \in [a,b]$.*

Let $\psi \in \mathbb{W}_0(\mathbb{B},\mathbb{R})$ and suppose to the contrary that $\varphi(x) \cap \psi(x) = \emptyset$ for every $x \in \overline{\mathbb{B}}$. We define $\eta : \overline{\mathbb{B}} \multimap \mathbb{R}$ by the formula:

$$\eta(x) = \varphi(x) - \psi(x).$$

In view of Proposition 1.6 and 1.9 we obtain that η is a w-map with $I_w(\eta) \neq 0$. Moreover, $0 \notin \eta(x)$ for every $x \in \overline{\mathbb{B}}$. Let $\gamma : [0,1] \to \overline{\mathbb{B}}$ be a path between x_0 and x_1 and let $\tilde{\eta} = \eta \circ \gamma$. It is easy to see that $\tilde{\eta}$ has the following properties:

(a) $\tilde{\eta}$ is a w-map with $I_w(\eta) \neq 0$,

(b) $\tilde{\eta}(0) \subset \mathbb{R}^-$ and $\tilde{\eta}(1) \subset \mathbb{R}^+$,

(c) $0 \notin \tilde{\eta}(x)$ for every $x \in [0,1]$.

But this contradicts Lemma 2.1. This ends the proof of essentiality of φ.

Example 2.2 (Essentiality of homeomorphism) Let U be an open and bounded subset of $\mathbb{E}$ such that $\overline{U} \in AR$ and let $f : \overline{U} \to f(\overline{U})$ be a homeomorphism such that $f(\overline{U})$ is a closed subset of $\mathbb{F}$. In addition assume that $f(U)$ is an open subset of $\mathbb{F}$ and $0 \in f(U)$. Then f is an essential w-map.

Indeed, let $\psi \in \mathbb{W}_0(U,\mathbb{F})$. Since $f(\overline{U}) \in AR$ there exists a retraction $r : \mathbb{F} \to f(\overline{U})$. Let us denote by $g : f(\overline{U}) \to \overline{U}$ an inverse function of f. Consider

$$\mathbb{B} = \{x \in \overline{U} \mid f(x) \in (t \cdot \psi(x)) \text{ for some } t \in [0,1]\}.$$

It is easy to see that $\mathbb{B}$ is closed in $\overline{U}$ and nonempty (since $0 \in f(U)$). Let $s : \mathbb{F} \to [0,1]$ be an Urysohn function such that $s(y) = 1$ for $y \in f(\mathbb{B})$ and $s(y) = 0$ for $y \in \mathbb{F} \setminus f(U)$. A definition of s is correct because $f(\mathbb{B})$ is closed in $\mathbb{F}$ and $f(\mathbb{B}) \subset f(U)$. Define $\eta : \mathbb{F} \multimap \mathbb{F}$ by the formula:

$$\eta(y) = s(y)\psi(g(r(y))),$$

for every $y \in \mathbb{F}$. It is easy to see that η is a compact w-map. Moreover, $I_w(\eta) \neq 0$. Indeed,

$$I_w(\eta) = I_w(s \circ \psi \circ g \circ r) = I_w(s)I_w(\psi)I_w(g)I_w(r) = 1 \cdot I_w(\psi) \cdot 1 \cdot 1 \neq 0.$$

Hence, in view of Theorem 1.1, we have a fixed point: $y \in \eta(y)$. If $y \in F \setminus f(U)$, then $s(y) = 0$ and $y = 0$ but $0 \in f(U)$ so we get a contradiction. Therefore we deduce that $y \in f(U)$. It follows that there exists a point $x \in U$ such that $f(x) = y$. Consequently $f(x) \in s(f(x))\psi(x)$. Then $x \in \mathbb{B}$ and hence $f(x) \in \psi(x)$. This completes the proof of essentiality of f. In particular we obtain:

Example 2.3 (Essentiality of linear isomorphism) Let $L : \mathbb{E} \to \mathbb{F}$ be a continuous linear isomorphism and let U be an open bounded subset of the origin in $\mathbb{E}$ such that $\overline{U} \in AR$. Then the restriction $\tilde{L} : \overline{U} \to \mathbb{F}$ of L to $\overline{U}$ is essential.

Let us enumerate several properties of the topological essentiality.

Proposition 2.1 (Existence) *If $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ is essential, then there exists $x \in \mathbb{U}$ such that $0 \in \varphi(x)$.*

Proof Indeed, let $\psi : \overline{\mathbb{U}} \multimap \mathbb{F}$ be defined by the formula: $\psi(x) = 0$, for every $x \in \overline{\mathbb{U}}$. It is easy to see that $\psi \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$. Now our claim follows from Definition 2.1. □

Proposition 2.2 (Compact perturbation) *If $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ is essential and $\eta \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$, then $\varphi + \eta \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ is an essential w-map.*

Proof Let $\psi \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$ and consider the map $\psi_1 : \overline{\mathbb{U}} \multimap \mathbb{F}$ given by

$$\psi_1(x) = \psi(x) - \eta(x).$$

In view of Proposition 1.6 we get that ψ_1 is a w-map. Moreover, it is easy to see that $\psi_1 \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$. Since $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ is essential, there exists a point $x \in \mathbb{U}$ such that:

$$\varphi(x) \cap \psi_1(x) \neq \emptyset.$$

Hence we obtain that there exists a point $x \in \mathbb{U}$ such that

$$(\varphi(x) + \eta(x)) \cap \psi(x) \neq \emptyset.$$

This completes the proof. □

Proposition 2.3 (Coincidence) *Assume that $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ is an essential w-map and $\eta \in \mathbb{W}_{\mathbb{C}}(\mathbb{U}, \mathbb{F})$. Let*

$$\mathbb{B} = \{x \in \overline{\mathbb{U}} \mid \varphi(x) \cap (t\eta(x)) \neq \emptyset \text{ for some } t \in [0, 1]\}.$$

If $\mathbb{B} \subset \mathbb{U}$, then φ and η have a coincidence.

Proof First observe that the essentiality of φ implies that $\mathbb{B}$ is nonempty. Moreover, it is easy to see that $\mathbb{B}$ is closed. Let $s : \overline{\mathbb{U}} \to [0,1]$ be an Urysohn function such that $s(x) = 1$ for $x \in \mathbb{B}$ and $s(x) = 0$ for $x \in \partial\mathbb{U}$ (we recall that due to our assumption: $\mathbb{B} \cap \partial\mathbb{U} = \emptyset$). We define the w-map $\psi : \overline{\mathbb{U}} \multimap \mathbb{F}$ as follows:

$$\psi(x) = s(x)\eta(x),$$

for every $x \in \overline{\mathbb{U}}$. It is clear that $\psi \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$ and since φ is an essential w-map we get that for some $x_0 \in \mathbb{U}$:

$$\varphi(x_0) \cap (s(x_0)\eta(x_0)) = \varphi(x_0) \cap \psi(x_0) \neq \emptyset.$$

This implies that $x_0 \in \mathbb{B}$ and hence $s(x_0) = 1$. Consequently, we get:

$$\varphi(x_0) \cap \eta(x_0) \neq \emptyset.$$

This completes the proof. □

Proposition 2.4 (Normalization) *Assume that* $0 \notin \partial\mathbb{U}$, $\overline{\mathbb{U}} \in AR$. *Then the inclusion map is an issential w-map if and only if* $0 \in \mathbb{U}$.

Proof ($\Rightarrow$) It follows immediately from the existence property.

($\Leftarrow$) Let $\psi \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$. Define the set $\mathbb{B}$ as follows:

$$\mathbb{B} = \{x \in \overline{\mathbb{U}} \mid x \in (t\psi(x)) \text{ for some } t \in [0,1]\}.$$

Then $\mathbb{B}$ is a closed nonempty subset of $\overline{\mathbb{U}}$ such that $0 \in \mathbb{B}$ and $\mathbb{B} \subset \mathbb{U}$. We consider an Urysohn function $s : \mathbb{E} \to [0,1]$ such that $s(x) = 1$ for $x \in \mathbb{B}$ and $s(x) = 0$ for $\mathbb{E}\backslash\mathbb{U}$. Since $\mathbb{E} \supset \overline{\mathbb{U}} \in AR$, there exists the retraction map $r : \mathbb{E} \to \overline{\mathbb{U}}$. Next, let $\varphi : \mathbb{E} \multimap \mathbb{E}$ be defined as follows:

$$\varphi(x) = s(x)\psi(r(x)).$$

In view of Proposition 1.3 and 1.7 we obtain that φ is a w-map with $I_w(\varphi) \neq 0$. Hence the Schauder fixed point theorem implies that φ has a fixed point; i.e. there exists $x_0 \in \mathbb{E}$ such that $x_0 \in \varphi(x_0)$. If $x_0 \notin \mathbb{U}$, then $s(x_0) = 0$ and $x_0 = 0$ but $0 \in \mathbb{U}$ so we get a contradiction. Hence we obtain that $x_0 \in \mathbb{U}$. Therefore

$$x_0 \in s(x_0)\varphi(x_0).$$

So, $x_0 \in \mathbb{B}$ and hence $x_0 \in \varphi(x_0)$. This completes the proof. □

Proposition 2.5 (Localization) *Let* $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ *be an essential w-map. Assume that* $\mathbb{V}$ *is an open subset of* $\mathbb{U}$ *such that* $\varphi_+^{-1}(\{0\}) \subset \mathbb{V}$ *and* $\overline{\mathbb{V}} \in AR$. *Then the restriction* $\varphi|_{\overline{\mathbb{V}}}$ *of* φ *to* $\overline{\mathbb{V}}$ *is an essential w-map.*

Proof It is easy to see that $\varphi|_{\overline{\mathbb{V}}}$ is a w-map. Next, from the existence property it follows that $\varphi_+^{-1}(\{0\})$ is nonempty. Let $\psi \in \mathbb{W}_0(\mathbb{V}, \mathbb{F})$ and let $\mathbb{A}$ be defined as follows:

$$\mathbb{A} = \{x \in \overline{\mathbb{V}} \mid \varphi(x) \cap (t\psi(x)) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

It is clear that $\mathbb{A} \subset \mathbb{V}$. Again let $s : \overline{\mathbb{U}} \to [0,1]$ be an Urysohn function such that $s(x) = 1$ for $x \in \mathbb{A}$ and $s(x) = 0$ for $x \notin \mathbb{V}$. Since $\overline{\mathbb{V}} \in AR$, there exists the retraction $r : \overline{\mathbb{U}} \to \overline{\mathbb{V}}$ such that $r(y) = y$ for every $y \in \overline{\mathbb{V}}$. We define the map $\eta : \overline{\mathbb{U}} \multimap \mathbb{F}$ by the formula:

$$\eta(x) = s(x) \cdot \psi(r(x)),$$

for every $x \in \overline{\mathbb{U}}$. In view of Proposition 1.3 and 1.7 we get that η is a w-map. Obviously $\eta \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$. Since φ is essential, there is a point $x_0 \in \mathbb{U}$ such that

$$\varphi(x_0) \cap \eta(x_0) \neq \emptyset.$$

It is easy to see that $x_0 \in \mathbb{V}$. Consequently

$$s(x_0) = 1 \text{ and } r(x_0) = x_0$$

and hence

$$\varphi|_{\overline{mV}}(x_0) \cap \psi(x_0) \neq \emptyset.$$

This completes the proof. □

Proposition 2.6 (w-homotopy) *Let $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ be an essential w-map. If $\mathbb{H} : \overline{\mathbb{U}} \times [0,1] \multimap \mathbb{F}$ is a compact w-map such that:*

(i) $\mathbb{H}(x, 0) = \{0\}$ for every $x \in \partial\mathbb{U}$,

(ii) $\{x \in \overline{\mathbb{U}} \mid \varphi(x) \cap \mathbb{H}(x,t) \neq \emptyset$ for some $t \in [0,1]\} \subset \mathbb{U}$.

Then the map $\varphi(\cdot) - \mathbb{H}(\cdot, 1)$ is an essential w-map.

Proof First let us observe that from the compact perturbation property we get that $\varphi(\cdot) - \mathbb{H}(\cdot, 0)$ is an essential w-map. Let $\psi \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$. We let:

$$\mathbb{B} = \{x \in \overline{\mathbb{U}} \mid \varphi(x) \cap (\psi(x) + \mathbb{H}(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

Since $(\psi(\cdot) + \mathbb{H}(\cdot, 0)) \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$ and φ is an essential w-map we obtain that $\mathbb{B}$ is a nonempty closed subset of $\overline{\mathbb{U}}$. It is clear that $\mathbb{B} \subset \mathbb{U}$. Let $s : \overline{\mathbb{U}} \to [0,1]$ be an Urysohn function such that $s(x) = 1$ for $x \in \mathbb{B}$ and $s(x) = 0$ for $x \in \partial\mathbb{U}$. We define $\eta : \overline{\mathbb{U}} \multimap \mathbb{F}$ as follows:

$$\eta(x) = \psi(x) + \mathbb{H}(x, s(x)).$$

Notice that η is a w-map and $\eta \in \mathbb{W}_0(\mathbb{U}, \mathbb{F})$ (see section 1). Since φ is essential, there exists a point $x_0 \in \mathbb{U}$ such that

$$\varphi(x_0) \cap \eta(x_0) \neq \emptyset.$$

Notice that $x_0 \in \mathbb{B}$ and therefore

$$\varphi(x_0) \cap (\psi(x_0) + \mathbb{H}(x_0, 1)) \neq \emptyset.$$

Consequently

$$(\varphi(x_0) - \mathbb{H}(x_0, 1)) \cap \psi(x_0) \neq \emptyset.$$

This completes the proof. □

Proposition 2.7 (Continuation) *Let $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ be an essential w-map. Assume that φ is proper, i.e. $\varphi_+^{-1}(\mathbb{K})$ is compact for every compact $\mathbb{K} \subset \mathbb{F}$. Assume further that $\eta : \overline{\mathbb{U}} \times [0,1] \multimap \mathbb{F}$ is a compact w-map such that $\eta(x, 0) = \{0\}$ for every $x \in \partial\mathbb{U}$. Then there exists $\varepsilon > 0$ such that the mapping $\varphi(\cdot) - \eta(\cdot, \lambda) : \overline{\mathbb{U}} \multimap \mathbb{F}$ is an essential w-map for every $\lambda \in [0, \varepsilon)$.*

Proof According to the w-homotopy property it is sufficient to show that there exists $\varepsilon > 0$ such that:

$$\varphi(x) \cap \eta(x, \lambda) = \emptyset,$$

for every $\lambda \in [0, \varepsilon)$ and $x \in \partial\mathbb{U}$. This condition it is easy to verify by contradiction. So, suppose to the contrary that for every $\varepsilon > 0$ there exists $\lambda \in [0, \varepsilon)$ and $x \in \partial\mathbb{U}$ such that

$$\varphi(x) \cap \eta(x, \lambda) \neq \emptyset.$$

Let $\varepsilon_n = \frac{1}{n}$, $n = 1, 2, \ldots$ Then there exist sequences x_n, y_n, λ_n such that:

$$\lambda_n \in [0, \frac{1}{n}), \quad y_n \in \varphi(x_n) \cap \eta(x_n, \lambda_n) \neq \emptyset, \quad x_n \in \partial\mathbb{U},$$

for $n = 1, 2, \ldots$ Since φ is compact and proper we may assume without loss of generality that

$$\lim x_n = x_0 \in \partial\mathbb{U} \quad \text{and} \quad \lim y_n = y_0.$$

It is clear that $\lim \lambda_n = 0$. In view of the upper semicontinuity of φ and η we get that $y_0 \in \varphi(x_0)$ and $y_0 \in \eta(x_0, 0)$. But $\eta(x_0, 0) = \{0\}$ because $x_0 \in \partial\mathbb{U}$. Consequently $y_0 = 0$ and hence $0 \in \varphi(x_0)$—a contradiction. This completes the proof. **Proof**

3 Applications

The topological essentiality has many applications in fixed point theory, analysis and other fields. In this paper we give a few examples.

Proposition 3.1 *Let $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U}, \mathbb{F})$ be an essential w-map and proper. If $\mathbb{D}$ is connected component of $\mathbb{F} \setminus \varphi(\partial\mathbb{U})$, which contains 0, then $\mathbb{D} \subset \varphi(\overline{\mathbb{U}})$.*

Proof The set $\varphi(\partial\mathbb{U})$ is a closed subset of $\mathbb{F}$, because any proper map is closed. Let $v \in \mathbb{D}$. We shall show that $v \in \varphi(\overline{\mathbb{U}})$. Since $\mathbb{F} \setminus \varphi(\partial\mathbb{U})$ is open, its components are open, and for open set in $\mathbb{F}$ connectedness is the same as arcwise connectedness. So, let $\sigma : [0,1] \to \mathbb{D}$ be a continuous curve with $\sigma(0) = 0$ and $\sigma(1) = v$. Define w-homotopy $\eta : \overline{\mathbb{U}} \times [0,1] \multimap \mathbb{F}$ by the formula:

$$\eta(x,t) = \sigma(t),$$

for every $(x,t) \in \overline{\mathbb{U}} \times [0,1]$. Now we can apply the w-homotopy property to deduce that:

$$\varphi(\cdot) - \eta(\cdot,1) : \overline{\mathbb{U}} \times [0,1] \multimap \mathbb{F}$$

is an essential w-map. Notice that

$$\varphi(x) - \eta(x,1) = \varphi(x) - \{v\},$$

for every $x \in \overline{\mathbb{U}}$. Hence from the existence property we obtain that $v \in \varphi(\mathbb{U})$. This ends the proof. □

Proposition 3.2 *Let $\varphi \in \mathbb{W}_{\partial\mathbb{U}}(\mathbb{U},\mathbb{F})$ be an essential w-map and $\psi \in \mathbb{W}_{\mathbb{C}}(\mathbb{U},\mathbb{F})$. If $\varphi(x) \cap \psi(x) = \emptyset$ for every $x \in \partial\mathbb{U}$, then at least one of the following conditions holds:*

(1) there exists $x \in \mathbb{U}$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$;

(2) there exists $\lambda \in (0,1)$ and $x \in \partial\mathbb{U}$ such that $\varphi(x) \cap (\lambda\psi(x)) \neq \emptyset$.

To see this it is enough to apply the w-homotopy property for φ and ψ, where $\mathbb{H}(x,t) = t \cdot \psi(x)$ for $x \in \overline{\mathbb{U}}$ and $t \in [0,1]$. Let us observe that if $\mathbb{E} = \mathbb{F}$, then from the above Proposition and the normalization property we obtain the following:

Proposition 3.3 (Nonlinear alternative) *Let $\psi \in \mathbb{W}_{\mathbb{C}}(\mathbb{U},\mathbb{F})$ and $0 \in \mathbb{U}$, then at least one of the following conditions is satisfied:*

(1) $Fix(\psi) \neq \emptyset$,

(2) there exists $x \in \partial\mathbb{U}$ and $\lambda \in (0,1)$ such that $x \in \lambda\psi(x)$.

Finally we would like to underline that also some other results remain true. For example we can prove the version of the Birkhoff–Kellogg theorem and Borsuk's theorem on antipodes. The proofs are similar to those obtained using the topological degree technique and therefore we leave them to the reader.

Acknowledgment I would like to thank Professor L. Górniewicz for valuable discussions and helpful suggestions.

References

[1] Conti, G., Pejsachowicz, J.: *Fixed point theorem for multivalued weighted maps.* Ann. Mat. Pura Appl. **126**, 4 (1980), 319–342.

[2] Darbo, G.: *Teoria dell'omologia in una categoria di maps plurivalenti ponderate.* Rend. Sem. Mat. della Univ. di Padova **28** (1958), 188–224.

[3] Darbo, G.: *Sulle coincidenze di mappe ponderate.* Rend. Sem. Mat. Univ. di Padova **29** (1959), 256–270.

[4] Darbo, G.: *Estensione alle mappe del teorema Lefshetz sui punti fissi.* Rend. Sem. Mat. Univ. Padova **31** (1961), 46–57.

[5] Dugundji, J., Granas, G.: *Fixed Point Theory.* Monograf. Mat. **61**, PWN, Warszawa, 1982.

[6] Górniewicz, L., Ślosarski, M.: *Topological essentiality and differential inclusions.* Bull. Austral. Math. Soc. **45** (1992), 177–193.

[7] Górniewicz, L.: Topological Fixed Point Theory of Multivalued Mappings. *Kluwer Acad. Pub.,* Vol. 495, 1999.

[8] Granas, A.: *The theory of compact vector field and some of its applications.* Dissertationes Math. (Warszawa) **30** (1962).

[9] Jerrard, R.: *Homology with multivalued functions applied to fixed points.* Trans. AMS **213** (1975), 407–428.

[10] Jodko-Narkiewicz, S.: *Topological degree of multivalued weighted mappings.* UMK Toruń 1989 (Thesis in Polish).

[11] Miklaszewski, D.: *Fixed points of symmetric product mappings.* UMK Toruń 1991 (Thesis in Polish).

[12] Pejsachowicz, J.: *Relation between the homotopy and the homology theory of weighted mappings.* Bolletino U.M.I. **15-B** (1978), 285–302.

[13] Pejsachowicz, J.: *The homotopy theory of weighted mappings.* Bolletino U.M.I. **14-B** (1977), 702–720.

[14] Skiba, R.: *On the Lefschetz fixed point theorem for multivalued weighted mappings.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **40** (2001), 201–214.

[15] Ślosarski, M.: *Certain applications of topological essentiality and charakterization of the set of fixed points to differential inclusions.* UMK Toruń 1993 (Thesis in Polish).