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Spectra of Weakly Associative Lattice Rings

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Abstract

Weakly associative lattice rings (*wal*-rings) are non-transitive generalizations of lattice ordered rings (*l*-rings) in which the identities of associativity of the lattice operations join and meet are replaced by the identities of weak associativity. The spectral topologies on the sets of straightening ideals of weakly associative lattice rings are introduced and studied.

Key words: Weakly associative lattice ring, straightening ideal, irreducible ideal, spectral topology, spectrum.

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A *weakly associative lattice (wa-lattice)* is an algebra $A = (A, \vee, \wedge)$ with two binary operations satisfying the identities

(I) $a \vee a = a$; $a \wedge a = a$.
(C) $a \vee b = b \vee a$; $a \wedge b = b \wedge a$.
(Abs) $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$.
(WA) $((a \wedge c) \vee (b \wedge c)) \vee c = c$; $((a \vee c) \wedge (b \vee c)) \wedge c = c$.

This notion has been introduced by E. Fried in [2] and by H. L. Skala in [11] and [12] as a non-associative generalization of a lattice. The identities of associativity of the operations $\vee$ and $\wedge$ are replaced by weaker conditions of weak associativity (WA). Nevertheless, similarly as for lattices, the properties of $\vee$ and $\wedge$ make possible to define also for *wa*-lattices a binary relation $\leq$ on A as follows:

$$\forall a, b \in A;\ a \leq b \iff_{df} a \wedge b = a.$$

The relation $\leq$ is reflexive and antisymmetric (i.e. $\leq$ is a so-called *semiorder* on A) and for each $x, y \in A$ there exist $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$ in A. Therefore we can equivalently view any *wa*-lattice as a special kind of a semiordered set.

A special case of a *wa*-lattice is a *tournament (totally semiordered set).* It is a semiordered set $(T, \leq)$ satisfying

$$\forall a, b \in T;\ a \leq b \text{ or } b \leq a.$$

If $(G, +)$ is a group and $(G, \vee, \wedge) = (G, \leq)$ is a *wa*-lattice then the system $G = (G, +, \leq)$ is called a *weakly associative lattice group (wal-group)* if G satisfies the following mutually equivalent conditions:

(M_+) $\forall a,\ b,\ c,\ d \in G;\ a \leq b \Longrightarrow c + a + d \leq c + b + d;$
$(\mathrm{D}_\vee)$ $\forall a,\ b,\ c,\ d \in G;\ c + (a \vee b) + d = (c + a + d) \vee (c + b + d);$
$(\mathrm{D}_\wedge)$ $\forall a,\ b,\ c,\ d \in G;\ c + (a \wedge b) + d = (c + a + d) \wedge (c + b + d).$

If for a *wal*-group G the *wa*-lattice $(G, \leq)$ is a tournament, then G is called a *totally semiordered group (to-group).*

For basic properties of *wal*-groups and *to*-groups see [5].

If $(R, +, \cdot)$ is an associative ring and $(R, \vee, \wedge) = (R, \leq)$ is a *wa*-lattice then the system $R = (R, +, \cdot, \leq)$ is called a *weakly associative lattice ring (wal-ring)* if R satisfies the following conditions

(M_+) $\forall a, b, c \in R;\ a \leq b \Longrightarrow a + c \leq b + c;$
$(\mathrm{M}_\cdot)$ $\forall a, b, c \in R;\ 0 \leq c \text{ and } a \leq b \Longrightarrow ac \leq bc \text{ and } ca \leq cb.$

If for a *wal*-ring R the *wa*-lattice $(R, \leq)$ is a tournament, then R is called a *totally semiordered ring (to-ring).*

(For basic properties of *wal*-rings see [10].) In contrast to lattice ordered rings (*l*-rings) and linearly ordered rings (*o*-rings) (see [1]), there are non-trivial finite *wal*-rings and *to*-rings.

If R is a *wal*-ring then $R^+ = \{x \in R;\ 0 \leq x\}$ is called the *positive cone* of R and its elements are *positive.*

Example 9 Let us consider the ring $\mathbb{Z}_3 = \{0, 1, 2\}$ with the addition and multiplication mod3. We denote $R = (R, +, \cdot) = (\mathbb{Z}_3, +, \cdot)$, $\mathbb{Z}_3^+ = R^+ = \{0, 1\}$. It is clear that $\mathbb{Z}_3^+$ is the positive cone of a total semiorder of the ring $\mathbb{Z}_3$.

The class $\mathcal{R}_{wal}$ of all *wal*-rings is a variety of algebras of type $\langle +, 0, -(\cdot), \cdot, \vee, \wedge \rangle$ of signature $\langle 2, 0, 1, 2, 2, 2 \rangle$. Some properties of the variety $\mathcal{R}_{wal}$ have been investigated in [10] and [9].

Subalgebras of *wal*-rings are called *wal-subrings.* That means if R is a *wal*-ring and $\emptyset \neq A \subseteq R$, then A is a *wal*-subring of R if A is both a subring and a *wa*-sublattice of R.

Let R be a *wal*-ring and I its ring ideal which is its convex *wa*-sublattice simultaneously. Then I is called a *wal-ideal* of R if it satisfies the following mutually equivalent conditions:

$(\mathrm{I_a})$ $\forall a, b \in I,\ x, y \in R;\ (x \leq a,\ y \leq b \implies \exists c \in I;\ x \vee y \leq c);$
$(\mathrm{I_b})$ $\forall a, b, c \in I,\ x, y \in R;\ x \leq a,\ y \leq b \implies (x \vee y) \vee c \in I.$

By [10], the *wal*-ideals of *wal*-rings coincide with the kernels of homomorphisms of *wal*-rings. The *wal*-ideals of any *wal*-ring R (ordered by set inclusion) form a complete lattice $\mathcal{I}(R)$ which is, by [10, Theorem 2.1.5], distributive. Moreover, by [10, Proposition 2.1.1], $\mathcal{I}(R)$ is a complete sublattice of the lattice $\mathcal{L}(R)$ of *wal*-ideals (i.e. normal convex *wal*-subgroups satisfying the conditions ($\mathrm{I_a}$) and ($\mathrm{I_b}$)) of the additive *wal*-group $(R,+)$. If $I_\gamma \in \mathcal{I}(R)$, $\gamma \in \Gamma$, then

$$\inf(I_\gamma; \gamma \in \Gamma) = \bigwedge_{\gamma\in\Gamma} I_\gamma = \bigcap_{\gamma\in\Gamma} I_\gamma,$$

$$\sup(I_\gamma; \gamma \in \Gamma) = \bigvee_{\gamma\in\Gamma} I_\gamma = \sum_{\gamma\in\Gamma} I_\gamma.$$

If I is a *wal*-ideal of R, we can define a semiorder on the factor ring R/I by

$$x + I \leq y + I \iff_{df} \exists a \in I;\ x + a \leq y.$$

Then R/I with this relation is a *wal*-ring.

A *wal*-ideal I of R is said to be *straightening* if it satisfies the following mutually equivalent conditions (see [10]):

(1) $x, y \in R,\ 0 \leq x \wedge y \in I \implies x \in I$ or $y \in I$;
(2) $x, y \in R,\ x \wedge y = 0 \implies x \in I$ or $y \in I$;
(3) R/I is a *to*-ring.

A *wal*-ideal I of R is called an *irreducible ideal of* R if it is a finitely meet-irreducible element in the lattice $\mathcal{I}(R)$ of *wal*-ideals of R, i.e. if it satisfies

(4) $\forall A,\ B \in \mathcal{I}(R);\ A \cap B = I \implies A = I$ or $B = I$.

By [10, Theorem 2.2.1], the condition (4) is equivalent to the following condition

(5) $\forall A,\ B \in \mathcal{I}(R);\ A \cap B \subseteq I \implies A \subseteq I$ or $B \subseteq I$.

By the same theorem, every straightening *wal*-ideal I of R satisfies the condition

(6) $\{A \in \mathcal{I}(R); I \subseteq A\}$ is a linearly ordered set.

It is obvious that every $I \in \mathcal{I}(R)$ which satisfies (6) is an irreducible ideal of R.

In contrast to l-rings where all conditions (1)–(6) are equivalent, there are irreducible ideals of *wal*-rings which are not straightening (see below).

A *wal*-ideal I of a *wal*-ring R is called *semimaximal* if there exists an element $a \in R$ such that I is a maximal *wal*-ideal of R with respect to the property "not containing a".

A *wal*-ideal $I \in \mathcal{I}(R)$ is semimaximal if and only if it is infinitely irreducible, i.e. if $I = \bigcap_{\gamma\in\Gamma} I_\gamma$, $(I_\gamma \in \mathcal{I}(R))$ implies the existence of a $\gamma_0 \in \Gamma$ such that $I = I_{\gamma_0}$. (See [10, Proposition 2.2.3]).

It is obvious that every semimaximal *wal*-ideal is irreducible.

In this paper, *spectra* of *wal*-rings, i.e. topological spaces of sets of their straightening *wal*-ideals, are studied. Spectra of abelian *wal*-groups have been searched in [8]. (Let us recall that each commutative *wal*-group can be studied as a *wal*-ring; it is sufficient to define multiplication on R by $ab = 0$ for any a, $b \in R$.) Spectra of f-rings have been investigated in [1]. (An f-ring is an l-ring isomorphic to a subdirect product of linearly ordered rings.)

Let R be a *wal*-ring. Let us denote by $\mathrm{Spec}(R)$ the set of proper straightening *wal*-ideals of R. Let be $M \subseteq R$. We define

$$S(M) = \{P \in \mathrm{Spec}(R);\ M \nsubseteq P\},$$
$$H(M) = \{P \in \mathrm{Spec}(R);\ M \subseteq P\}.$$

If $M = \{a\}$ we will denote $S(a) = S(\{a\})$ and $H(a) = H(\{a\})$.

Let $I(M)$ be the *wal*-ideal of R generated by M for any $M \subseteq R$. It is obvious that $M \subseteq P$ if and only if $I(M) \subseteq P$ where $P \in \mathrm{Spec}(R)$. Therefore $S(M) = S(I(M))$ and $H(M) = H(I(M))$ and we will consider only $S(I)$ and $H(I)$ for $I \in \mathcal{I}(R)$ and $S(a)$ and $H(a)$ for $a \in R$.

Lemma 10 *Let R be a wal-ring. Then*
(1) $S(0) = \emptyset,\ S(R) = \mathrm{Spec}(R)$;
(2) $\forall I, J \in \mathcal{I}(R);\ S(I \cap J) = S(I) \cap S(J)$;
(3) $\forall I_\gamma \in \mathcal{I}(R);\ S(\bigvee_{\gamma\in\Gamma} I_\gamma) = \bigcup_{\gamma\in\Gamma} S(I_\gamma)$;
(4) $\forall a, b \in R^+;\ S(a \vee b) = S(a) \cup S(b)$;
(5) $\forall a, b \in R^+;\ S(a \wedge b) = S(a) \cap S(b)$.

Proof (1) Obvious.

(2) Let $I, J \in \mathcal{I}(R)$, $P \in \mathrm{Spec}(R)$. Since P satisfies the condition (5), we have $I \cap J \nsubseteq P$ if and only if $I \nsubseteq P$ and $J \nsubseteq P$, hence $S(I \cap J) = S(I) \cap S(J)$.

(3) Let $I_\gamma \in \mathcal{I}(R)$, $\gamma \in \Gamma$, and $P \in \mathrm{Spec}(R)$. Let $\bigvee_{\gamma\in\Gamma} I_\gamma \nsubseteq P$. Then there exists $\gamma_0 \in \Gamma$ such that $I_{\gamma_0} \nsubseteq P$. The converse implication holds too, so $S(\bigvee_{\gamma\in\Gamma} I_\gamma) = \bigcup_{\gamma\in\Gamma} S(I_\gamma)$.

(4) Let $a, b \in R^+$, $P \in \mathrm{Spec}(R)$. If $P \in S(a) \cup S(b)$ then $a \notin P$ or $b \notin P$. If $a \vee b \in P$ then $0 \leq a,\ b \leq a \vee b$ and from the convexity of P we get that $a \in P$ and $b \in P$, a contradiction. Therefore $S(a) \cup S(b) \subseteq S(a \vee b)$.

Conversely, let $Q \in S(a \vee b)$. Then $a \vee b \notin Q$. If $a, b \in Q$, then $a \vee b \in Q$, a contradiction. Hence $a \notin Q$ or $b \notin Q$, i.e. $Q \in S(a) \cup S(b)$, hence $S(a \vee b) \subseteq S(a) \cup S(b)$.

(5) Let $a, b \in R^+$, $P \in \mathrm{Spec}(R)$. If $P \in S(a) \cap S(b)$ then $a \notin P$ and $b \notin P$. But P is a straightening *wal*-ideal, hence, if $0 \leq a \wedge b \in P$ then by [10], $a \in P$ or $b \in P$, a contradiction. Therefore $S(a \wedge b) \subseteq S(a) \cap S(b)$.

Conversely, let $Q \in S(a \wedge b)$, i.e. $a \wedge b \notin Q$. If $a \in Q$ then, because $0 \leq a \wedge b$, $a \wedge b \leq a$, the convexity of Q implies $a \wedge b \in Q$, a contradiction. Hence $a \notin Q$. Similarly we can prove that $b \notin Q$. Therefore $S(a \wedge b) \subseteq S(a) \cap S(b)$. □

Now the following theorem is an immediate consequence.

Theorem 11 *If R is a wal-ring and* $\mathrm{Spec}(R)$ *is the set of all proper straightening wal-ideals of R, then the sets S(I), where I is an arbitrary wal-ideal in R, form a topology of* $\mathrm{Spec}(R)$.

Definition The topology of $\mathrm{Spec}(R)$ with the open sets $S(I)$, where $I \in \mathcal{I}(R)$, is called the *spectral topology* of *wal*-ring R. The corresponding topological space is called the *spectrum* of R.

Let us recall that for l-rings, straightening and irreducible ideals coincide. Now we will show that for *wal*-rings this is not true in general, but that there are *wal*-rings not being l-rings for which every irreducible ideal is straightening.

Example 12 a) (See also [10].) Let R be the direct product $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z} = (\mathbb{Z}, +, \cdot)$ is semiordered with the positive cone $\mathbb{Z}^+ = \{0, 1, 2, 4, 6, \ldots\}$. As a direct product of *wal*-rings, R is a *wal*-ring. Denote $I = \{(x, 0);\ x \in \mathbb{Z}\}$. Let us show that I is a *wal*-ideal of R. By the definition of operations in the direct product R, it is easily seen that I is a ring ideal and a *wa*-sublattice. We check that it is a convex ideal. Let $a = (a_1, 0)$, $b = (b_1, 0) \in I$, $x = (x_1, x_2) \in R$ and hold $a \leq x$, $x \leq b$. Then $a_1 \leq x_1$, $0 \leq x_2$ and $x_1 \leq b_1$, $x_2 \leq 0$. $\mathbb{Z}$ is a convex set and from the above it follows $x_2 = 0$. Therefore $x \in I$.

It remains to verify that the condition $(\mathrm{I_b})$ is satisfied. Let $a = (a_1, 0)$, $b = (b_1, 0)$, $c = (c_1, 0) \in I$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in R$, and let hold $x \leq a$, $y \leq b$. Then $x_1 \leq a_1$, $x_2 \leq 0$ and $y_1 \leq b$, $y_2 \leq 0$. There exists $d_1 \in \mathbb{Z}$ such that $(x_1 \vee y_1) \vee c_1 = d_1$. Hence $(x \vee y) \vee c = ((x_1 \vee y_1) \vee c_1,\ (x_2 \vee y_2) \vee 0) = (d_1, 0) \in I$. It follows that I is a *wal*-ideal of R.

I is not a straightening *wal*-ideal because, for example, $(1, 4) \wedge (4, 1) = (0, 0)$ but neither $(1, 4)$ nor $(4, 1)$ belongs to I.

Let $A \in \mathcal{I}(R)$, let I be a proper *wal*-ideal of A and let $(a_1, a_2) \in A \setminus I$. Then $a_2 \neq 0$ and $(0, a_2) = (a_1, a_2) - (a_1, 0) \in A$. Since the convex *wal*-ideal of $\mathbb{Z}$ generated by a_2 is equal to $\mathbb{Z}$, we get $(x_1, x_2) = (x_1, 0) + (0, x_2) \in A$ for any element $(x_1, x_2) \in R$, hence $A = R$.

b) Let $R = \mathbb{Z}_9$, where $\mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ with the addition and multiplication mod9 and with $\mathbb{Z}_9^+ = \{0, 1, 3, 4, 7\}$. The ring $R = \mathbb{Z}_9$ is a *to*-ring hence every its *wal*-ideal is straightening. The ring R has a unique non-trivial ring ideal [3] (the principal ideal generated by 3). This ideal is the kernel of the *wal*-homomorphism $f : \mathbb{Z}_9 \longrightarrow \mathbb{Z}_3$ such that $f :\ 0, 3, 6 \longmapsto 0;\ 1, 4, 7 \longmapsto 1;\ 2, 5, 8 \longmapsto 2$. Therefore [3] is a *wal*-ideal of R. It is a unique non-trivial *wal*-ideal of R and so it is irreducible. Therefore R is a *wal*-ring (not being l-ring) in which irreducible and straightening *wal*-ideals coincide.

Theorem 13 *If R is a wal-ring in which every its irreducible wal-ideal is straightening, then the mapping $S : I \longmapsto S(I)$ is an isomorphism of the lattice $\mathcal{I}(R)$ onto the lattice of all open sets in* $\mathrm{Spec}(R)$.

Proof Let R be a *wal*-ring. By Lemma 10, S is a surjective lattice homomorphism. Further by [10, Corollary 2.2.6], every *wal*-ideal is an intersection of semimaximal *wal*-ideals. As every semimaximal *wal*-ideal is irreducible, it holds

$$I = \bigcap \{P;\ P \in H(I)\}, \qquad \text{for any } I \in \mathcal{I}(R).$$

That is why, if $I, J \in \mathcal{I}(R)$ and $S(I) = S(J)$, then

$$I = \bigcap\{P;\ P \in H(I)\} = \bigcap\{Q;\ Q \in H(J)\} = J. \qquad \square$$

Let R be any *wal*-ring and $a \in R$. Then by the *absolute value* of a it will be meant the element $|a| = (a \vee 0) \vee (-a \vee 0)$. It holds:

Proposition 14 *If R is a wal-ring and $a \in R$ then $I(a) = I(|a|)$.*

Proof Let $I \in \mathcal{I}(R)$ and $|a| \in I$. Then $0 \leq a \vee 0$, $a \vee 0 \leq |a|$, hence from the convexity of I we have $a \vee 0 \in I$. In the same way, $-a \vee 0 \in I$. By [5, Proposition 1.5], $a = (a \vee 0) - (-a \vee 0)$, hence $a \in I$.

Conversely, let $a \in I \in \mathcal{I}(R)$. Then $|a| = (a \vee 0) \vee (-a \vee 0) \in I$ too. $\square$

Remark 15 There exist *wal*-rings such that their positive cones are their *wa*-sublattices but also others which fail this property.

a) It is obvious that for every *to*-ring (and also for every representable *wal*-ring) R, its positive cone R^+ is a *wa*-sublattice of R.

b) Let us consider $R = (\mathbb{Z}, +, \cdot)$ with $R^+ = \{0, 1, 2, 4, \ldots, 2n, \ldots\}$. Then $1, 4 \in R^+$ but $1 \vee 4 = 5 \notin R^+$.

Corollary 16 *If R^+ is a wa-sublattice of R then every principal wal-ideal in R is generated by a positive element.*

Theorem 17 *If R is a wal-ring such that R^+ is a wa-sublattice of R, then the sets $S(a)$, where $a \in R$, form a basis of open sets of the spectrum of the wal-ring R which is stable under finite unions and intersections.*

Proof By Lemma 10, we get

$$S(I) = S(\bigvee_{a \in I} I(a)) = \bigcup_{a \in I} S(a) \quad \text{for any } I \in \mathcal{I}(R).$$

Hence the sets $S(a)$ form a basis in $\mathrm{Spec}(R)$. The second assertion is a consequence of Lemma 10 and Proposition 14. $\square$

Theorem 18 *a) If R is a wal-ring such that every its irreducible wal-ideal is straightening, then $S(a)$ is compact in* $\mathrm{Spec}(R)$ *for every $a \in R$.*

b) If, moreover, R^+ is a wa-sublattice of R and B is an open compact set in $\mathrm{Spec}(R)$ *then $B = S(a)$ for some $a \in R$.*

Proof a) Let R be a *wal*-ring, $a \in R$, $I_\gamma \in \mathcal{I}(R)$, $\gamma \in \Gamma$. Put

$$S(a) \subseteq \bigcup_{\gamma \in \Gamma} S(I_\gamma) = S(\bigvee_{\gamma \in \Gamma} I_\gamma).$$

Then by Theorem 13, $a \in \bigvee_{\gamma \in \Gamma} I_\gamma$. By [10, Proposition 2.1.1], $\bigvee_{\gamma \in \Gamma} I_\gamma = \sum_{\gamma \in \Gamma} I_\gamma$, from this it follows that there exist $\gamma_1, \dots, \gamma_k \in \Gamma$ such that

$$a \in \sum_{i=1}^{k} I_{\gamma_i} = \bigvee_{i=1}^{k} I_{\gamma_i}.$$

Therefore

$$S(a) \subseteq S(\bigvee_{i=1}^{k} I_{\gamma_i}) = \bigcup_{i=1}^{k} S(I_{\gamma_i}).$$

b) Let B be an open compact set. Then $B = \bigcup_{i=1}^{n} S(a_i)$, where $a_i \in B$. If R^+ is a wa-sublattice of R we can suppose (by Corollary 16) that $a_i \in R^+$, and so by Lemma 10, $\bigcup_{i=1}^{n} S(a_i) = S((\dots((a_1 \vee a_2) \vee a_3) \vee \dots) \vee a_n)$. □

Corollary 19 *Let R be a wal-ring for which every its irreducible wal-ideal is straightening and R^+ is wa-sublattice of R. Then* Spec(R) *is compact if and only if R contains an element a such that $I(a)$ (i.e. wal-ideal generated by the element a) is equal to R.*

Theorem 20 *Let R be a wal-ring, $P, Q \in$* Spec(R) *and $P \| Q$. Then P and Q have in* Spec(R) *disjoint neighborhoods.*

Proof Let us suppose $P, Q \in \mathrm{Spec}(R)$, $P \| Q$. Since every *wal*-ideal is generated by its positive cone, there exist $0 < a \in P \backslash Q$ and $0 < b \in Q \backslash P$. We will denote $u = a - (a \wedge b)$, $v = b - (a \wedge b)$. By [5, Proposition 1.5], $u \wedge v = 0$. Assume $u \in Q$. Since $0 \leq a \wedge b$, $a \wedge b < b$, we get $a \wedge b \in Q$, and from this $a = u + (a \wedge b) \in Q$, a contradiction. That means $u \notin Q$. Similarly we can prove that $v \notin P$. That is why $P \in S(v)$ and $Q \in S(u)$. As $u \wedge v = 0$, we get $S(u) \cap S(v) = S(u \wedge v) = \emptyset$. □

It is evident that the following theorem holds.

Theorem 21 *Let R be a wal-ring and* $\mathbf{x} \subseteq$ Spec(R) *a set of pairwise non-comparable straightening wal-ideals of R. Then the spectral topology of* $\mathbf{x}$ *is a* T_2*-topology.*

Proposition 22 *If R is a wal-ring and if P is an irreducible wal-ideal of R, then there exists a minimal irreducible wal-ideal which is contained in P.*

Proof Let R be a *wal*-ring, $\{P_\alpha;\ \alpha \in \Gamma\}$ be a collection of irreducible *wal*-ideals of R. Let $\{P_\alpha;\ \alpha \in \Gamma\}$ be linearly ordered by set inclusion and $P = \bigcap_{\alpha \in \Gamma} P_\alpha$. If $A, B \in \mathcal{I}(R)$ and $A \cap B \subseteq P$, then $\forall \alpha \in \Gamma$; $A \cap B \subseteq P_\alpha$, hence $\forall \alpha \in \Gamma$; $A \subseteq P_\alpha$ or $B \subseteq P_\alpha$. Suppose the existence of $\beta \in \Gamma$ such that $A \subseteq P_\beta$, $B \not\subseteq P_\beta$. Then $A \subseteq P_\gamma$ for every $P_\gamma \subseteq P_\beta$, thus $A \subseteq \bigcap_{\alpha \in \Gamma} P_\alpha = P$. Therefore the set of all irreducible *wal*-ideals ordered by set inclusion is inductive, hence every irreducible *wal*-ideal contains a minimal irreducible *wal*-ideal. □

The following theorem is now immediate consequence of Theorem 21 and Proposition 22.

Theorem 23 *Let R be a wal-ring such that every its irreducible wal-ideal is straightening. Then the set* $\mathrm{m}(R)$ *of all minimal straightening wal-ideals of R is non-empty and the spectral topology of* $\mathrm{m}(R)$ *is a* T_2*-topology.*

If $\mathbf{x} \subseteq \mathrm{Spec}(R)$, put

$$\mathcal{D}\mathbf{x} = \bigcap\{P;\ P \in \mathbf{x}\}.$$

Theorem 24 *a) The closed sets in the spectrum of a wal-ring R are just all $H(I)$, where $I \in \mathcal{I}(R)$.*
b) If $\boldsymbol{x} \subseteq \mathrm{Spec}(R)$, then its closure is $\overline{\boldsymbol{x}} = H(\mathcal{D}\boldsymbol{x})$.

Let us recall that a *wal*-ring R is called *representable* if R is isomorphic to a subdirect product of *to*-rings.

By [9, Theorem 2.5], the class $\mathcal{RR}_{wal}$ of all representable *wal*-rings is a variety of *wal*-rings. It holds (see [9, Proposition 2.2]) that a *wal*-ring R is representable if and only if the intersection of all its straightening *wal*-ideals is equal to $\{0\}$.

It follows from this:

Theorem 25 *If R is a representable wal-ring and $\boldsymbol{x} \subseteq \mathrm{Spec}(R)$ then $\boldsymbol{x}$ is dense if and only if*

$$\bigcap\{P;\ P \in \boldsymbol{x}\} = \{0\}.$$

Let R be a *wal*-ring and $0 \neq a \in R$. Let us denote by $\mathrm{V}(a)$ the set of all semimaximal *wal*-ideals, maximal with respect to the property "not containing a". (For $a = 0$ the set $\mathrm{V}(a) = \emptyset$.) By [10, Proposition 2.2.5], $\mathrm{V}(a) \neq \emptyset$ for each $a \neq 0$, and by [10, Proposition 2.2.3], every $C \in \mathrm{V}(a)$ is irreducible in R. Moreover, let us assume that every irreducible *wal*-ideal of R is straightening. Then $\mathrm{V}(a) \subseteq \mathrm{Spec}(R)$. Let $P \in S(a)$. Then by [10, Theorem 2.2.1], the set of all *wal*-ideals of R containing P is linearly ordered, and by [10, Proposition 2.2.5], there exists a *wal*-ideal in $\mathrm{V}(a)$ that contains P. Hence there exists exactly one *wal*-ideal $M_P \in \mathrm{V}(a)$ such that $P \subseteq M_P$.

Let us denote by $\psi_a : S(a) \longrightarrow \mathrm{V}(a)$ the mapping such that $\psi_a : P \longmapsto M_P$.

Theorem 26 *If R is a wal-ring such that every its irreducible wal-ideal is straightening and $a \in R$, then the mapping ψ_a is continuous.*

Proof Let $a \in R$, $P \in S(a)$ and let U be a neighborhood of M_P in $\mathrm{V}(a)$. We can suppose that $U = S(b) \cap \mathrm{V}(a)$ for some $b \in R$. If $Q \in \mathrm{V}(a)\backslash S(b)$ then there exist neighborhoods U_Q of Q and V_Q of M_P such that $U_Q \cap V_Q = \emptyset$, which follows from Theorem 20. Let Q runs over $\mathrm{V}(a)\backslash S(b)$. Then the corresponding U_Q form a covering of $S(a)\backslash S(b)$. By Theorem 18, $S(a)$ is compact in $\mathrm{Spec}(R)$. Moreover, $S(a)\backslash S(b)$ is closed in $S(a)$, hence $S(a)\backslash S(b)$ is also compact. Thus there exist $n \in \mathbb{N}$ and $Q_1, \dots, Q_n \in S(a)\backslash S(b)$ such that $S(a)\backslash S(b) \subseteq U_{Q_1} \cup \dots \cup U_{Q_n}$. Let us denote $C = S(a)\backslash(U_{Q_1} \cup \dots \cup U_{Q_n})$. We get $V_{Q_1} \cap \dots \cap V_{Q_n} \subseteq C$ hence C is a neighborhood of M_P which is closed in $S(a)$, and $C \cap \mathrm{V}(a) \subseteq U$. Therefore

$C \subseteq {\psi_a}^{-1}(C \cap \mathrm{V}(a)) \subseteq {\psi_a}^{-1}(U)$. Furthermore, C is a neighborhood of M_P, thus it is also a neighborhood of P. □

Theorem 27 *Let R be a wal-ring such that every its irreducible wal-ideal is straightening. If $a \in R$, then the set $V(a)$ is a compact T_2-space.*

Proof By Theorem 21, $\mathrm{V}(a)$ is a T_2-space. Further, $\mathrm{V}(a)$ is by Theorem 26 the image of the compact set $S(a)$ in the continuous mapping ψ_a, hence $\mathrm{V}(a)$ is also compact. □

Theorem 28 *Let I be a wal-ideal of a wal-ring R. Then the mapping $f : H(I) \longrightarrow \mathrm{Spec}(R/I)$ such that $f(P) = P/I$ for every $P \in H(I)$ is a homeomorphism of the space $H(I)$ onto the spectrum $\mathrm{Spec}(R/I)$.*

Proof Let R be a *wal*-ring, $I \in \mathcal{I}(R)$. Consider the mapping $f : H(I) \longrightarrow \mathrm{Spec}(R/I)$ such that $f(P) = P/I$. By [10, Theorem 1.4.5], $P/I \in \mathcal{I}(R/I)$ and $(R/I)/(P/I)$ is isomorphic to R/P. Since P is a straightening *wal*-ideal of R, R/P is a totally semiordered *wal*-ring, it follows that $(R/I)/(P/I)$ is also a totally semiordered *wal*-ring. It means that P/I is a straightening *wal*-ideal of R/I. Moreover, $P/I \neq R/I$ therefore $P/I \in \mathrm{Spec}(R/I)$.

Conversely, let $Q \in \mathrm{Spec}(R/I)$. We denote $P = \{x \in R;\ x + I \in Q\}$. Then $P \in \mathcal{I}(R)$, $P \neq R$, $I \subseteq P$, $Q = P/I$. At the same time, *wa*-lattices R/P and $(R/I)/Q$ are isomorphic, hence R/P is totally semiordered and it means that $P \in H(I)$. Then f is a bijective mapping $H(I)$ onto $\mathrm{Spec}(R/I)$ preserving inclusions and in this way any set intersections too.

Let $\mathbf{x} \subseteq H(I)$. Then $f(\overline{\mathbf{x}}) = \{f(P);\ P \supseteq \bigcap(Q;\ Q \in \mathbf{x}\}$. The mapping f preserves intersections, hence

$$f(P) \supseteq f(\bigcap(Q;\ Q \in \mathbf{x})) = \bigcap(f(Q);\ Q \in \mathbf{x}).$$

It follows that

$$f(\overline{\mathbf{x}}) = \{f(P);\ f(P) \supseteq \bigcap(f(Q);\ Q \in \mathbf{x}\}.$$

Moreover,

$$\mathcal{D}f(\mathbf{x}) = \bigcap(T;\ T \in f(\mathbf{x})) = \bigcap(f(Q);\ Q \in H(I),\ f(Q) \in f(\mathbf{x})),$$

thus

$$\overline{f(\mathbf{x})} = \{f(Z) \in \mathrm{Spec}(R/I);\ \mathcal{D}f(\mathbf{x}) \subseteq f(Z)\}.$$

Hence $f(\overline{\mathbf{x}}) = \overline{f(\mathbf{x})}$, therefore f is a homeomorphism of $H(I)$ onto $\mathrm{Spec}(R/I)$. □

Theorem 29 *Let I be a wal-ideal of a wal-ring R. Then the mapping $g : S(I) \longrightarrow \mathrm{Spec}(I)$ such that $g(P) = P \cap I$ for each $P \in S(I)$, is a homeomorphism of the space $S(I)$ onto some subspace of the spectrum $\mathrm{Spec}(I)$.*

Proof Let R be a *wal*-ring, $I \in \mathcal{I}(R)$, $P \in S(I)$. By [10, Theorem 2.1.2], $(P+I) \in \mathcal{I}(R)$, $P \in \mathcal{I}(P+I)$ and $I/P \cap I$ is isomorphic to $(P+I)/P$. Since P is a straightening *wal*-ideal of R, it is a straightening *wal*-ideal of $(P+I)$, too, and hence $(P+I)/P$ is a totally semiordered *wal*-ring. It follows that $I/P \cap I$ is also totally semiordered, hence $P \cap I$ is a straightening *wal*-ideal of I. Moreover, $P \cap I \neq I$ thus $P \cap I \in \mathrm{Spec}(I)$.

Let $P, Q \in S(I)$ be such that $g(P) = g(Q)$. That means $P \cap I = Q \cap I$, then $P \cap I \subseteq Q$, hence $P \subseteq Q$ or $I \subseteq Q$. By the assumption, $I \nsubseteq Q$, from this $P \subseteq Q$. Similarly $Q \subseteq P$, thus g is an injection.

We proceed to show that g is a homeomorphism. Suppose $\mathbf{x} \subseteq S(I)$. Then $g(\mathbf{x}) = \{g(Q);\, Q \in \mathbf{x}\}$. Let $P \in \overline{\mathbf{x}} \cap S(I)$. Then $P \supseteq \mathcal{D}\mathbf{x}$, hence $P \cap I \supseteq \mathcal{D}\mathbf{x} \cap I = \bigcap(Q \cap I;\, Q \in \mathbf{x})$. It means that for the closure $\overline{g(\mathbf{x})}$ of the set $g(\mathbf{x})$ in $\mathrm{Spec}(I)$ we have $g(P) = P \cap I \in \overline{g(\mathbf{x})}$.

Conversely, let $P \in S(I)$ be such that $g(P) \in \overline{g(\mathbf{x})}$. Then $P \cap I \supseteq \mathcal{D}\mathbf{x} \cap I$, thus $\mathcal{D}\mathbf{x} \cap I \subseteq P$.

Since $I \nsubseteq P$, we have $\mathcal{D}\mathbf{x} \subseteq P$, and it means that $P \in \overline{\mathbf{x}}$. □

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