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On Two Points Boundary Value Problems for Ordinary Nonlinear Differential Equations of the Fourth Order in the Colombeau Algebra

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Abstract

The existence and uniqueness of the two points boundary value problems for nonlinear differential equations of the fourth order in the Colombeau algebra $\mathcal{G}(\mathbb{R})$ are considered.

Key words: Generalized ordinary differential equations, boundary value problems, generalized functions, distributions, Colombeau algebra.

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1 Introduction

For fixed $i = 1, 2, 3$ we will consider the following problems

$$x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)) \tag{1.1}$$

$$L_{ij}(x) = d_j, \qquad j = 1, 2, 3, 4, \tag{1.2_i}$$

where

$$L_{11}(x) = x(0), \quad L_{12}(x) = x'(0), \quad L_{13}(x) = x''(0), \quad L_{14}(x) = x'''(0),$$
$$L_{21}(x) = x(0), \quad L_{22}(x) = x(T), \quad L_{23}(x) = x'(0), \quad L_{24}(x) = x'(T),$$
$$L_{31}(x) = x(0), \quad L_{32}(x) = x(T), \quad L_{33}(x) = x''(0), \quad L_{34}(x) = x''(T)$$

and $0 < t < \infty$.

We assume that f is an element of the Colombeau algebra $\mathcal{G}(\mathbb{R}^5)$, d_j for $j = 1, 2, 3, 4$ are elements of the algebra $\overline{\mathbb{R}}$ of generalized real numbers, $x(0)$, $x'(0)$, $x''(0)$, $x'''(0)$, $x(T)$, $x'(T)$, $x''(T)$ are understood as the values of the generalized functions x, x', x'', x''' at the point 0 and T respectively. The elements d_j and f are given. The multiplication, the differentiation, the sum, the composition of generalized functions and the equality are meant in the Colombeau algebra sense. We prove theorems on the existence and uniqueness of solutions of the problems (1.1), (1.2_i). In certain cases they generalize some of the results given in [2], [7], [12], [15], [17] and [20].

2 Notation

Here we recall some basic definitions which are needed later on. For more details concerning generalized functions, generalized real numbers as well as for the proof of the assertions mentioned in this section, see [3].

Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with the compact support. For $q \in \mathbb{N}$ we denote by $\mathcal{A}_q(\mathbb{R})$ the set of all functions $\overline{\varphi} \in \mathcal{D}(\mathbb{R})$ with the following properties:

$$\int_{-\infty}^{\infty} \overline{\varphi}(t)\, dt = 1, \quad \int_{-\infty}^{\infty} t^k \overline{\varphi}(t)\, dt = 0, \quad k = 1, 2, \ldots, q.$$

We put

$$\mathcal{A}_q(\mathbb{R}^n) = \{\varphi(t_1, \ldots, t_n) = \prod_{r=1}^{n} \overline{\varphi}(t_r) : \overline{\varphi} \in \mathcal{A}_q(\mathbb{R})\}, \quad \mathbb{N}_0^n = \underbrace{\mathbb{N}_0 \times \ldots \times \mathbb{N}_0}_{n\text{-times}},$$

where $\mathbb{N}_0$ denotes the set of all non-negative integer numbers.

Furthemore, $\mathcal{E}[\mathbb{R}^n]$ is the set of all functions $R : \mathcal{A}_1(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}$ such that $R(\varphi, \cdot) \in C^\infty(\mathbb{R}^n)$ for every $\varphi \in \mathcal{A}_1(\mathbb{R}^n)$.

For $R \in \mathcal{E}[\mathbb{R}^n]$, $\varphi \in \mathcal{A}_1(\mathbb{R}^n)$, $t \in \mathbb{R}^n$ and $m \in \mathbb{N}_0^n$

$$D_m(R, t) = \frac{\partial^{|m|}}{\partial_{t_1}^{m_1} \cdots \partial_{t_n}^{m_n}} R(\varphi, t).$$

where $m = |m_1| + \ldots + |m_n|$. (In particular, $D_0 R(\varphi, t) = R(\varphi, t)$). Furthemore, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varepsilon > 0$ we define

$$\varphi_\varepsilon(t_1, \ldots, t_n) = \frac{1}{\varepsilon^n} \prod_{r=1}^{n} \overline{\varphi}\left(\frac{t_r}{\varepsilon}\right).$$

$R \in \mathcal{E}[\mathbb{R}^n]$ is said to be moderate, if for every compact subset K of $\mathbb{R}^n$ and every $m \in \mathbb{N}_0^n$ there is $N \in \mathbb{N}$ with the following property: for every $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_m R(\varphi_\varepsilon, t)| \leq c\varepsilon^{-N} \qquad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

The set of all moderate elements of $\mathcal{E}[\mathbb{R}^n]$ is denoted by $\mathcal{E}_M[\mathbb{R}^n]$.

By Γ we denote the set of functions $\alpha : \mathbb{N} \to \mathbb{R}^+$ which are increasing and such that $\lim_{q\to\infty} \alpha(q) = \infty$. Furthemore, we define an ideal $\mathcal{N}[\mathbb{R}^n]$ in $\mathcal{E}_M[\mathbb{R}^n]$ as follows: $R \in \mathcal{N}[\mathbb{R}^n]$ if for every compact subset K of $\mathbb{R}^n$ and every $m \in \mathbb{N}_0^n$ there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ with the property: for every $q \geq N$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_m R(\varphi_\varepsilon, t)| \leq c\varepsilon^{\alpha(q)-N} \qquad \text{if } \varepsilon \in (0, \varepsilon_0).$$

The algebra $\mathcal{G}(\mathbb{R}^n)$ (the Colombeau algebra) is defined as the quotient algebra of $\mathcal{E}_M[\mathbb{R}^n]$ with respect to $\mathcal{N}[\mathbb{R}^n]$, i.e.

$$\mathcal{G}(\mathbb{R}^n) = \frac{\mathcal{E}_M[\mathbb{R}^n]}{\mathcal{N}[\mathbb{R}^n]}.$$

Its elements are called generalized functions. For $R \in \mathcal{E}_M[\mathbb{R}^n]$, the corresponding class $G \in \mathcal{G}(\mathbb{R}^n)$ (i.e. $G = R + \mathcal{N}[\mathbb{R}^n]$) is denoted by $[R]$. Vice versa, if $G \in \mathcal{G}(\mathbb{R}^n)$, then its representative in $\mathcal{E}_M[\mathbb{R}^n]$ is usually denoted by R_G.

$\mathcal{E}_0$ is the set of the functions mapping $\mathcal{A}_1(\mathbb{R})$ into $\mathbb{R}$ and $\mathcal{E}_M$ is the set of all moderate elements of $\mathcal{E}_0$, i.e.

$$\begin{aligned} \mathcal{E}_M = \{R \in \mathcal{E}_0 :\ & \text{there is } N \in \mathbb{N} \text{ such that for every } \overline{\varphi} \in \mathcal{A}_N(\mathbb{R}) \\ & \text{there are } c > 0 \text{ and } \eta_0 > 0 \text{ such that} \\ & |R(\overline{\varphi}_\varepsilon| \leq c\varepsilon^{-N} \text{ for } \varepsilon \in (0, \eta_0)\}. \end{aligned}$$

The ideal $\mathcal{N}$ of $\mathcal{E}_M$ is defined by

$$\begin{aligned} \mathcal{N} = \{R \in \mathcal{E}_0 :\ & \text{there are } N \in \mathbb{N},\ \alpha \in \Gamma \text{ such that for any} \\ & q \geq N,\ \overline{\varphi} \in \mathcal{A}_q(\mathbb{R}) \text{ there are } c > 0 \text{ and } \eta_0 > 0 \text{ such that} \\ & |R(\overline{\varphi}_\varepsilon)| \leq c\varepsilon^{\alpha(q)-N} \text{ for } \varepsilon \in (0, \eta_0)\} \end{aligned}$$

and

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{N}}.$$

It is known that $\overline{\mathbb{R}}$ is an algebra, it is not a field. The elements of $\overline{\mathbb{R}}$ are called generalized real numbers.

If $R \in \mathcal{E}_M[\mathbb{R}]$ and $G = [R] \in \mathcal{G}(\mathbb{R})$, then for any $t \in \mathbb{R}$ the map $Y : \overline{\varphi} \to R(\overline{\varphi}, t) \in \mathbb{R}$ is defined on $\mathcal{A}_1(\mathbb{R})$ and belongs to $\mathcal{E}_M$. Furthemore, Y depends only on G and t and we denote it by $G(t)$. We say that $G(t)$ is the value of the generalized function G at the point t. $G \in \mathcal{G}(\mathbb{R})$ is said to be a constant generalized function on $\mathbb{R}$ if it admits a representative $R(\overline{\varphi}, t)$ which does not depend on t. With any $Z \in \overline{\mathbb{R}}$ we associate a constant generalized function $Z \in \mathcal{G}(\mathbb{R})$ which admits $R_Z(\overline{\varphi}, t) = Z(\overline{\varphi})$ as its representative.

If $f \in C^\infty(\mathbb{R})$ and $G \in \mathcal{G}(\mathbb{R})$, then $f(R_G)$ does not belong to $\mathcal{E}_M[\mathbb{R}]$ in general (see [3]). Now we will define composition of generalized functions. To this aim we will need the algebra $\tilde{\mathcal{G}}_\tau(\mathbb{R}^m \times \mathbb{R}^n)$ defined by

$$\tilde{\mathcal{G}}_\tau(\mathbb{R}^m \times \mathbb{R}^n) = \frac{\tilde{\mathcal{E}}_\tau[\mathbb{R}^m \times \mathbb{R}^n)}{\tilde{\mathcal{N}}_\tau[\mathbb{R}^m \times \mathbb{R}^n]},$$

where $\tilde{\mathcal{E}}_\tau[\mathbb{R}^m \times \mathbb{R}^n] = \{R \in \mathcal{E}[\mathbb{R}^m \times \mathbb{R}^n]$: such that, for every compact K of $\mathbb{R}^m$ and every $s \in \mathbb{N}_0^{m+n}$ there is $p \in \mathbb{N}$ with the following property: for every $\varphi \in \mathcal{A}_p(\mathbb{R}^m \times \mathbb{R}^n)$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t\in K}\sup_{x\in\mathbb{R}^n} \frac{|D_s R(\varphi_\varepsilon, t, x)|}{(1+|x|)^p} \le c\varepsilon^{-p} \qquad \text{for all } \varepsilon \in (0, \varepsilon_0)\}$$

and $\tilde{\mathcal{N}}_\tau[\mathbb{R}^m \times \mathbb{R}^n] = \{R \in \mathcal{E}[\mathbb{R}^m \times \mathbb{R}^n]$: such that, for every compact K of $\mathbb{R}^m$ and every $s \in \mathbb{N}_0^{m+n}$ there are $p \in \mathbb{N}$ and $\alpha \in \Gamma$ with the following property: for every $q \ge p$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^m \times \mathbb{R}^n)$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t\in K}\sup_{x\in\mathbb{R}^n} \frac{|D_s R(\varphi_\varepsilon, t, x)|}{(1+|x|)^p} \le c\varepsilon^{\alpha(q)-N} \qquad \text{for all } \varepsilon \in (0, \varepsilon_0)\}.$$

Let $g = [R_g(\varphi, u_1, \ldots, u_n)] \in \tilde{\mathcal{G}}_\tau(\mathbb{R}^n)$ and let $f_i = [R_{f_i}(\overline{\varphi}, t)] \in \mathcal{G}(\mathbb{R})$, where $i = 1, \ldots, n$ and $\varphi(t_1, \ldots, t_n) = \prod_{r=1}^{n} \overline{\varphi}(t_r)$. Then $R_g(\varphi, R_{f_1}(\overline{\varphi}, t), \ldots, R_{f_n}(\overline{\varphi}, t)) \in \mathcal{E}_M[\mathbb{R}]$ and we define

$$h = g(f_1, \ldots, f_n) = [R_g(\varphi, R_{f_1}(\overline{\varphi}, t), \ldots, R_{f_n}(\overline{\varphi}, t)] = [R_h(\overline{\varphi}, t)] \qquad \text{(see [8])}.$$

The class $[R_h(\overline{\varphi}, t)]$ does not depend on the choices of R_g and R_{f_i}. All this is rather straightforward from the definitions.

Throughout the paper $[0, T]$ stands for the compact interval $0 \le t \le T$. By $C^3[0, T]$ we denote the set of all real functions with the continuous 3rd derivative in $[0, T]$. Finally, for $x \in C^3[0, T]$ and $n \in \{0, 1, 2, 3\}$ we put

$$||D_n(x)||_{[0,T]} = \max_{t\in[0,T]} |D_n x(t)| \quad \text{and} \quad ||x||_{[0,T],3} = \sum_{i=0}^{3} ||D_i(x)||_{[0,T]}.$$

We say that $x \in \mathcal{G}(\mathbb{R})$ is a solution of the equation (1.1) if there is $\eta \in \mathcal{N}[\mathbb{R}]$ such that for any representative R_x of x the relations

$$D_4 R_x(\overline{\varphi}, t) - R_f(\varphi, t, R_x(\overline{\varphi}, t), D_1 R_x(\overline{\varphi}, t), D_2 R_x(\overline{\varphi}, t), D_3 R_x(\overline{\varphi}, t)) = \\ = \eta(\overline{\varphi}, t) \tag{2.1}$$

are satisfied for all $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$ and $t \in \mathbb{R}$. (Here $\varphi(t_1, \ldots, t_5) = \prod_{j=1}^{5} \overline{\varphi}(t_j)$).

The generalized function $p \in \mathcal{G}(\mathbb{R})$ is called of m-type if it has a representative $R_p(\overline{\varphi}, t)$ with the following property: for every compact interval $[-a, a]$ there is $N \in \mathbb{N}$ such that for every $\overline{\varphi} \in \mathcal{A}_N(\mathbb{R})$ there are constants $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t\in[-a,a]} \left| \int_0^t |R_p(\overline{\varphi}_\varepsilon, s)| ds \right| \leq c \qquad \text{for } \varepsilon \in (0, \varepsilon_0). \tag{2.2}$$

The generalized function f in $\tilde{G}_\tau(\mathbb{R} \times \mathbb{R}^n)$ is called of the semilinear type if it has a representative $R_f(\varphi, t, u_1, \ldots, u_n)$ with the following property: there are elements $p_j \in \mathcal{G}(\mathbb{R})$, $(j = 1, \ldots, n+1)$ and $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R} \times \mathbb{R}^n)$ there is constant $\varepsilon_0 > 0$ such that

$$|R_f(\varphi_\varepsilon, t, u_1, \ldots, u_n)| \leq \sum_{r=1}^{n} |R_{p_r}(\overline{\varphi}_\varepsilon, t)||u_r| + |R_{p_{n+1}}(\overline{\varphi}_\varepsilon, t)|, \tag{2.3}$$

for all $\varepsilon \in (0, \varepsilon_0)$, where p_r are of m-type and $\varphi(t_1, \ldots, t_{n+1}) = \prod_{j=1}^{n+1} \overline{\varphi}(t_j)$.

The generalized function f in $\tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^n)$ is called the Lipschitz type if it has a representative $R_f(\varphi, t, u_1, \ldots, u_n)$ with the following property: there are elements $p_r \in \mathcal{G}(\mathbb{R})$ and $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R} \times \mathbb{R}^n)$ there is constant $\varepsilon_0 > 0$ such that

$$|R_f(\varphi_\varepsilon, t, u_1, \ldots, u_n) - R_f(\varphi_\varepsilon, t, \overline{u_1}, \ldots, \overline{u_n})| \leq \sum_{r=1}^{n} |R_{p_r}(\overline{\varphi}_\varepsilon, t)|\, |u_r - \overline{u_r}|, \tag{2.4}$$

for all $\varepsilon \in (0, \varepsilon_0)$, where p_r are of m-type and $\varphi(t_1, \ldots, t_{n+1}) = \prod_{j=1}^{n+1} \overline{\varphi}(t_j)$.

Remark 2.1 Any generalized functions of the Lipschitz type is also of the semilinear type.

3 Main results

Theorem 3.1 *Let $f \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^4)$ and let ^{f} be the Lipschitz type. Then the problem (1.1)-(1.2$_1$) has a unique solution $x \in \mathcal{G}(\mathbb{R})$.*

Proof The proof of Theorem 3.1 is similar to that of Theorem 3.1 in the paper [8]. Indeed, let R_f and R_{d_j} be representative of f and d_j respectively. From the classical theory of differential equations we have that there is a unique solution $x(\overline{\varphi}_\varepsilon, t) \in C^\infty(\mathbb{R})$ to

$$\begin{cases} x^{(4)}(t) = R_f(\varphi_\varepsilon, t, x, x', x'', x''') \\ L_{1j}(x(\overline{\varphi}_\varepsilon, 0) = R_{d_j}(\overline{\varphi}_\varepsilon), \quad \overline{\varphi} \in \mathcal{A}_N(\mathbb{R}),\ j = 1, 2, 3, 4. \end{cases} \tag{3.1}$$

We will prove that $x(\overline{\varphi}_\varepsilon, t)$ belongs to $\mathcal{E}_M[\mathbb{R}]$. We have

$$\begin{aligned} x(\overline{\varphi}_\varepsilon, t) = {} & \frac{1}{6} \int_0^t (t-s)^3 R_f(\varphi_\varepsilon, s, x(\overline{\varphi}_\varepsilon, s), D_1 x(\overline{\varphi}_\varepsilon, s), D_2 x(\overline{\varphi}_\varepsilon, s), D_3 x(\overline{\varphi}_\varepsilon, s))\, ds \\ & + W_3(\overline{\varphi}_\varepsilon, t), \end{aligned} \tag{3.2}$$

where

$$W_3(\overline{\varphi}_\varepsilon, t) = R_{d_1}(\overline{\varphi}_\varepsilon) + R_{d_2}(\overline{\varphi}_\varepsilon)t + R_{d_3}(\overline{\varphi}_\varepsilon)\frac{t^2}{2} + R_{d_4}(\overline{\varphi}_\varepsilon)\frac{t^3}{6}\,.$$

Using (2.3), (3.2) and the Gronwall inequality we obtain that for every $\overline{\varphi} \in \mathcal{A}_N(\mathbb{R})$ there are $c_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t\in[-a,a]} |x(\overline{\varphi}_\varepsilon, t)| \le c_0\,\varepsilon^{-N} \qquad \text{for } \varepsilon \in (0, \varepsilon_0). \tag{3.3}$$

Hence, by (3.2) and (3.3) for every $r \in \mathbb{N}_0$ there is $N_r \in \mathbb{N}$ such that

$$\sup_{t\in[-a,a]} |D_r(\overline{\varphi}_\varepsilon, t)| \le c_r\,\varepsilon^{-N} \qquad \text{for } \varepsilon \in (0, \varepsilon_r) \text{ and } \overline{\varphi} \in \mathcal{A}_{N_r}(\mathbb{R}).$$

Now, if we extend the definition of $\overline{x}(\overline{\varphi}_\varepsilon, t)$ on the whole $\mathcal{A}_1(\mathbb{R})$, by setting $x(\overline{\varphi}, t) \equiv 0$ in the cases that $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R}) \setminus \mathcal{A}_N(\mathbb{R})$ we get $x(\overline{\varphi}, t) \in \mathcal{E}_M[\mathbb{R}]$. Denoting by x the class of $x(\overline{\varphi}, t)$ in $\mathcal{G}(\mathbb{R})$ we conclude that x is a solution of the problem (1.1), (1.2_1). Let $y \in \mathcal{G}(\mathbb{R})$ be another solution of the problem (1.0), (1.2_1). Then

$$\begin{aligned} R_y(\overline{\varphi}_\varepsilon, t) &= \\ = \frac{1}{6}\int_0^t (t-s)^3 (R_f(\varphi_\varepsilon, s, &R_y(\overline{\varphi}_\varepsilon, s), D_1R_y(\overline{\varphi}_\varepsilon, s), D_2R_y(\overline{\varphi}_\varepsilon, s), D_3R_y(\overline{\varphi}_\varepsilon, s)) \\ &+ R_\eta(\overline{\varphi}_\varepsilon, s))ds + W_3(\overline{\varphi}_\varepsilon, t) + R_{\overline{\eta}}(\overline{\varphi}_\varepsilon, t), \end{aligned} \tag{3.4}$$

$L_{1j}(R_y(\overline{\varphi}_\varepsilon, t)) = R_{d_j}(\overline{\varphi}_\varepsilon) + \eta_j(\overline{\varphi}_\varepsilon)$, where $R_\eta, R_{\overline{\eta}} \in \mathcal{N}[\mathbb{R}]$, $\eta_j \in \mathcal{N}$, $j = 1, 2, 3, 4$.

According to (2.4), (3.2), (3.4) and the Gronwall inequality we have that there are $\tilde{c_0} > 0$, $N \in \mathbb{N}$ and $\varepsilon_0' > 0$ such that

$$\sup_{t\in[-a,a]} |x(\overline{\varphi}_\varepsilon, t) - R_y(\overline{\varphi}_\varepsilon, t)| \le \tilde{c_0}\,\varepsilon^{\alpha(q)-N'} \tag{3.5}$$

for all $q \ge N'$, $\overline{\varphi} \in \mathcal{A}_N(\mathbb{R})$ and $\varepsilon \in (0, \varepsilon_0')$. Similarly, the relations (3.2)–(3.5) yield the existence of constants $c_r > 0$, $N_r' \in \mathbb{N}$ and $\varepsilon_r' > 0$ such that

$$\sup_{t\in[-a,a]} |D_r x(\overline{\varphi}_\varepsilon, t) - D_r R_y(\overline{\varphi}_\varepsilon, t)| \le \tilde{c_r}\varepsilon^{\alpha(q)-N_r'}$$

is true for $q \ge N_r'$, $\overline{\varphi} \in \mathcal{A}_q(\mathbb{R})$ and $\varepsilon \in (0, \varepsilon_r')$. We see that

$$x(\overline{\varphi}, t) - R_y(\overline{\varphi}, t) \in \mathcal{N}[\mathbb{R}]$$

which completes the proof of Theorem 3.1. □

Remark 3.1 If

$$f(t, u_1, u_2, u_3, u_4) = \sum_{r=1}^{4} p_r(t)\frac{1}{1 + u_1^2 + u_2^2 + u_3^2 + u_4^2} + p_5(t),$$

where $p_i \in \mathcal{G}(\mathbb{R})$, $i = 1, \ldots, 5$ and p_r are of m-type for $r = 1, \ldots, 4$, then f is the Lipschitz type.

Corollary 3.1 *Let $f \in \tilde{\mathcal{G}}_T(\mathbb{R} \times \mathbb{R}^4)$ and let f be of the semilinear type. Then the problem (1.1)–(1.2$_1$) has a solution in $\mathcal{G}(\mathbb{R})$*

By the same way we can prove the following

Corollary 3.2 *Let $f_i \in \tilde{\mathcal{G}}_T(\mathbb{R} \times \mathbb{R}^n)$ and let f_i be of the Lipschitz type. Then the problem*

$$x_i'(t) = f_i(t, x_1(t), \dots, x_n(t)) \tag{1.1$'$}$$

$$x_i(t_0) = d_i, \quad d_i \in \overline{\mathbb{R}}, \ t_0 \in \mathbb{R}, \ i = 1, \dots, n \tag{1.2$_1$$'$}$$

has a unique solution $x = (x_1, \dots, x_n) \in (\mathcal{G}(\mathbb{R}))^n$.

Corollary 3.3 *Let $f \in \tilde{\mathcal{G}}_T(\mathbb{R} \times \mathbb{R}^n)$ and let f_i be of the semilinear type. Then the problem (1.1)$'$–(1.2$_1$)$'$ has a solution in $(\mathcal{G}(\mathbb{R}))^n$.*

Now we will formulate two theorems on the existence and uniquenes of the solutions of the problem (1.1)–(1.2$_2$) and (1.1), (1.2$_3$). To this aim we will need the following hypothesis:

Hypothesis (H_i) There is $N \in \mathbb{N}$ such that for every $\overline{\varphi} \in \mathcal{A}_N(\mathbb{R})$ there exist $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the condition

$$a_i I_{0\varepsilon}(p_1, p_2, p_3, p_4) \le 1 - \gamma_0 \tag{3.6}$$

is satisfied with

$$I_{0\varepsilon}(p_1, p_2, p_3, p_4) = \sum_{r=1}^{4} \int_0^T |R_{p_r}(\overline{\varphi}_\varepsilon, t)| \, dt$$

and

$$a_i = \begin{cases} \frac{T^3}{192} + \frac{(39\sqrt{13}-138)T^2}{162} + \frac{4}{27}T + 1, & \text{if i} = 2 \\ \frac{T^3}{48} + \frac{T^2\sqrt{3}}{27} + \frac{T}{4} + 1, & \text{if i=3.} \end{cases} \tag{3.6$_i$}$$

Theorem 3.2 *Let $i \in \{2, 3\}$ and let (H_i) be true. Furthemore, suppose that all the assumptions of Theorem 3.1 are fulfilled. Then the problem (1.1), (1.2$_i$), has in $\mathcal{G}(\mathbb{R})$ exactly one solution.*

In the proof of Theorem 3.2 we will make use of the Green function $G_i(t, s)$, $i = 2, 3$ of the boundary value problem

$$x^{(4)}(t) = 0, \qquad L_{i_1}(x) = L_{i_2}(x) = L_{i_3}(x) = L_{i_4}(x) = 0$$

which are respectively defined by the following presciptions:

$$G_2(t, s) = \begin{cases} k_1(t, s), & \text{if } 0 \le t \le s \le T \\ k_2(t, s), & \text{if } 0 \le s \le t \le T, \end{cases} \tag{3.7$_2$}$$

$$G_3(t, s) = \begin{cases} l_1(t, s), & \text{if } 0 \le t \le s \le T \\ l_2(t, s), & \text{if } 0 \le s \le t \le T, \end{cases} \tag{3.7$_3$}$$

where

$$\begin{aligned}
k_1(t,s) &= \left(\frac{-s^3}{3T^3} + \frac{1}{2T^2}s^2 - \frac{1}{6}\right)t^3 + \left(\frac{1}{2T^2}s^3 - \frac{s^2}{T} + \frac{1}{2}s\right)t^2,\\
k_2(t,s) &= \left(\frac{-s^3}{3T^3} + \frac{1}{2T^2}s^2\right)t^3 + \left(\frac{1}{2T^2}s^3 - \frac{s^2}{T}\right)t^2 + \frac{1}{2}s^2 t - \frac{1}{6}s^3,\\
l_1(t,s) &= \frac{1}{6}\left(-1 + \frac{s}{T}\right)t^3 - t\left(\frac{-s^3}{6T} + \frac{s^2}{2} - \frac{s}{3}T\right),\\
l_2(t,s) &= \frac{t^3 s}{6T} - \frac{st^2}{2} + t\left(\frac{s^3}{6T} + \frac{s}{3}T\right) - \frac{s^3}{6}.
\end{aligned}$$

The properties of the functions G_i needed later on are described by the following lemma:

Lemma 3.1 *Let $i \in \{2,3\}$. Then*

$$\begin{aligned}
&\sup_{t,s\in[0,T]} |G_i(t,s)| = a_{i0}, && \sup_{t,s\in[0,T]} \left|\frac{\partial G_i}{\partial t}(t,s)\right| = a_{i1},\\
&\sup_{t,s\in[0,T]} \left|\frac{\partial^2 G_i(t,s)}{\partial t^2}\right| = a_{i2}, && \sup_{t,s\in[0,T]} \left|\frac{\partial^3 G_i(t,s)}{\partial t^3}\right| = a_{i3} = 1,
\end{aligned} \tag{3.8}$$

where

$$a_{i_0} = \begin{cases} \frac{T^3}{192} & \text{for } i=2\\ \frac{T^3}{48} & \text{for } i=3, \end{cases} \qquad a_{i_1} = \begin{cases} \frac{(39\sqrt{13}-138)T^2}{162} & \text{for } i=2\\ \frac{T^2\sqrt{3}}{27} & \text{for } i=3, \end{cases} \qquad a_{i2} = \begin{cases} \frac{4T}{27} & \text{for } i=2\\ \frac{T}{4} & \text{for } i=3. \end{cases}$$

Remark 3.2 Notice that

$$a_{20} = \left|G_2\left(\tfrac{T}{2},\tfrac{T}{2}\right)\right|, \quad a_{30} = \left|G_3\left(\tfrac{T}{2},\tfrac{T}{2}\right)\right|, \quad a_{21} = \left|\tfrac{\partial G_2}{\partial t}\left(\tfrac{(5-\sqrt{13})T}{6}, \tfrac{T(\sqrt{13}-1)}{6}\right)\right|,$$

$$a_{31} = \left|\tfrac{\partial G_3}{\partial t}\left(0, \tfrac{(3-\sqrt{3})T}{3}\right)\right|, \quad a_{22} = \left|\tfrac{\partial^2 G_2}{\partial t^2}\left(0,\tfrac{T}{3}\right)\right| \quad \text{and} \quad a_{32} = \left|\tfrac{\partial^2 G_3}{\partial t^2}\left(\tfrac{T}{2},\tfrac{T}{2}\right)\right|.$$

Proof of Theorem 3.2 Let $i \in \{2,3\}$ be given and let R_f and R_{d_j} be representative of f and d_j respectively. We consider the problem

$$x^{(4)}(t) = R_f(\varphi_\varepsilon, t, x(t), x'(t), x''(t), x'''(t)) \quad L_{ij}(x) = R_{d_j}(\overline{\varphi}_\varepsilon),\ j = 1,2,3,4. \tag{3.9}$$

It is easy to see that the problem (3.9) is equivalent to the problem of determining the fixed point of the operator T_i:

$$(T_i(x))(t) = \int_0^T G_i(t,s) R_f(\varphi_\varepsilon, s, x(s), x'(s), x''(s), x'''(s))\, ds + W_i(t), \tag{3.10}$$

where $x \in C^\infty[0,T]$, $W_i(t) = A_i(\overline{\varphi}_\varepsilon)t^3 + B_i(\overline{\varphi}_\varepsilon)t^2 + C_i(\overline{\varphi}_\varepsilon)t + E_i(\overline{\varphi}_\varepsilon)$; $A_i, B_i, C_i, E_i \in \mathcal{E}_M$ and $L_{ij}(W_i(t)) = R_{d_j}(\overline{\varphi}_\varepsilon)$.

By (2.4), Lemma 3.1 and (3.10) for all $x, y \in C^3[0,T]$ we have

$$||T_i(x) - T_i(y)||_{[0,T]} \leq a_{i0} I_{0\varepsilon}(p_1,p_2,p_3,p_4)||x-y||_{[0,T],3}, \quad (3.11)$$

$$||(T_i(x))' - (T_i(y))'||_{[0,T]} \leq a_{i1} I_{0\varepsilon}(p_1,p_2,p_3,p_4)||x-y||_{[0,T],3}, \quad (3.12)$$

$$||(T_i(x))'' - (T_i(y))''||_{[0,T]} \leq a_{i2} I_{0\varepsilon}(p_1,p_2,p_3,p_4)||x-y||_{[0,T],3} \quad (3.13)$$

and

$$||(T_i(x))''' - (T_i(y))'''||_{[0,T]} \leq a_{i3} I_{0\varepsilon}(p_1,p_2,p_3,p_4)||x-y||_{[0,T],3}, \quad (3.14)$$

where $I_{0\varepsilon}$ is defined by (3.6) and $x, y \in C^3_{[0,T]}$. Now, taking into account the relations (3.11)–(3.14), it follows that

$$||T_i(x) - T_i(y)||_{[0,T],3} \leq a_i I_{0\varepsilon}(p_1,p_2,p_3,p_4)||x-y||_{[0,T],3}. \quad (3.15)$$

Hence for fixed $\overline{\varphi} \in \mathcal{A}_N(\mathbb{R})$ and $\varepsilon \in (0,\varepsilon_0)$ we get that T_i is a contractive operator. In view of the contractivity of T_i the problem (3.10) has a solution. Let $x(\overline{\varphi}_\varepsilon, t) \in C^\infty[0,T]$ be a solution of the problem (3.9). We extend the definition of $x(\overline{\varphi}, t) \equiv 0$ on the whole $\mathcal{A}_1(\mathbb{R})$ by setting $x(\overline{\varphi}, t) \equiv 0$ in the cases that $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R}) \setminus \mathcal{A}_N(\mathbb{R})$. By the relations (2.4), (3.6) and (3.15) we conclude that for every $r \in \mathbb{N}_0$ there are constants $\overline{c}_r > 0, \varepsilon_r > 0$ and $N_r \in \mathbb{N}$ such that

$$||D_r x(\overline{\varphi}_\varepsilon, t)||_{[0,T],3} \leq \overline{c}_r \varepsilon^{-N_r} \quad (3.16)$$

is true for $\overline{\varphi} \in \mathcal{A}_{N_r}(\mathbb{R})$ and $\varepsilon \in (0,\varepsilon_r)$. Furthemore, if $t_0 \in (0,T)$, then (3.16) implies that

$$D_n x(\varphi_\varepsilon, t_0) \in \mathcal{E}_M \qquad \text{for } n = 0,1,2,3.$$

On the other hand $z = x(\overline{\varphi}_\varepsilon, t)$ is a solution of the problem

$$\begin{cases} z^{(4)}(t) = R_f(\varphi_\varepsilon, t, z(t), z'(t), z''(t), z'''(t)) \\ z(t_0) = x(\overline{\varphi}_\varepsilon, t_0), \quad z'(t_0) = D_1 x(\overline{\varphi}_\varepsilon, t_0), \quad z''(t_0) = D_2 x(\overline{\varphi}_\varepsilon, t_0), \\ z'''(t_0) = D_3 x(\overline{\varphi}_\varepsilon, t_0). \end{cases}$$

So, by virtue of Theorem 3.1 we obtain $x(\overline{\varphi}, t) \in \mathcal{E}_M[\mathbb{R}]$. If we define x as the class of $x(\overline{\varphi}, t)$ in $\mathcal{G}(\mathbb{R})$, then x is a solution of the problem (1.1), (1.2$_i$). To obtain uniqueness, assume that there are two solutions x and y with representatives $x(\overline{\varphi}, t)$ and $R_y(\overline{\varphi}, t)$. Then

$$\begin{aligned} R_y(\overline{\varphi}_\varepsilon, t) = \int_0^T & G_i(t,s)(R_f(\varphi_\varepsilon, s, R_y(\overline{\varphi}_\varepsilon, s), D_1 R_y(\overline{\varphi}_\varepsilon, s), D_2 R_y(\overline{\varphi}_\varepsilon, s), \\ & D_3 R_y(\overline{\varphi}_\varepsilon, s)) + \eta(\overline{\varphi}_\varepsilon, s))\, ds + W_i(t) + \overline{\eta}(\overline{\varphi}_\varepsilon, t), \end{aligned} \quad (3.17)$$

where $\eta, \overline{\eta} \in \mathcal{N}[\mathbb{R}]$.

Therefore, there are $\tilde{\varepsilon}_0 > 0$, $\tilde{c}_0 > 0$, $\tilde{N}_1 \in \mathbb{N}$ such that for $q \geq N_1$, $\overline{\varphi} \in \mathcal{A}_{\tilde{N}_1}(\mathbb{R})$ and $\varepsilon \in (0, \tilde{\varepsilon}_0)$ the following inequality is true

$$\|x(\overline{\varphi}_\varepsilon, t) - R_y(\overline{\varphi}_\varepsilon, t)\|_{[0,T],3} \leq$$
$$\leq a_i I_{0\varepsilon}(p_1, p_2, p_3, p_4)\|x(\overline{\varphi}_\varepsilon, t) - R_y(\overline{\varphi}_\varepsilon, t)\|_{[0,T],3} + c_0\varepsilon^{\alpha(q)-\tilde{N}_1}. \tag{3.18}$$

If $t_0 \in (0, T)$, then (3.18) inplies that

$$x(\overline{\varphi}_\varepsilon, t_0) - R_y(\overline{\varphi}_\varepsilon, t_0) \in \mathcal{N} \quad \text{and} \quad D_m x(\overline{\varphi}_\varepsilon, t_0) - D_m R_y(\overline{\varphi}_\varepsilon, t_0) \in \mathcal{N}$$

for $m = 1, 2, 3$. Using Theorem 3.1 we get $x = y$, which completes the proof of Theorem 3.2. □

Corollary 3.4 *Let $i \in \{2, 3\}$ and let (H_i) be true. Furthemore, suppose that all the assumptions of Corollary 3.1 are fulfilled. Then the problem (1.1), (1.2$_i$) has a solution $x \in \mathcal{G}(\mathbb{R})$.*

Remark 3.3 The generalized function $R_\delta(\overline{\varphi}, t) = \overline{\varphi}(t)$, $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$ is of m-type. Furhermore, if

$$R_{p_j}(\overline{\varphi}_\varepsilon, t) = \frac{b_j \overline{\varphi}_\varepsilon(t)}{\int_{-\infty}^{\infty} |\overline{\varphi}(t)|\, dt},$$

where $b_j, t \in \mathbb{R}$, $a_i\left(\sum_{j=1}^4 |b_j|\right) < 1$, a_i are defined by (3.6$_i$) and $p_j = [R_{p_j}]$, then hypothesis (H_i) is satisfied.

Corollary 3.5 *Let $i \in \{2, 3\}$, $p_r \in L_{loc}(\mathbb{R}), r = 1, 2, 3, 4$ and*

$$\gamma = a_i \left(\sum_{r=1}^{4} \int_0^T |p_r(t)|\, dt \right) < 1, \tag{3.19}$$

with a_i given by (3.6$_i$). Then the problem (1.1), (1.2$_i$) has a unique solution in the Caratheodory sense.

4 Relations between Carathéodory's and Colombeau's concepts of solutions of differential equations

Remark 4.1 It is known that every distribution is moderate (see [3]). In general, the composition in $\mathcal{G}(\mathbb{R})$ does not coincide with the usual composition of continuous functions (see [3]). As a consequence, solutions of ordinary diferential equations in the Carathéodory sense and in the Colombeau sense are different (in general). To "repair" the consistency problem for composition we give the definition introduced by J. F. Colombeau in [3]. A generalized function $u \in \mathcal{G}(\mathbb{R})$ is said to admit a member $w \in \mathcal{D}'(\mathbb{R})$ as the associated distribution, if it has a representative R_u with the following property: for every $\Psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that for every $\overline{\varphi} \in \mathcal{A}_N(\mathbb{R})$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_u(\overline{\varphi}_\varepsilon, t)\Psi(t)\, dt = w(\Psi).$$

If $p \in L_{loc}(\mathbb{R})$, we define

$$R_p(\overline{\varphi}, t) = \int_{-\infty}^{\infty} f(t-u)\overline{\varphi}(u)\, du = (p * \overline{\varphi})(t), \quad \overline{\varphi} \in \mathcal{A}_1(\mathbb{R}).$$

Obviously, $R_p \in \mathcal{E}_M[\mathbb{R}]$ and p is of m-type.

In this section we assume that a function $f : \mathbb{R}^5 \to \mathbb{R}$ satisfies condition $(\tilde{C})$ in $\mathbb{R}^5$ i.e.

$$\begin{gathered}\text{the function } f(t, u_1, u_2, u_3, u_4) \text{ is Lebesgue measurable}\\ \text{with respect to } t \text{ for all fixed } (u_1, u_2, u_3, u_4).\end{gathered} \tag{4.1}$$

$$\begin{gathered}\text{the function } f(t, u_1, u_2, u_3, u_4) \text{ is continuous}\\ \text{with respect to } (u_1, u_2, u_3, u_4) \text{ for every } t \in \mathbb{R},\end{gathered} \tag{4.2}$$

$$|f(t, u_1, u_2, u_3, u_4) - f(t, \overline{u_1}, \overline{u_2}, \overline{u_3}, \overline{u_4})| \le \sum_{i=1}^{4} p_i(t)|u_i - \overline{u_i}|, \tag{4.3}$$

$$|f(t, 0, 0, 0, 0)| \le p_5(t), \tag{4.4}$$

where p_r are non negative functions and $p_r \in L_{loc}(\mathbb{R})$ for $r = 1, 2, 3, 4, 5$.

Furthemore, we assume that $d_j \in \mathbb{R}$ for $j = 1, 2, 3, 4$.

Remark 4.2 If f has the property $(\tilde{C})$, then we define

$$\begin{aligned} R_f(\varphi, t, u_1, u_2, u_3, u_4) &= \int_{\mathbb{R}^5} f(t-\tau_0, u_1-\tau_1, u_2-\tau_2, u_3-\tau_3, u_4-\tau_4)\varphi(\tau)\, d\tau \\ &= f * \varphi, \end{aligned} \tag{4.5}$$

where $\varphi(\tau) = \overline{\varphi}(\tau_0)\overline{\varphi}(\tau_1)\overline{\varphi}(\tau_2)\overline{\varphi}(\tau_3)\overline{\varphi}(\tau_4)$, $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$ and $d\tau = d\tau_0 d\tau_1 d\tau_2 d\tau_3 d\tau_4$.

From the properties of the convolution we have

$$D_s(f * \varphi) = f * (D_s\varphi) \qquad \text{(see [1])}.$$

Hence we get $f \in \tilde{\mathcal{G}}_T(\mathbb{R}\times\mathbb{R}^4)$ and R_f satisfies all the assumptions of Theorem 3.1.

Theorem 4.1 *Let $x = [R_x(\varphi, t)]$ be a solution of the problem (1.1), (1.2$_1$) in the Colombeau sense and let $\tilde{x}$ be a solution of the problem (1.1), (1.2$_1$) in the Carathéodory sense. Then x admits an associated distribution which equals $\tilde{x}$.*

We start with auxiliary lemmas.

Lemma 4.1 *Let B be a compact subset of $\mathbb{R}^4$, let $[a, b]$ be an arbitrary compact interval and let $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$. Furthermore, let*

$$f_\varepsilon(t, u) = \int_{-\infty}^{\infty} f(t - \varepsilon\tau_0, u)\, \overline{\varphi}(\tau_0)\, d\tau_0, \quad \overline{\varphi} \in \mathcal{A}_1(\mathbb{R}). \tag{4.5$'$}$$

Then $\int_a^t f_\varepsilon(s, u) - f(s, u))\, ds$ converges uniformly to 0 for $\varepsilon \to 0^+$ on the set $[a, b] \times B$.

Lemma 4.2 *Let $B \subset \mathbb{R}^4$ be a compact subset of $\mathbb{R}^4$, let $x_\varepsilon^{(i)} : [a,b] \to \mathbb{R}$ and let $(x_\varepsilon(t), x'_\varepsilon(t), x''_\varepsilon, x'''_\varepsilon(t)) \in B$ for $i = 0,1,2,3$; $t \in [a,b]$, $\varepsilon \in (0, \varepsilon_0)$. Furthemore, let the system of functions $\{x'''_\varepsilon\}$ be equicontinuous. Then*

$$\int_a^t (f_\varepsilon(s, x_\varepsilon(s), x'_\varepsilon(s), x''_\varepsilon(s), x'''_\varepsilon(s)) - f(s, x_\varepsilon(s), x'_\varepsilon(s), x''_\varepsilon(s), x'''_\varepsilon(s)))\, ds$$

converges uniformly to 0 on the interval $[a,b]$.

Lemmas 4.1–4.2 are consequences of the Lebesgue theorem, Lemma 3.1 and Remark 3.1 from the paper [10].

Proof of Theorem 4.1 Let R_f be defined by (4.5) and let $R_x(\overline{\varphi}_\varepsilon, t)$ be a solution of the problem (3.1). We consider the integral equation (3.2), where $R_{d_j}(\overline{\varphi}_\varepsilon) = d_j$ for $j = 1,2,3,4$. Evidently

$$R_x(\overline{\varphi}_\varepsilon, t) = \frac{1}{6}\int_0^t \int_{\mathbb{R}^5} (t-s)^3((k_{1\varepsilon}(s) - k_{2\varepsilon}(s)) + (k_{2\varepsilon}(s) - k_3(s)))\varphi(\tau)\, d\tau\, ds$$
$$+ \frac{1}{6}\int_0^t (t-s)^3 k_3(s) ds + W_3(t), \tag{3.2$'$}$$

where

$$\begin{aligned} k_{1\varepsilon} &= f(s - \varepsilon\tau_0, x(\overline{\varphi}_\varepsilon, s) - \varepsilon\tau_1, D_1 x(\overline{\varphi}_\varepsilon, s) - \varepsilon\tau_2, D_2 x(\overline{\varphi}_\varepsilon, s) - \varepsilon\tau_3, \\ &\quad D_3 x(\overline{\varphi}_\varepsilon, s) - \varepsilon\tau_4), \\ k_{2\varepsilon} &= f(s - \varepsilon\tau_0, x(\overline{\varphi}_\varepsilon, s), D_1 x(\overline{\varphi}_\varepsilon, s), D_2 x(\overline{\varphi}_\varepsilon, s), D_3 x(\overline{\varphi}_\varepsilon, s)), \\ k_3(s) &= f(s, x(\overline{\varphi}_\varepsilon, s), D_1 x(\overline{\varphi}_\varepsilon, s), D_2 x(\overline{\varphi}_\varepsilon, s), D_3 x(\overline{\varphi}_\varepsilon, s)) \end{aligned}$$

and

$$W_3(t) = d_1 + d_2 t + d_3 \frac{t^2}{2} + \frac{d_4}{6} t^3.$$

Let $[a,b]$ be an arbitrary compact interval. Using the relations (4.2), (4.3), (3.2) and the Gronwall inequality we conclude that the systems of functions $\{D_i R_x(\overline{\varphi}_\varepsilon, t)\}$ are equibounded and equicontinuous for fixed $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$, $i = 0,1,2,3$; $\varepsilon \in (0, \varepsilon_0)$ and $t \in [a,b]$. Thus, by virtue of the relations (3.2)$'$, (4.2)–(4.3) and Lemma 4.2 for $i = 0,1,2,3$ and for every fixed $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$, we get

$$\lim_{\varepsilon \to 0} D_i x(\overline{\varphi}_\varepsilon, t) = \tilde{x}^{(i)}(t)$$

uniformly on $[a,b]$ and $\tilde{x}$ is a solution of (1.1), (1.2) in the Carathéodory sense. On the other hand, $x = [x(\overline{\varphi}, t)]$ is a generalized solution of (1.1), (1.2$_1$). This proves Theorem 4.1. □

Theorem 4.2 *Let the assumptions of Corollary 3.5 be satisfied and let $\tilde{x}$ be a solution of the problem (1.1), (1.2$_i$) in the Carathéodory sense. Furthemore, let $x = [x(\overline{\varphi}, t)]$ be a solution of the problem (1.2$_i$) generalized in the Colombeau sense. Then x admits an associated distribution which equals $\tilde{x}$.*

Proof Let $i \in \{2,3\}$. We consider the operator $T_{i\varepsilon}$ defined as follows

$$\begin{aligned} T_{i\varepsilon}(x) &= \int_0^T \int_{\mathbb{R}^5} G_i(t,s)((\overline{k_{1\varepsilon}}(s) - \overline{k_{2\varepsilon}}(s)) + (\overline{k_{2\varepsilon}}(s) - \overline{k_3}(s)))\varphi(s)\,d\tau\,ds \\ &\quad + \int_0^T G_i(t,s)\overline{k_3}(s)\,ds + W_i(t), \end{aligned} \tag{4.6}$$

where $x \in C^3[0,T]$,

$$\begin{aligned} \overline{k_{1\varepsilon}}(s) &= f(s-\varepsilon\tau_0, x(s)-\varepsilon\tau_1, x(s)-\varepsilon\tau_2, x''(s)-\varepsilon\tau_3, x'''(s)-\varepsilon\tau_4), \\ \overline{k_{2\varepsilon}}(s) &= f(s-\varepsilon\tau_0, x(s), x'(s), x''(s), x'''(s)), \\ \overline{k_3}(s) &= f(s, x(s), x'(s), x''(s), x'''(s)) \end{aligned}$$

and $W_i(t)$ is a polynomial of the order ≤ 3 such that $L_{ij}(W_i) = d_j$ for $j = 1,2,3,4$. We denote

$$\beta = 1 + a_i \int_0^T p_5(t)\,dt, \qquad r_0 = \beta(1-\gamma)^{-1},$$

where γ is defined by (3.19). Define now a set U as follows

$$U = \{x \in C^3[0,T] : \; \|x - W_i\|_{[0,T],3} \le r\}, \quad \text{where } r \ge r_0.$$

Obviously the set U is nonempty, convex and closed in the space $(C^3[0,T], \|\cdot\|_{[0,T],3})$. In virtue of (4.2)–(4.3), (4.6) and Lemma 4.2 for fixed $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$, fixed $\varepsilon \in (0,\varepsilon_0)$ and $x \in U$ we have

$$\|T_{i\varepsilon}(x) - W_i\|_{[0,T],3} \le \gamma\|x\|_{[0,T],3} + \beta \le r. \tag{4.7}$$

By (4.3) and (4.6)–(4.7), for every $x \in U$ the function $T_{i\varepsilon}(x)$ is continuous and its values belong to U. The relations (4.6)–(4.7) also imply that the set $T_{i\varepsilon}(U)$ consists of equicontinuous and equibounded functions and therefore it is relatively compact. It follows from the Schauder theorem that the problem

$$\begin{cases} x^{(4)}(t) = R_f(\varphi_\varepsilon, t, x(t), x'(t), x''(t), x'''(t)), \\ L_{ij}(x) = d_j, \quad j = 1,2,3,4 \end{cases}$$

has for any $\varepsilon \in (0,\varepsilon_0)$ a solution in the class $C^3[0,T]$. Using arguments similar to those in the proof of Theorem 3.2 we get that the problem (1.1), (1.2_i) has a solution $x(\overline{\varphi}_\varepsilon, t)$ in $C^\infty(-\infty,\infty)$ and $x(\overline{\varphi}_\varepsilon, t) \in \mathcal{E}_M[\mathbb{R}]$.

Now, taking into account the relations (4.6)–(4.7) and Lemma 4.2, we conclude that the systems of functions $\{D_r x(\overline{\varphi}_\varepsilon, t)\}$ are equibounded and equicontinuous in $(C^3[0,T], \|\cdot\|_{[0,T],3})$ for $\varepsilon \in (0,\varepsilon_0)$ and $r = 0,1,2,3$. There exists a subsequence $\{x(\overline{\varphi}_{\varepsilon_v}, t)\}$ of $\{x(\overline{\varphi}_\varepsilon, t)\}$ convergent in $(C^3[0,T], \|\cdot\|_{[0,T],3})$ to $\overline{x} \in C^3[0,T]$. Lemma 4.2 implies that $\overline{x}$ is a solution of the problem (1.1), (1.2_i) in the set $C^3[0,T]$. On the other hand $\tilde{x}$ is the unique solution of (1.1), (1.2_i) in the Carathéodory sense and $\tilde{x} \in U$. So, $\overline{x}(t) = \tilde{x}(t)$ for $t \in [0,T]$. Let $t_0 \in (0,T)$

and let $\lim_{v\to\infty}\left(D_r x(\overline{\varphi}_{\varepsilon_v}, t_0)\right) = d_{r+1}$ for $r = 0, 1, 2, 3$. Then $z = x(\overline{\varphi}_\varepsilon, t)$ is a solution of the Cauchy problem

$$\begin{cases} z^{(4)}(t) = R_f(\varphi_\varepsilon, t, z(t), z'(t), z''(t), z'''(t)) \\ z^{(r)}(t_0) = (D_r x(\overline{\varphi}_\varepsilon, t_0)), \quad r = 0, 1, 2, 3. \end{cases}$$

By proceding as in the proof of Theorem 4.1 we can prove, that for every $\overline{\varphi} \in \mathcal{A}_1(\mathbb{R})$ and $r = 0, 1, 2, 3$,

$$\lim_{\varepsilon\to 0} D_r x(\overline{\varphi}_\varepsilon, t) = \tilde{x}^{(r)}(t)$$

almost uniformly on $\mathbb{R}$. This proves Theorem 4.2.

Remark 4.3 The concept of generalized solutions of ordinary differential equations can be considered also in ofter ways. See e.g. [4]–[6], [8]–[11], [14], [16], [18]–[19], [21].

Remark 4.4 The definition of Colombeau generalized functions on a given open subset $\mathbb{R}^n$ is analogous to the definition used in this paper (see [3]). It is not difficult to observe that if reformulated our assumptions in a proper way, the results of this paper would remain true also in the case when the generalized functions are considered on some open subset $(a, b) \times \mathbb{R}^4$ of $\mathbb{R}^5$.

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