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Tilings and Isoperimetrical Shapes I: Square Lattice

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Abstract

In this paper, we consider a variant of the isoperimetric inequalities in lattices. We characterize polyominoes in 2-dimensional square lattices which maximize area for a fixed perimeter.

From this characterization, we can give the 'explicit' values of the smallest perimeter of a polyomino with fixed area. This last problem corresponds to the usual isoperimetrical problem.

Moreover, we propose to apply our characterization in order to solve a Golomb-type pentomino problem for the plane. *Find the minimum density of unit square to be placed on the plane so as to exclude all polyominoes of a fixed area.*

If we consider polyominoes of area 5 and we consider the chessboard instead of the plane, then the 'previous problem is known as a Pentomino Exclusion Problem due to Golomb.

Our characterization allows one to solve this problem for some values of the fixed area.

Key words: Tilings, isoperimetrical inequality, Pentomino Exclusion Problem.

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1 Introduction

In this paper, a *polyomino* is a finite set (not necessary connected) of unit squares centered on the grid $\mathbb{Z}^n$. The *interior-boundary* $\delta_{int}(P)$ of a polyomino P is the set of squares of P having a common edge with the 'exterior' of P. The *exterior boundary* $\delta_{ext}(P)$ of a polyomino P is the interior boundary of the complement of P. The *perimeter* of a polyomino P is $|\delta_{int}(P)|$. For a given polyomino P, $\Delta(P)$ denotes the area of P.

Consider the adjacency relations α and β, which define what are usually called 8-connectivity and 4-connectivity respectively in Discrete Geometry, between unit squares in $\mathbb{Z}^2$: we have $C\alpha C'$ (resp. $C\beta C'$) iff C and C' have a common vertex (resp. edge).

Here, we are interested in the following question (Q): *Suppose we are given a positive integer n. What are the polyominoes of perimeter n with maximum area?*

This question is related to an isoperimetrical problem (P): *What is the least perimeter of a polyomino of a given area?* This last question was first settled for any n by D.-L. Wang and P. Wang [7] in $\mathbb{Z}_+^n$; an alternative proof is proposed by B. Bollobás and I. Leader [2] in order to generalize Harper's theorem. Nevertheless, these results don't yield any information on the 'shape' of the optimal polyominoes (for instance uniqueness of the minimal shapes for specific values of the area).

L. Alonso and R. Cerf [1] solved the question (Q) for another kind of perimeter: the length (in $\mathbb{Z}^2$) or the area (in $\mathbb{Z}^3$) of the boundary.

Notice that in the continuous case the two questions (Q) and (P) are equivalent (by similarity).

In Section 2, we solve (Q) for the 2-dimensional case. In Section 4, we propose an application of this result in order to solve a Golomb-type Pentomino Exclusion problem (G) for the plane: *Find the minimum density of unit squares to be placed on the plane so as to exclude all polyominoes of a fixed area.*

If we consider polyominoes of area 5 and we consider the chessboard instead of the plane then the previous problem is known as the Pentomino Problem due to Golomb [5].

Our characterization of isoperimetrical shapes allows to solve this problem for some values of the fixed area.

2 Maximizing the area for a given perimeter

The proofs of the results mentioned in this section are in the next section.

We need to introduce some preliminary definitions (see Figure 1). For a given polyomino P, we can associate the graph $G(P) = (V, E)$ defined by $V = \{p \text{ center of a unit square in } P\}$ and $E = \{UV | U\alpha V\}$. A vertex v of $G(P)$ can be seen as a unit square of $\mathbb{Z}^2$, so for brevity sometimes v should be seen as the unique corresponding unit square. Moreover, $G(\mathbb{Z}^2)$ is usually called in graph theoretical language the *total infinite complete grid graph* which can be defined as a total product of two infinite paths. Then:

- The *interior-shape* (respectively *exterior-shape*) of a polyomino P is the graph $G_{int}(P) = G(\delta_{int}(P))$ (resp. $G_{ext}(P) = G(\delta_{ext}(P))$);
- An *isolable* unit square u of a polyomino P is a unit square such that the polyomino $Q = P - u + v$, where $v \in \mathbb{Z}^2 - (P \cup \delta_{ext}(P))$, satisfies $|\delta_{int}(P)| = |\delta_{int}(Q)|$.
- The *undress* operator applied to a polyomino P gives the polyomino $UD(P) = P - \delta_{int}(P)$. For $q > 1$, we denote $UD^q(P) = UD(UD^{q-1}(P))$ and by convention $UD^1(P) = UD(P)$.

For convenience, we let P_n be a polyomino with perimeter n which maximizes the area.

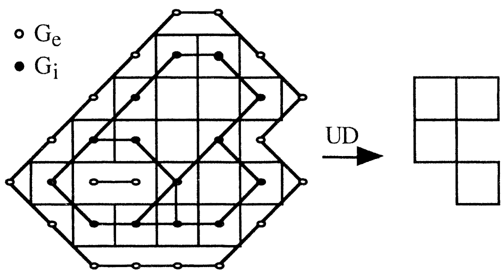

Figure 1: A polyomino P, the graphs $G_{int}(P)$ and $G_{ext}(P)$, and the polyomino $UD(P)$.

Theorem 1 *For $n = 1, 2$ and 3, a P_n consists of n distinct unit squares. For $n = 4$, the P_n is a cross (see Figure 2). For $n = 5$, a P_n consists of a disjoint union of a cross and a unit square.*

If $n = 4q + r \geq 6$ and $0 \leq r \leq 3$, then $C_r = UD^q(P_n)$ where C_0 is the cross, C_1 is the fish, C_2 is the twins or the domino, and C_3 is the stair (see Figure 2).

Theorem 2 *The area of a P_n with $n = 4q + r > 0$ with $0 \leq r \leq 3$, is given by the following function:*

$$\Delta(P_n) = \begin{cases} 2q^2 + 2q + 1 & \text{if } r = 0, \\ 2q^2 + 3q + 1 & \text{if } r = 1, \\ 2q^2 + 4q + 2 & \text{if } r = 2, \\ 2q^2 + 5q + 3 & \text{if } r = 3. \end{cases}$$

In order to describe the shape of an optimal polyomino, we need some additional definitions.

A *cycle* in a graph G is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1}$ (the indices have to be read modulo k) is an edge of G for all $i = 1, \ldots, k$. A *chord* in a cycle $v_0, \ldots, v_k$ is an edge $v_i v_j$ with $j \neq i + 1$.

Observe that in G there are four type of edges (see Figure 3):

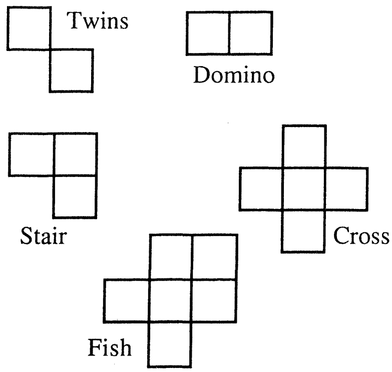

Figure 2: Extremal configurations.

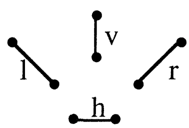

Figure 3: Type of edges in G.

- A *left* edge of G, denoted by l, is an edge uv where $u = (x_u, y_u)$ and $v = (x_u - 1, y_u + 1)$;
- A *right* edge of G, denoted by r, is an edge uv where $u = (x_u, y_u)$ and $v = (x_u + 1, y_u + 1)$;
- A *horizontal* edge of G, denoted by h, is an edge uv where $u = (x_u, y_u)$ and $v = (x_u + 1, y_u)$;
- A *vertical* edge of G, denoted by v, is an edge uv where $u = (x_u, y_u)$ and $v = (x_u, y_u + 1)$.

Observe that a cycle of G can be completely characterized by a circular word (by a clockwise orientation of the cycle) on the alphabet $\{l, r, h, v\}$.

Theorem 3 *If $n = 4q + r \geq 6$ and $0 \leq r \leq 3$, then $G_{int}(P_n)$ is a cycle (possibly with chords) where the cycle obtained after removing the chords of $G_{int}(P_n)$ can be described, up to rotation, by the words:*

$$\begin{array}{rl} l^q r^q l^q r^q & \textit{if } r = 0 \\ h l^{q-1} v r^q l^q r^q & \textit{if } r = 1 \\ h l^{q-1} r^{q-1} h l^{q-1} r^{q-1} \textit{ or } l^q r^{q-1} l^q r^{q-1} & \textit{if } r = 2 \\ h l^{q-1} v r^{q-1} l^q r^{q-1} & \textit{if } r = 3. \end{array}$$

3 Proofs of main results

Let P be a polyomino of perimeter n which maximizes area. Let $\mathbf{G} = G(P)$, $G_{int} = G_{int}(P)$ and $G_{ext} = G_{ext}(P)$. We claim that:

$$\text{We may assume that any connected component } G \text{ of cardinality at least 2 has no isolable unit square.} \tag{1}$$

Indeed, in the opposite case, let v be an isolable unit square of a connected component of G, and let u be a unit square of $\mathbb{Z}^2 - (P \cup \delta_{ext}(P))$. Then the polyomino $Q = P - v + u$ has the same area and the same perimeter as P.

From (1), we assume that any connected component $\mathbf{G}$ of P has no isolable unit square. Using this assumption, we study the geometrical shape of a connected component of G.

We consider now the four types of configuration described in Figure 4:

- A *oblique 3-path* is a path $(a, u, b) \in \mathbf{G}$ such that, up to rotation, $a = (x_u - 1, y_u + 1), u = (x_u + 1, y_u + 1)$ and $b = (x_u, y_u)$ and $(x_u, y_u + 1) \notin P$;
- A *semi-oblique 3-path* is a path $(a, u, b) \in \mathbf{G}$ such , up to rotation, that $a = (x_u - 1, y_u + 1), u = (x_u, y_u)$ and $b = (x_u + 1, y_u)$ and $(x_u, y_u + 1) \notin P$;
- A *right 3-path* is a path $(a, u, b) \in \mathbf{G}$ such that , up to rotation, $a = (x_u - 1, y_u), u = (x_u, y_u)$ and $b = (x_u + 1, y_u)$ and $(x_u, y_u + 1) \notin P$;
- A *boundary triangle* is a complete graph on 3 vertices $\{a, u, b\}$ belonging to G_{int}.

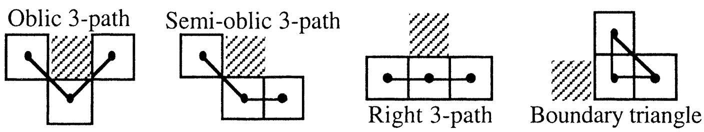

Figure 4: Forbiden configurations.

Observe that if there is neither an oblique nor a semi-oblique 3-path in $\mathbf{G}$ and if P has no hole then G_{int} is a 'convex' cycle with possible chord (that is, a convex polygon).

Lemma 1 *We may assume that any connected component of G_{int} with no isolable unit square has neither hole, nor oblique 3-path, nor semi-oblique 3-path, nor right 3-path, nor boundary triangle.*

Proof Let C be any connected component of $\mathbf{G}$. We assume that C has no isolable unit square (by Claim (1)). This implies that $G_{int}(C)$ has no vertex cutset. We claim that:

$$C \text{ has no hole.} \tag{2}$$

Indeed, if C has a hole H then we construct a polyomino Q by filling this hole and adding possible unit squares of $\mathbb{Z}^2 - (P \cup H)$ in order to have the same perimeter as P. So we obtain a contradiction to the maximality of P, since $\Delta(Q) \geq \Delta(P) + \Delta(H) > \Delta(P)$.

Assume that there exists either an oblique or a semi-oblique or a right 3-path (a, u, b) in $G_{int}(C)$ with $u = (x_u, y_u)$ and $(x_u, y_u + 1) \notin P$.

If there exists another path $(v_1 = a, v_2, \ldots, v_k = b)$ from a to b where $\forall i\ v_i \neq u$ in $G(C)$ then by (2), the unit squares $(x_u, y_u - 1), (x_u - 1, y_u)$ and $(x_u + 1, y_u)$ belong to C.

Let $v = (x_u, y_u + 1)$. We construct the polyomino Q obtained from $P + v$ by adding possible unit squares of $\mathbb{Z}^2 - (P + v)$ in order to have same perimeter than P. By previous remarks the polyomino Q has same perimeter than P and $\Delta(Q) \geq \Delta(P) + 1$, which contradicts the maximality of P.

Now suppose that u is a vertex-cutset. First, observe that we may assume that C is connected in β adjacency (for otherwise we move the β-connected components of C in order to obtain a new polyomino with only components β connected and with same area and perimeter than P).

By β-adjacency and since C has no hole, we have that $(x_u - 1, y_u)$ and $(x_u + 1, y_u)$ belong to C. And so since u is a vertex-cutset, $(x_u, y_u - 1)$ does not belong to C, and $C - u$ has two β-connected components C_1 and C_2 with $(x_u - 1, y_u) \in C_1$ and $(x_u + 1, y_u) \in C_2$.

We construct a polyomino Q obtained from P, by translation of C_2 (each unit square of C_2 translated by the vector $(-1, 0)$), keeping C_1 and all the other components of P and adding possible unit squares in $\mathbb{Z}^2 - (P \cup \delta_{ext}(P))$ (at least one which replaces u) in order to have the same perimeter than P.

Observe that $|\delta_{int}(C_1 \cup C_2)| \geq |\delta_{int}(C)| - 1$ if and only if $(x_u - 1, y_u) \in \delta_{int}(C_1) \cap UD(C)$ or $(x_u + 1, y_u) \in \delta_{int}(C_2) \cap UD(C)$. And now, remark that $|\delta_{int}(C_1 \cup C_2'| \leq |\delta_{int}(C)| - 1$ where C_2' denotes the translated of C_2 by the vector $(-1, 0)$, since $(x_u + 1, y_u) + (-1, 0) = u$ and since $u \notin C_1 \cup C_2$. Hence Q has same perimeter and area greater or equal than P.

Assume now that $\{a, u, b\}$ with $a = (x_u, y_u - 1), u = (x_u, y_u)$ and $b = (x_u + 1, y_u)$ induces a triangle in $G_{int}(C)$. Since u belongs to $G_{int}(C)$ then without loss of generality, we may assume that $(x_u - 1, y_u) \notin \mathbf{G}$. But now, since $G_{int}(C)$ has no right 3-path, then $(x_u, y_u + 1) \notin \mathbf{G}$. Thus, u represents an isolable unit square of C, which contradicts the hypothesis. $\square$

Let C be a connected component of G with no isolable unit square. Checking for the small values of $|\delta_{int}(C)|$, we can verify that if $|\delta_{int}(C)| > 1$ then $|\delta_{int}(C)| \geq 4$. Moreover, from Lemma 1, the interior boundary of C is a convex cycle with possible chords. Observe that by Lemma 1, the word describing $G_{int}(C)$ does not contain hh, vv, hv or vh. Thus, up to rotation, there exist four positive integers p, q, s, t such that (see Figure 5):

$$G_{int}(C) - \{\text{chords}\} \equiv \begin{cases} (S_1)\ l^p r^q l^s r^t, \text{ or} \\ (S_2)\ h l^p r^q l^s r^t, \text{ or} \\ (S_3)\ h l^p v r^q l^s r^t, \text{ or} \\ (S_4)\ h l^p r^q h l^s r^t, \text{ or} \\ (S_5)\ h l^p v r^q h l^s r^t, \text{ or} \\ (S_6)\ h l^p v r^q h l^s v r^t. \end{cases}$$

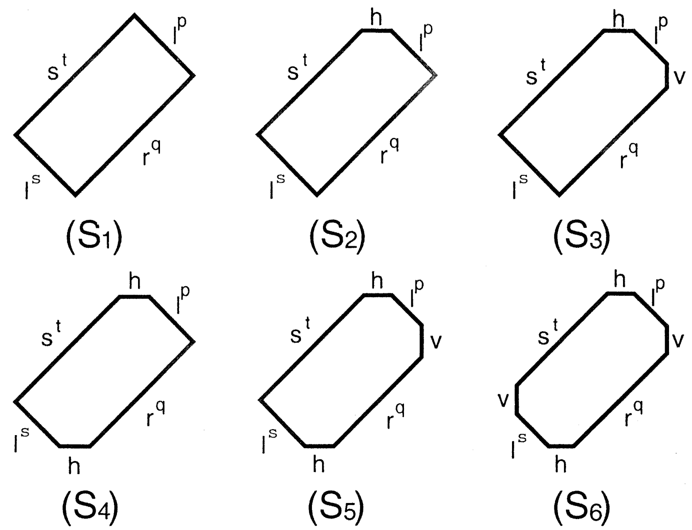

Figure 5: Optimal shapes.

Lemma 2 *For any polyomino of boundary C described by a word of type (S_k) for some $k \in \{1, \ldots, 6\}$ where p, q, s and t are integers greater than* 1, *we have* $|\delta_{int}(C)| = |\delta_{int}(UD(C))| + 4$.

Proof Assume that C is of type (S_k) for some $k \in \{1, \ldots, 6\}$, and let p, q, s and t be positive integers of the word associated to (S_k). We claim that:

$$UD(C) \text{ is of type } S_k \text{ with } p-1, q-1, s-1 \text{ and } t-1. \tag{3}$$

Indeed, observe that an h or v path in C remains an h or v path respectively in $UD(C)$; and a l^u or r^u path becomes a l^{u-1} or a r^{u-1} path in $UD(C)$.

If p, q, s and t are greater than 1 then the paths of $UD(C)$ described in the proof of (3) are distinct, so the lemma follows from (3). □

A *nice shape* (S) is one of the following shapes, up to rotation, for some $a > 1$:

$$(S) \equiv \begin{cases} (S_1)\ l^a r^a l^a r^a, \text{ or} \\ (S_1)'\ l^a r^{a-1} l^a r^{a-1}, \text{ or} \\ (S_3)\ h l^{a-1} v r^a l^a r^a, \text{ or} \\ (S_3)'\ h l^{a-1} v r^{a-1} l^a r^{a-1}, \text{ or} \\ (S_4)\ h l^a r^a h l^a r^a. \end{cases}$$

Observe that if $a = 1$, then (S_1) is a Cross, $(S_1)'$ is the Twins, (S_3) is a Fish, $(S_3)'$ is a Stair, (S_4) is a Domino.

Lemma 3 *The area of a polyomino P with a nice shape (S) and for some $a > 1$ is given by:*

$$\Delta(P) = \begin{cases} 2a^2 + 2a + 1 \ \textit{if}\ (S) \equiv (S_1) \\ 2a^2 + a \ \textit{if}\ (S) \equiv (S_3) \\ 2a^2 + 3a + 1 \ \textit{if}\ (S) \equiv (S_3)' \\ 2a^2 \ \textit{if}\ (S) \equiv (S_1)' \ \textit{or}\ (S_4) \end{cases}$$

Proof First observe that as (3) in the proof of Lemma 2, the polyominoes described in Figure 2 are obtained by undressing from one of the polyominoes of a nice shape.

Now Lemma 3 follows by simple computation from Lemma 2. □

Now, we are ready to settle one of the keys of our main results.

Lemma 4 *Let C be a connected component of* **G** *without isolable unit square and $|C| > 1$. The polyomino $C_i = G_{int}(C) - \{chords\}$ has a nice shape for some $a > 1$.*

Proof Let C be a connected component of a $G(P_n) = \mathbf{G}$. To eliminate (S_2) and (S_5) shapes, we use a *coloring* argument: consider the bicoloring (as a chessboard) of $\mathbb{Z}^2$. Remark that a l^u or r^u path is monocolored; and the two extremes of a h or v path have distinct colors. Then a shape with an odd number of h and v can not describe the boundary of a polyomino.

For $(S_1), (S_3), (S_4)$ and (S_6) shapes, using elementary geometrical arguments we can see that:

— For $(S_1), (S_4)$ and (S_6), we must have $p = s$ and $q = t$,

— for (S_3), we must have $p + 1 = s$ and $q = t$.

We may assume that $\min\{p, q, s, t\} = 1$. For otherwise, we prove Lemma 4 for $UD^{k-1}(C_i)$ where $k = \min\{p, q, s, t\}$, and conclude with Lemma 2.

To eliminate the (S_6) shape, we may assume (by symmetry) that the type of C_i is (S_6) and $q \geq p = 1$. Then $\Delta(C) = 3(q+1) + 2(q+2)$ and $|C_i| = 2(q+1)+4$. We choose the polyomino C' of a nice shape $(S) \equiv (S_1)$ with $a = \frac{q+3}{2}$ if q is odd or C' has nice shape $(S) \equiv (S_1)'$ with $a = \frac{q+4}{2}$ if q is even. In any case, we obtain by Lemma 3, that $\Delta(C') > \Delta(C)$ and $|\delta_{int}(C')| = |C_i|$.

To achieve the proof of Lemma 4, we consider the two following cases:

Case 1: $p < q$.

We obtain that C_i is described by the word $lr^q lr^q$ if C_i is of type (S_1), $hlvr^q l^2 r^q$ if C_i is of type (S_3), $hlr^q hlr^q$ if C_i is of type (S_4), $hlvr^q hlvr^q$ if C_i is of type (S_6).

If the type of C_i is (S_1) and $q > p + 1 = 2$ then $\Delta(C) = 2(q+1) + q$ and $|C_i| = 2(q+1)$. If q is odd then we choose the polyomino C' of nice shape $(S) \equiv (S_1)$ with $a = \frac{q+1}{2}$, otherwise we choose the polyomino C' of a nice shape $(S) \equiv (S_1)'$ with $a = \frac{q+2}{2}$. In any case, we obtain by Lemma 3, that $\Delta(C') > \Delta(C)$ and $|\delta_{int}(C')| = |C_i|$.

If the type of C_i is (S_4) and $q > p = 1$ then $\Delta(C) = 2(q+1) + 2q + 2$ and $|C_i| = 2(q+1) + 2$. We conclude as in the previous case, with nice shape $(S) \equiv (S_1)$ and $a = \frac{q+2}{2}$ if q is even and nice shape $(S) \equiv (S_1)'$ and $a = \frac{q+3}{2}$ if q is odd.

If the type of C_i is (S_3) and $q > p = 2$ then $\Delta(C) = 5(q+1)$ and $|C_i| = 2(q+1)+3$. We conclude as in the previouses cases, with nice shape $(S) \equiv (S_3)$ and $a = \frac{q+3}{2}$ if q is odd and nice shape $(S) \equiv (S_3)'$ and $a = \frac{q+4}{2}$ if q is even.

Case 2 : $p > q$.

Remark that by symetry with case 1, we have only to check in the case when the type of C_i is $(S_3) \equiv hl^{p-1} vrl^p r$ and $p > q + 1 = 2$. Then $\Delta(C) = 2(p+1) + 2(p+2)$ and $|C_i| = p+1+p+2+2 = 2(p+2)+1$. We conclude as in the previouses cases, with nice shape $(S) \equiv (S_3)$ and $a = \frac{p+3}{2}$ if p is odd and nice shape $(S) \equiv (S_3)'$ and $a = \frac{p+4}{2}$ if p is even.

In any cases, we obtain a polyomino with same perimeter but with largest area, which contradicts the maximality of P_n. □

Lemma 5 *For any $n \geq 6$, $G(P_n)$ is connected.*

Proof Let n be the smallest integer such that there exists a non connected P_n. Assume on the contrary that $n \geq 6$. Let $C_1, \ldots, C_t$ be $t \geq 2$ connected components of $G(P_n)$.

If there does not exist some C_i which is not a square then it easy to see that $n \leq 3$. Hence, since $n \geq 6$, we may assume now that C_i has no isolable unit square for $i = 1, \ldots, a$ for some $a > 0$ (perhaps $a = t$); and C_j is a square for $j = a+1, \ldots, t$. Thus, by Lemma 4, each C_i's for $i \leq a$, can be described by a nice shape (S). Let $p_i = |\delta_{int}(C)|$. Remark that $p_i \geq 4$ $\forall i \leq a$. Without loss of generality, we may assume that $p_1 + p_2 + p_3 \geq 6$ (with by convention $p_3 = 0$ if $t = 2$). Then by Lemma 3, $\Delta(P_{p_1+p_2+p_3}) \geq 2(p_1+p_2+p_3)^2$ and $\Delta(P_{p_1}) + \Delta(P_{p_2}) + \Delta(P_{p_3}) \leq 2(p_1^2 + p_2^2 + p_3^2) + 3(p_1+p_2+p_3) + 2$. And, so $\Delta(P_{p_1+p_2+p_3}) > \Delta(P_{p_1}) + \Delta(P_{p_2}) + \Delta(P_{p_3})$, which contradicts the maximality of P_n. □

Theorem 1 follows from Lemmas 2 and 5.
Theorem 2 follows from Lemmas 3 and 5.
Theorem 3 follows from Lemmas 4 and 5. □

4 Applications

We will use here the notations and definitions given in Introduction and in Section 2.

4.1 Isoperimetrical problem (P)

From the results of Section 2, we can solve the problem (P) in $\mathbb{Z}^2$ with sharped values of perimeter. We denote P_n a polyomino of perimeter n which maximizes its area.

Theorem 4 *Any polyomino P of area Δ satisfies $|\delta_{int}(P)| \geq n$ where n is defined by $\Delta(P_{n-1}) < \Delta \leq \Delta(P_n)$.*

Moreover, for any value of $\Delta \geq 5$ there exist polyominoes of perimeter n.

Proof The inequality is clear from the definition of P_n.

For small values of n, it is clear that there exist such polyominoes as defined in the theorem. Assume now that $n - 1 \geq 6$. By Theorem 2, P_{n-1} has a nice shape (S) (see Section 3). Let $u = \Delta - \Delta(P_{n-1})$ and let $n = 4q + r$ with $0 \leq r \leq 3$. We may assume that $u > 0$.

If $(S) \equiv (S_1)$ then by Theorem 2, $u < q$. The polyomino with shape described by $hl^{u-1}vl^{q-u}\ r^q l^q r^q$ has the desired property (see Figure 6).

If $(S) \equiv (S_3)$ then by Theorem 2, $u < q + 1$. The polyomino with shape described by $hl^q l^1 r^u h r^{q-u-1} l^q r^q$ has the desired property (see Figure 6).

If $(S) \equiv (S_4)$ then by Theorem 2, $u < q + 1$. The polyomino with shape described by $hl^q r^q r^1 l^u v l^{q-u} r^q$ has the desired property (see Figure 6).

If $(S) \equiv (S_3)'$ then by Theorem 2, $u < q + 1$. The polyomino with shape described by $hl^{q-1} r^q l^q r^u h r^{q-u-1}$ has the desired property (see Figure 6). □

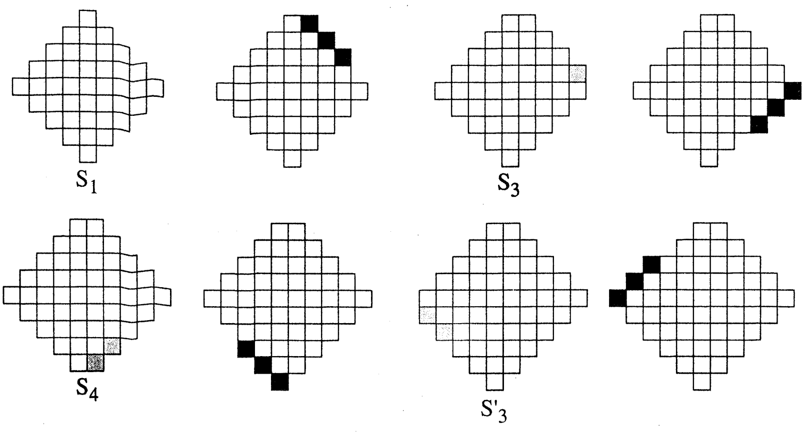

Figure 6: Extremal polyomino (when $u = 3$ in the proof of Theorem 4).

Observe that by Lemma 2, we can establish an analogue of Theorem 4 for the exterior boundary:

Theorem 5 *Any polyomino P of area Δ satisfies $|\delta_{ext}(P)| \geq n+4$ where n is defined by $\Delta(P_{n-1}) < \Delta \leq \Delta(P_n)$.*

Moreover, for any value of $\Delta \geq 1$ there exist polyominoes of (exterior) perimeter $n+4$. □

4.2 Golomb type problem (G)

We denote by (G_Δ) the problem: *Find the minimum density of unit square to be placed on the plane so as to exclude all polyominoes of area $> \Delta$.*

admissible solution of (G_Δ) problem is a set $\mathcal{S}$ of squares centered on $\mathbb{Z}^2$ such that any connected component in β adjacency of $\mathbb{R}^2 - \mathcal{S}$ has area less than or equal to Δ. We color 'black' the squares belonging to an admissible solution $\mathcal{S}$ and 'white' the others.

We now need a measure called 'density', of an admissible solution of (G_Δ) in order to compare two admissible solutions. If T is a finite subset of $\mathbb{Z}^2$ then a natural way to define the density of $\mathcal{S}$ relative to T is $\frac{|\mathcal{S}\cap T|}{|T-\mathcal{S}|}$. We show now a way to extend this definition to the infinite case:

For an admissible solution $\mathcal{S}$ of (G_Δ), observe that if we remove one 'crossing edge' of each K_4 (complete graph on 4 vertices) in $G(\mathcal{S})$, then the resulting **plane** graph $G'(\mathcal{S})$ defines a tiling of $\mathbb{R}^2$ (see Figure 7) where the tiles are the faces of $G'(\mathcal{S})$. For a face (or a tile) $\langle C\rangle$ of $G'(\mathcal{S})$ there corresponds a unique polyomino C where $\delta_{ext}(C) \subset \mathcal{S}$.

Some of these tiles correspond to some connected components (in β connectivity) of $\mathbb{Z}^2 - \mathcal{S}$. Some others are triangles corresponding to 3 mutually adjacent elements of $\mathcal{S}$ (in this case $C = \emptyset$).

Let $D \subset \mathbb{R}^2$. The density of an admissible solution $\mathcal{S}$ of (G_Δ) relative to D is

$$d(\mathcal{S}, D) = \frac{\text{'black' area of } \bar{D}}{\text{'white' area of } \bar{D}},$$

where $\bar{D}$ is the union of all faces of G'($\mathcal{S}$) which intersect D.

Notice that $\bar{D}$ defines a polyomino P with some unit square in $\mathcal{S}$; moreover all the squares in the interior boundary of P belong to $\mathcal{S}$. Moreover, observe that $d(\mathcal{S}, D)$ is well-defined since each face of $G'(\mathcal{S})$ defines a polyomino with bounded area.

Let B_r be a ball of $\mathbb{R}^2$ of radius r. Then

$$\underline{d}(\mathcal{S}) = \liminf_{r\to\infty} d(\mathcal{S}, B_r)$$

and

$$\bar{d}(\mathcal{S}) = \limsup_{r\to\infty} d(\mathcal{S}, B_r),$$

are called the *lower* and *upper* density respectively.

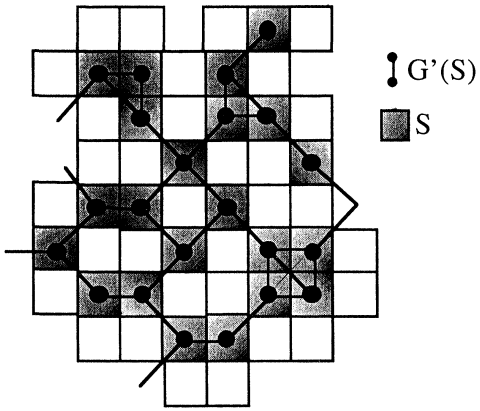

Figure 7: $\mathcal{S}$ and $G'(\mathcal{S})$.

If these two values coincide, their common value is called *density* $d(\mathcal{S}, D)$. This kind of definition of density is more or less standard (see for example [6]). We are ready now to state our result on Golomb-type problem:

Theorem 6 *Let $n = 4q + r \geq 5$ with $0 \leq r \leq 3$ be an integer such that $\Delta \geq \Delta(P_n)$. If $q > 1$ and $\Delta - \Delta(P_n) \leq \lceil \frac{q}{2} \rceil$ then an optimal solution $\mathcal{S}$ of (G_Δ) satisfies:*

$$\underline{d}(\mathcal{S}) \geq \frac{\frac{|\delta_{ext}(P_n)|}{2} - 1}{\Delta(P_n)}.$$

Moreover, for any $\Delta \geq \Delta(P_n)$, we have:

$$\bar{d}(\mathcal{S}) \leq \begin{cases} \dfrac{\frac{|\delta_{ext}(P_n)|}{2} - 1}{\Delta(P_n)} & \textit{if } r = 0, 2 \\[2ex] \dfrac{\frac{|\delta_{ext}(P_n)|+1}{2} - 1}{\Delta(P_n)} & \textit{if } r = 1, 3 \end{cases}.$$

Proof Let $\mathcal{S}$ be an optimal solution of (G_Δ).

Let $D \subset \mathbb{R}^2$ and let

$$\bar{D} = \cup_{\{\langle C \rangle \in G'(\mathcal{S}) \ | \ \langle C \rangle \cap D \neq \emptyset\}} \langle C \rangle.$$

First we claim that:

$$\text{We may assume that any } \langle C \rangle \text{ has no hole.} \tag{4}$$

If $\langle C \rangle$ has a hole then move it close to the exterior boundary of $\langle C \rangle$ in order to obtain a new face $\langle C' \rangle$ with no hole.

If we do that for any $\langle C\rangle$ having a hole then we obtain a new admissible solution of (G_Δ) with the same density.

Now we assume that any face of $G'(\mathcal{S})$ has no hole. Using the structure of $\mathbb{Z}^2$, we claim that:

$$d(\mathcal{S},D)=\frac{\sum_{\langle C\rangle\in\bar{D}}\frac{|\delta_{ext}(C)|}{2}-1}{\sum_{\langle C\rangle\in\bar{D}}|C|}. \tag{5}$$

If P is the polyomino defined by $\bar{D}$ then Pick's theorem gives that the area of $\bar{D}$ is given by $|UD(P)|+\frac{|\delta_{int}(P)|}{2}-1$ and the area of each $\langle C\rangle$ is given $|C|+\frac{|\delta_{ext}(C)|}{2}-1$, since by assumption each $\langle C\rangle$ has no hole. Now by additivity of the area and since $\bar{D}$ is partitionate by $\cup_{\langle C\rangle\in\bar{D}}\langle C\rangle$, we have that $\sum_{\langle C\rangle\in\bar{D}}\frac{|\delta_{ext}(C)|}{2}-1$ is equal to the number of squares in $\mathcal{S}\cap P$ not in the interior boundary of P, plus half of the number of squares in the interior boundary of P which corresponds to the 'black' area of $\bar{D}$. And $\sum_{\langle C\rangle\in\bar{D}}|C|$ is the number of squares in P not in $\mathcal{S}$, which corresponds to the 'white' area of $\bar{D}$.

We now prove the lower bounds on $\underline{d}(\mathcal{S})$. From (5), we have:

$$d(\mathcal{S},D)\geq\min_{\langle C\rangle\in\bar{D}}\frac{\frac{|\delta_{ext}(C)|}{2}-1}{|C|}. \tag{6}$$

Let $\langle C\rangle$ be a face of $G'(\mathcal{S})$.
If $|C|\leq\Delta(P_n)$ then let $p=|\delta_{ext}(C)|$.
If $p\geq n+4$ then

$$\frac{\frac{p}{2}-1}{|C|}\geq\frac{\frac{n+4}{2}-1}{\Delta(P_n)}.$$

If $p<n+4$ then by Theorem 5, $|C|\leq\Delta(P_{p-4})$. Let $s=n+4-p$. Now by Theorem 2, $|C|\leq\Delta(P_{p-4})\leq\Delta(P_n)-qs$ and $\Delta(P_n)\leq q(n+2)$ if $q>1$. Hence, if $q>1$ then

$$\frac{\frac{p}{2}-1}{|C|}\geq\frac{\frac{n+4-s}{2}-1}{\Delta(P_n)-qs}\geq\frac{\frac{n+4}{2}-1}{\Delta(P_n)}.$$

If $\Delta\geq|C|\geq\Delta(P_n)$ then by Theorem 5 we have that $|\delta_{ext}(C)|\geq n+5$. So

$$\frac{\frac{|\delta_{ext}(C)|}{2}-1}{|C|}\geq\frac{\frac{n+5}{2}-1}{\Delta}.$$

If $\Delta-\Delta(P_n)\leq\lceil\frac{q}{2}\rceil$, then by an easy computation based on the values of $\Delta(P_n)$ given in Theorem 2, we have that

$$\frac{\frac{n+5}{2}-1}{\Delta}\geq\frac{\frac{n+4}{2}-1}{\Delta(P_n)}=\frac{\frac{|\delta_{ext}(P_n)|}{2}-1}{\Delta(P_n)}.$$

In each case, we obtain by (6), that if $q>1$ and $\Delta-\Delta(P_n)\leq\lceil\frac{q}{2}\rceil$ then

$$\underline{d}(\mathcal{S})\geq\frac{\frac{|\delta_{ext}(P_n)|}{2}-1}{\Delta(P_n)}.$$

We prove now the upper bounds on $\bar{d}(\mathcal{S})$. By (5), we have:

$$d(\mathcal{S}, D) \leq \max_{\langle C\rangle \in \bar{D}} \frac{\frac{|\delta_{ext}(C)|}{2} - 1}{|C|}. \tag{7}$$

Now assume that $r = 0, 2$. In these case the shapes $(S_1), (S_1)'$ and (S_4) are respectively a square, a rectangle and an hexagon. In each case, we know tilings of $\mathbb{Z}^2$ by these shapes and, by (7), the density of such tilings is precisely

$$\frac{\frac{|\delta_{ext}(P_n)|}{2} - 1}{\Delta(P_n)}.$$

Now assume that $r = 1, 3$. In these case the shapes (S_3) and $(S_3)'$ are described by the words $hl^{a-1}vr^al^ar^a$ and $hl^{a-1}vr^{a-1}l^ar^{a-1}$. In each case, we can not tile $\mathbb{Z}^2$ with tiles of these shapes. This is why we obtain only a lower bound of the upper density in these cases. We can easily find a tiling of $\mathbb{Z}^2$ by tiles with shape $hl^ahr^al^ar^a$ and $hl^ahr^{a-1}l^ar^{a-1}$ (see Figure 8). Observe that, by (7), such a tiling has density

$$\frac{\frac{|\delta_{ext}(P)|+1}{2} - 1}{\Delta(P_n)}.$$ □

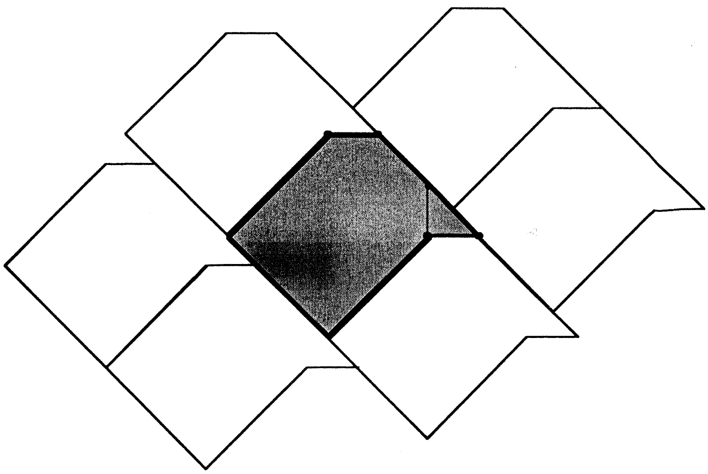

Figure 8: Tilings with optimal polyominoes.

A direct consequence of the proof of Theorem 6 is that when $r = 0, 2$, $q > 1$ and when $\Delta - \Delta(P_n) \leq \lceil \frac{q}{2} \rceil$ then the density of an optimal solution of (G_Δ) exists and is equal to

$$\frac{\frac{|\delta_{ext}(P_n)|}{2} - 1}{\Delta(P_n)}.$$

Moreover this density is independant of the position of the ball B_r and we should choose another domain instead of a ball to define density.

5 Conclusion

It would be interesting to adapt the technique developed in Section 3 in order to solve some isoperimetrical problems in other lattices such as hexagonal or triangular lattices.

We proved that there exists a family of 5 polyominoes in $\mathbb{Z}^2$ such that any polyomino in $\mathbb{Z}^2$ of maximum area for a fixed perimeter can be obtained (when the perimeter is large enough) from an element of this family by an undressing procedure.

We can ask what happens in higher dimensions. But note that already in the 3-dimensional case, the family seems to be infinite. Indeed, observe that in any dimension n the 'dressing' (r times) of an n-cube corresponds to the sphere of radius r in $\mathbb{Z}^n$. Moreover, probably a sphere is in any dimension an optimal shape for the problem (Q). But, unfortunately the difference of the perimeters between two spheres of consecutive radii depends on the smallest radius of these two spheres and it is not as in 2-dimensional case a constant ($+4$ in $\mathbb{Z}^2$).

The Golomb type problem has been investigated in the finite case (a finite subgrid $k \times n$ of $\mathbb{Z}^2$) [3, 4, 5]. Note that Theorem 6 allows one to obtain some asymptotic results on Golomb's problem for some appropriate values of k.

The density defined in previous section allows us to compare tilings of $\mathbb{R}^2$ when the tiles are polygons with vertices belonging to a lattice.

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