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On the Lefschetz Fixed Point Theorem for Multivalued Weighted Mappings

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Abstract

The class of multivalued weigthed mappings has been introduced by G. Darbo, and rediscovered independently by R. Jerrard. It is well known that the Lefschetz fixed point theorem holds true for such mappings from a compact polyhedron into itself (see [10], [13]). In our paper we extend this result to so-called compact absorbing contraction weighted maps of arbitrary metric ANR's.

Key words: Fixed point, weighted mappings, Darbo homology, Lefschetz number.

2000 Mathematics Subject Classification: 54C60, 55M20

1 Introduction

First we recall some well known notions and introduce necessary notations (for details see [6] or [5]). We shall use letters $\psi, \varphi, \ldots$, to denote multivalued mappings. The single valued maps will be denoted by $f, g, w, \ldots$ In what follows by a space we understand a Hausdorff topological space. Let X, Y be two spaces. We shall say that $\psi : X \to Y$ is a multivalued map, if for each $x \in X$ nonempty, finite subset $\psi(x) \subset Y$ is given. A map $\psi : X \to Y$ is called upper semicontinuous (u.s.c.), if for each open set $U \subset Y$ a set: $\psi^{-1}(U) = \{x \in X; \psi(X) \subset Y\}$ is open in X. Given space X, Y and Z, maps $\psi : X \to Y$ and $\varphi : Y \to Z$, we define the composition $\varphi \circ \psi$ by $\varphi \circ \psi(x) := \bigcup_{y \in \psi(x)} \varphi(y)$ for each $x \in X$. If

φ, ψ are multivalued maps (u.s.c.), then $\varphi \circ \psi$ is also a multivalued map (u.s.c.). We say that $x \in X$ is a fixed point of $\psi : X \to X$, if $x \in \psi(x)$. By a topological pair we shall understand a pair (X, X_0), where X is a space and X_0 is a subset of X. A pair of the form $(X, \emptyset)$ will be identified with the space X. We shall write $\psi : (X, A) \to (Y, B)$, if $\psi : X \to Y$ and $\psi(A) \subset B$.

2 Weighted mappings

Following [10], [11] we recall the notion of a weighted mapping and its properties.

Definition 2.1 A weighted mapping from X to Y with coefficients in a commutative ring with unity Ω (or simply a w-map) is a pair $\psi = (\sigma_\psi, w_\psi)$ satisfying the following conditions:

- $\sigma_\psi : X \to Y$ is a multivalued upper semicontinuous mapping;
- $w_\psi : X \times Y \to \Omega$ is a function with the following properties:
  - $w_\psi(x, y) = 0$ for any $y \notin \sigma_\psi(x)$
  - if U is an open subset of Y and $x \in X$ is such that $\sigma_\psi(x) \cap bd\, U = \emptyset$, then there exists an open neighbourhood V of the point x such that:

$$\sum_{y \in U} w_\psi(x, y) = \sum_{y \in U} w_\psi(z, y)$$

for every $z \in V$, where $bd\, U$ denotes the boundary of U in Y.

Note 2.1 For our comfort a multivalued weighted mapping from X to Y, i.e. $\psi = (\sigma_\psi, w_\psi)$, we shall denote, in short, by $\psi : X \to Y$. So, by $\psi(x)$ we shall mean $\sigma_\psi(x)$ for every $x \in X$ and, consequently, $\psi^{-1}(U)$ for every subset of Y stands for the counter image $\sigma_\psi^{-1}(U)$ of U under σ_ψ, etc. The mapping σ_ψ from the above definition will be called a support of ψ. By a weight of ψ we shall understand a function w_ψ, i.e. $w_\psi : X \times Y \to \Omega$.

Note 2.2 Each continuous map $f : X \to Y$ can be considered as a weighted one by assigning the coefficient 1 to each $f(x)$. We shall use also the same notation for this map. Now, we shall give some example of a w-map.

Example 2.1 Let $\psi : X \to Y$ be an u.s.c. map such that for all $x \in X$ $\psi(x)$ consist of 1 or exactly n points (with n fixed). A weight $w_\psi : X \times Y \to \mathcal{Z}$ ($\mathcal{Z}$ denotes a ring of integers) we define by the formula:

$$w_\psi(x, y) = \begin{cases} 0 & \text{if } y \notin \psi(x); \\ \text{n} & \text{if } \{y\} = \psi(x); \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that a pair $\psi = (\sigma_\psi = \psi, w_\psi)$ is a w-map. For more examples see [13].

Note 2.3 There exists a multivalued u.s.c. mapping ψ which has only trivial weight! (i.e. for every $x \in X$ and $y \in Y$ $w_\psi(x,y) = 0$). It is enough to define this map as follows:

$$\psi(x) = \begin{cases} \{1\} & \text{if } 0 \leq x < \frac{1}{2}; \\ \{0,1\} & \text{if } x = \frac{1}{2}; \\ \{0\} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let us underline, once more, that for the above mapping one cannot define a nontrivial weight (see Definition 2.1). Moreover, this w-map has no fixed point. The above example shows us that mappings with trivial weight are not too interesting for the fixed point theory. Nonetheless, we shall make use of the above w-maps (we need the zero element in an Ω-module, see Proposition 2.3). Below we shall list important and well known properties of w-maps ([10], [11]).

Proposition 2.1 *If $\psi, \varphi : X \to Y$ are w-maps, then $\psi \cup \varphi = (\sigma_{\psi\cup\varphi}, w_{\psi\cup\varphi})$ is also one, where $\sigma_{\psi\cup\varphi} : X \to Y$ and $w_{\psi\cup\varphi} : X \times Y \to \Omega$ are defined by the formulas: $\sigma_{\psi\cup\varphi}(x) = \sigma_\psi(x) \cup \sigma_\varphi(x)$ and $w_{\psi\cup\varphi}(x,y) = w_\psi(x,y) + w_\varphi(x,y)$ for every $x \in X$ and $y \in Y$.*

Proposition 2.2 *If $\psi : X \to Y$ is a w-map and $\alpha \in \Omega$, then $\alpha \cdot \psi = (\sigma_{\alpha\cdot\psi}, w_{\alpha\cdot\psi})$ is also one, where $\sigma_{\alpha\cdot\psi} : X \to Y$ and $w_{\alpha\cdot\psi} : X \times Y \to \Omega$ are defined as follows: $\sigma_{\alpha\cdot\psi}(x) = \sigma_\psi(x)$ and $w_{\alpha\cdot\psi}(x,y) = \alpha \cdot w_\psi(x,y)$ for every $x \in X$ and $y \in Y$.*

Note 2.4 By $\mathcal{W}(X,Y)$ we shall understand the class of all w-maps. Let us define $\sim$ an equivalence relation on $\mathcal{W}(X,Y)$ as follows: $\psi \sim \varphi \Leftrightarrow w_\psi = w_\varphi$. The class of equivalence classes we shall denote by $<X,Y> := \mathcal{W}(X,Y)/\sim$.

Now we are able to formulate the following:

Proposition 2.3 *The quotient set $<X,Y>$ admits the structure of the Ω-module.*

Definition 2.2 Given w-maps $\psi : X \to Y$ and $\varphi : Y \to Z$. By their composition we understand a w-map $\varphi \circ \psi : X \to Z$ whose support $\sigma_{\varphi\circ\psi}$ is the composition of σ_ψ and σ_φ, but a weight $w_{\varphi\circ\psi} : X \times Z \to \Omega$ is defined by the formula:

$$w_{\varphi\circ\psi}(x,z) = \sum_{y \in Y} w_\psi(x,y) \cdot w_\varphi(y,z)$$

for every $x \in X$ and $z \in Z$.

Now as a simple consequence of Definition 2.2 we obtain:

Proposition 2.4 *The multivalued weighted mappings over Ω and Hausdorff spaces form a category: $\mathcal{C}_\Omega$.*

Observe that the category *Top* of Hausdorff topological spaces and continuous (single valued) mappings is a subcategory of $\mathcal{C}_\Omega$.

3 The Darbo homology

In this section we briefly describe the Darbo homology theory on $\mathcal{C}_\Omega$ that restricted to the subcategory of finite polyhedra and continuous maps coincides with the singular homology with coefficients in Ω (see [2], [10], [13]). Here this homology will be defined with very little change from the original construction. We adopt approach from [11]. We shall use the Darbo homology theory in the next section in order to express the Lefshetz number of weighted mappings. So, let us proceed to the construction.

By $\triangle_m$ we shall denote the m-dimensional simplex. One can prove that $\triangle_m = \mathrm{conv}(\{e_0, \ldots, e_m\})$, where

$$e_0 = (0, \ldots, 0),\ e_1 = (0, 1, 0, \ldots, 0),\ \ldots,\ e_k = (0, \ldots, 0, \underbrace{1}_{k\text{-th}}, 0, \ldots, 0),$$

for $k = 1, \ldots, m$, are points of an *Euclidean* m-dimensional space $\mathbb{R}^m$. Let us denote by X any Hausdorff topological space. From Section 2 we obtain for each $n \geq 0$ the Ω-module $<\triangle_n, X>$. Moreover, we put:

$$\mathcal{C}_n(X) := <\triangle_n, X>.$$

Hence we obtain the graded Ω-module:

$$\mathcal{C}(X) = \{\mathcal{C}_n(X)\}_{n \geq 0}\,.$$

Now we shall define a homomorphism $\delta_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$, where $n \geq 1$. By d_n^i we shall understand a linear map from $\triangle_{n-1}$ to $\triangle_n$ which is uniquely determined by its values on the vertices:

$$d_n^i(e_j) = \begin{cases} e_j & \text{when } j < i \\ e_{j+1} & \text{when } j \geq i\,. \end{cases}$$

Of course it is a w-map. So, the boundary map $\delta : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ is given by

$$\delta_n([s]) = [\,\bigcup_{i=0}^{n} (-1)^i \cdot s \circ d_n^i\,]_\sim$$

for all equivalence classes $[s] \in \mathcal{C}_n(X)$.

Fact that the above homomorphism is well defined follows easily from Section 2. One can show that $\delta_n \circ \delta_{n+1} = 0$ (comp. [14]), where $n \geq 1$.

Next we put:

$$\mathcal{Z}_n(X) = \mathrm{Ker}\,\delta_n\,, \quad \mathcal{B}_n(X) = \mathrm{Im}\,\delta_{n+1} \quad \text{and} \quad \mathcal{H}_n(X) = \mathcal{Z}_n(X)/\mathcal{B}_n(X).$$

So, we get the graded module of X over Ω:

$$\mathcal{H}_*(X) = \{\mathcal{H}_n(X)\}_{n \geq 0}\,.$$

It will be called the *Darbo* homology module of X with coefficients in Ω. One can easily see that a w-map $\psi : X \to Y$ induces functorially a homomorphism $\psi_* : \mathcal{H}_*(X) \to \mathcal{H}_*(Y)$. It is well known how to define:

$$\mathcal{H}_*(X, A) \quad \text{and} \quad \psi_* : \mathcal{H}_*(X, A) \to \mathcal{H}_*(Y, B)$$

for a topological pair (X, A) and a w-map $\psi : (X, A) \to (Y, B)$.

It turn out that in $\mathcal{C}_\Omega$ one can define also the notion of the homotopy be the interval $[0, 1]$. Given two w-maps ψ and φ from X to Y, we say that ψ is w-homotopic to φ ($\psi \sim_w \varphi$) if there exists a w-map $\mathbb{H}$ from $X \times I$ to Y such that:

$$w_{\mathbb{H}}((x, 0), y) = w_\psi(x, y) \quad \text{and} \quad w_{\mathbb{H}}((x, 1), y) = w_\varphi(x, y).$$

Let us underline that we do not demand in order to:

$$\sigma_{\mathbb{H}}(x, 0) = \sigma_\psi(x) \quad \text{and} \quad \sigma_{\mathbb{H}}(x, 1) = \sigma_\varphi(x).$$

It is easy to see that w-homotopy is an equivalence relation in $\mathcal{C}_\Omega$ which extends the usual homotopy relation on the category of tpological (Hausdorff) spaces and continuous (single valued) maps. So we obtain the covariant functor:

$$\mathcal{H} : \mathcal{C}_\Omega \to \mathcal{M}_\Omega$$

from the category Hausdorff topological spaces and multivalued weighted mappings over Ω to the category of graded Ω-modules which satisfies the *Eilenberg–Steenrod* axioms for a homology theory with compact carriers and coefficients in a commutative ring with unity Ω (see [2], [10]).

4 The Lefschetz number

In this section all the vector spaces are taken over $\mathbb{Q}$ and all maps between such spaces are linear. First we shall recall the notion of the ordinary trace. Let $f : E \to E$ be an endomorphism of a finite dimensional vector space E and let $e_1, \ldots, e_n$ be a basis for E. Then for every e_i we can write

$$f(e_i) = \sum_{j=1}^{n} a_{i,j} e_j \,.$$

Hence we have the matrix $A = [a_{i,j}]_{i,j=1}^n$ of f. The trace of A is given by the formula:

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{i,i} \,.$$

By the trace of an endomorphism of a finite dimensional vector space $f : E \to E$, writen $\operatorname{tr}(f)$, we shall understand the trace of the matrix of f with respect to some basis for E. The above definition is correct, i.e. it does not depend on the choice of the basis for E. Now we shall collect the important and well known properties of the defined trace $\operatorname{tr}(f)$ (see [6] or [8]).

Property 4.1 *Assume that in the category of finite dimensional vector spaces the followong diagram commutes:*

$$\begin{array}{ccc} E' & \xrightarrow{u} & E'' \\ f' \downarrow & \overset{v}{\swarrow} & \downarrow f'' \\ E' & \xrightarrow{u} & E'' \end{array} .$$

Then $\operatorname{tr}(f') = \operatorname{tr}(f'')$, *or equivalently,* $\operatorname{tr}(vu) = \operatorname{tr}(uv)$.

Property 4.2 *Given a commutative diagram of finite dimensional vector spaces with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \end{array} .$$

Then we have $\operatorname{tr}(f) = \operatorname{tr}(f') + \operatorname{tr}(f'')$.

Let $E = \{E_n\}_{n \geq 0}$ be a graded vector of finite type, i.e. $\dim E_n < \infty$ for all n and $E_n = 0$ for almost n. If $f = \{f_n\}_{n \geq 0}$ is an endomorphism of degree zero (i.e. $f_n : E_n \to E_n$) of the space E, then the *Lefschetz number* is defined by:

$$\lambda(f) = \sum_n (-1)^n tr(f_n).$$

It is well known that one can generalize the Lefschetz number. First we have to generalize the notion of the trace. Let $f : E \to E$ be an endomorphism of arbitary vector space E. By $f^{(n)} : E \to E$ we denote the n-th iterate of f. Let us note that the kernels

$$\operatorname{Ker} f \subset \operatorname{Ker} f^{(2)} \subset \ldots \subset \operatorname{Ker} f^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E. Next, let us define the set $\mathcal{N}(f)$ by the formula:

$$\mathcal{N}(f) = \{x \in E;\ f^{(n)}(x) = 0 \text{ for some } n\}.$$

It is clear that:

$$\mathcal{N}(f) = \bigcup_{n \geq 1} \operatorname{Ker} f^{(n)}.$$

Let us note also that f maps $\mathcal{N}(f)$ into itself and hence, consequently, we get the induced endomorphism $\tilde{f} : \widetilde{E} \to \widetilde{E}$, where $\widetilde{E} = E/\mathcal{N}(f)$ is the factor space. It is easy to see that $\tilde{f} : \widetilde{E} \to \widetilde{E}$ is a monomorphism. Now we are able to formulate the Leray trace.

Definition 4.1 Let $f : E \to E$ be an endomorphism of a vector space E. Assume also that $\dim \widetilde{E} < \infty$. By the *Leray* trace $\operatorname{Tr}(f)$ of f we understand the ordinary trace of $\tilde{f}$, i.e. we let $\operatorname{Tr}(f) = \operatorname{tr}(\tilde{f})$.

Note 4.1 For such the trace Property 4.1 and 4.2 hold true. Moreover, one can show the following:

Property 4.3 *Let $f : E \to E$ be an endomorphism. If* $\dim E < \infty$ *then* $\operatorname{Tr}(f) = \operatorname{tr}(f)$.

We are now ready to define the generalized Lefschetz number. Let $f = \{f_n\}_{n\geq 0}$ be an endomorphism of degree zero of a graded vector space $E = \{E_n\}_{n\geq 0}$. We say that f is the *Leray* endomorphism provided that the graded vector space $\widetilde{E} = \{\widetilde{E}_n\}_{n\geq 0}$ is of finite type. If f is the Leray endomorphism , then we can define the generalized Lefschetz number $\Lambda(f)$ of f by putting:

$$\Lambda(f) = \sum_n (-1)^n \operatorname{Tr}(f_n).$$

Now from Property 4.3 we have the following:

Property 4.4 *Let $f : E \to E$ be an endomorphism of degree zero. If E is a graded vector space of finite type then*

$$\Lambda(f) = \lambda(f).$$

The next two properties immediately follows from Property 4.1 and 4.2, respectively (see Note 4.1).

Property 4.5 *Assume that in the category of graded vector spaces the following diagram commutes:*

$$\begin{array}{ccc} E' & \xrightarrow{u} & E'' \\ f' \downarrow & \overset{v}{\swarrow} & \downarrow f'' \\ E' & \xrightarrow{u} & E''. \end{array}$$

Then if any of the the maps f' or f is the Leray endomorphism, then so is the other and in that case

$$\Lambda(f') = \Lambda(f'').$$

Property 4.6 *Let*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & E'_n & \longrightarrow & E_n & \longrightarrow & E''_n & \longrightarrow & E'_{n-1} & \longrightarrow \cdots \\ & \downarrow f'_n & & \downarrow f_n & & \downarrow f''_n & & \downarrow f'_{n-1} & \\ \cdots \longrightarrow & E'_n & \longrightarrow & E_n & \longrightarrow & E''_n & \longrightarrow & E'_{n-1} & \longrightarrow \cdots \end{array}$$

be a commutative diagram of vector spaces in which the rows are exact. If of the following endomorphism

$$f = \{f_n\}_{n\geq 0},\ f' = \{f'_n\}_{n\geq 0},\ f'' = \{f''_n\}_{n\geq 0}$$

are the Leray endomorphism then so is the third, and, moreover, in that case we have:

$$\Lambda(f'') + \Lambda(f') = \Lambda(f).$$

Finaly this section we recall the notion of a weakly nilpotent endomorphism.

Definition 4.2 A linear map $f : E \to E$ of a vector space E into itself is called *weakly nilpotent* if for every $x \in X$ there exists a natural number $n = n_x$ such that: $f^{n_x}(x) = 0$

From the above definition we deduce that $f : E \to E$ is weakly nilpotent if and only if $\mathcal{N}(f) = E$. Let us note also the following:

Property 4.7 *If $f : E \to E$ is weakly nilpotent then* $\mathrm{Tr}(f)$ *is well defined and* $\mathrm{Tr}(f) = 0$.

We say that an endomorphism $f = \{f_n\}_{n \geq 0} : E \to E$ is weakly nilpotent if and only if $f_n : E_n \to E_n$ is weakly nilpotent for every n, where $E = \{E_n\}_{n \geq 0}$ is a graded vector space. From Property 4.7 we get:

Property 4.8 *Any weakly nilpotent endomorphism $f : E \to E$ of a graded vector space is a Leray endomorphism and $\Lambda(f) = 0$.*

5 Main results

In the rest of this paper all spaces are assumed to be metric. We assume also that Ω is the field of rational numbers $\mathbb{Q}$. This section is organized as follows. To begin with, we shall prove the Lefschetz Fixed Point Theorem for compact w-maps on ANR's. The second part is devoted to extension of the above result to *compact absorbing contraction weighted* maps ($\mathbb{CAC}_w$-maps) from any *ANR* into itself.

Before we give the proof of the first fact, let us recall a few notions and their properties. Let $\psi : X \to X$ be a w-map. If the induced homomorphism $\psi_* : \mathcal{H}_*(X) \to \mathcal{H}_*(X)$ is a Leray endomorphism, then ψ is called a *Lefschetz* w-map and for such ψ we can define the Lefschetz number $\Lambda(\psi)$ of ψ by putting:

$$\Lambda(\psi) = \Lambda(\psi_*).$$

Clearly, if ψ and φ are w-homotopic then $\Lambda(\psi) = \Lambda(\varphi)$. Let us remark that if φ has a trivial weight then φ is a Lefschetz w-map and $\Lambda(\varphi) = 0$. We shall say that a w-map $\varphi : X \to X$ has a fixed point provided there exists $x_0 \in X$ such that $x_0 \in \varphi(x_0)$. Applying the Darbo homology functor and Property 4.5 we get:

Property 5.1 *Assume that in the category $C_{\mathbb{Q}}$ the following diagram commutes*

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X'' \\ \psi\downarrow & \overset{\theta}{\swarrow} & \downarrow\varphi \\ X' & \xrightarrow{\alpha} & X''. \end{array}$$

Then, if one of the w-maps ψ or φ is a Lefschetz w-map, then so is the other and in that case $\Lambda(\psi) = \Lambda(\varphi)$.

We shall say that a w-map $\psi : X \to Y$ is a compact w-map if the closure $\overline{\psi(X)}$ of $\psi(X)$ in Y is a compact set. We shall need also some elementary facts concerning absolute neighbourhood retracts (*ANR*-spaces, see [1], [6]).

Definition 5.1 Let X and Y be metric spaces. By an embedding of a space X into Y we shall mean any homeomorphism $h : X \to Y$ from X into Y such that $h(X)$ is a closed subset of Y.

The following notions are especially important in our considerations.

Definition 5.2 A continuous mapping $r : X \to Y$ is called an r-map provided there exists a continuous map $s : Y \to X$ such that $r \circ s = id_Y$.

Definition 5.3 We say that a subset A of X is a retract of X if there exists a retraction $r : X \to A$, i.e. a continuous map satisfying the following condition: $r(x) = x$ for every $x \in A$.

It is easy to see that if A is a retract of X then A is a closed subset of X.

Definition 5.4 A subset Y of X is called a neighbourhood retract of X if there exists an open subset $U \subset X$ such that $Y \subset U$ and Y is a retract of U.

Definition 5.5 A space X is said to be an absolute neighbourhood retract provided for any space Y and for any embedding $h : X \to Y$ the set $h(X)$ is a neighbourhood retract of Y.

We shall use the notation: $X \in ANR$. One can show the useful facts:

Proposition 5.1 *$X \in ANR$ if and only if it is an r-image of some open subset U of some normed space E.*

Proposition 5.2 *If $X \in ANR$ and U is an open subset of X, then: $U \in ANR$.*

In particular, we shall say that X is an *Euclidean neighbourhood retract* ($X \in ENR$) if and only if X is an r-image of some open subset U of $\mathbb{R}^n$. Now, we may state the first main theorem of this section.

Theorem 1 (The Lefschetz Fixed Point Theorem) *Let $X \in ANR$ and let $\varphi : X \to X$ be a compact w-map ($\varphi \in \mathbb{K}_w(X)$) then:*

1. φ is a Lefschetz w-map;

2. $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.

Proof It will be given in several steps.

Step 1. We consider the following special case:

1. U is an open subset of $\mathbb{R}^n$

2. φ is a compact w-map from U into itself.

From condition 2 we have that the closure $\overline{\varphi(U)}$ of $\varphi(U)$ in U is a compact set. It is well known that there exists a compact polyhedron X such that $\overline{\varphi(U)} \subset$

$X \subset U$. Consider the following diagram in which all the arrows represent either the obvious inclusions or the contractions of the w-map φ:[1]

$$\begin{array}{ccc} X & \xrightarrow{i} & U \\ {\scriptstyle\widetilde{\varphi}}\downarrow & \overset{\widehat{\varphi}}{\swarrow} & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow{i} & U\,. \end{array}$$

From Section 2 it follows that the above diagram commutes in the category $\mathcal{C}_{\mathbb{Q}}$. Since the graded vector space $\mathcal{H}(X) = \{\mathcal{H}_n(X)\}_{n\geq 0}$ of a compact polyhedron X is of finite type[2], then $\widetilde{\varphi}$ is a lefschetz w-map. Consequently, from Property 5.1 we infer that φ is a Lefschetz w-map and, moreover, $\Lambda(\varphi) = \Lambda(\widetilde{\varphi})$. Let us assume now that $\Lambda(\varphi) \neq 0$. (Hence, of course, $\Lambda(\widetilde{\varphi}) \neq 0$). Then, in view of [10] and [13], we get that $\widetilde{\varphi}$ has a fixed point and hence φ has also a fixed point.

Note It is obvious that if in Step 1 we replace $\mathbb{R}^n$ by any n-dimensional normed space E^n, then the same result remain true.

Step 2. We replace the assumptions 1 and 2 by the weaker one, namely: φ is a compact w-map from X into itself, where $X \in ENR$.

Since X is an ENR-space, then it is an r-image of some open subset U of $\mathbb{R}^n$, i.e. there are the continuous maps $r : U \to X$ and $s : X \to U$ such that $r \circ s = id_X$. Let us consider the following commutative diagram (recall that in the category $\mathcal{C}_{\mathbb{Q}}$):

$$\begin{array}{ccc} X & \xrightarrow{s} & U \\ {\scriptstyle\varphi}\downarrow & \overset{\varphi\circ r}{\swarrow} & \downarrow{\scriptstyle s\circ\varphi\circ r} \\ X & \xrightarrow{s} & U\,. \end{array}$$

Since $s\circ\varphi\circ r$ is a compact w-map, then from Step 1 we obtain that $s\circ\varphi\circ r : U \to U$ is a Lefschetz w-map. Consequently, from Property 5.1 we have that φ is a Lefschetz w-map and $\Lambda(\varphi) = \Lambda(s\circ\varphi\circ r)$. So if $\Lambda(\varphi) \neq 0$, then $\Lambda(s\circ\varphi\circ r) \neq 0$. Now, in view of Step 1, we get that there exists x_0 such that $x_0 \in s\circ\varphi\circ r(x_0)$. Hence $r(x_0)$ is a fixed point of φ.

Step 3. In this step $X \in ANR$ and $\varphi : X \to X$ is still a compact w-map. We shall need the weight version of the Schauder Approximation Theorem (see [11]):

Lemma 5.1 *Let U be an open subset of a normed space E and let $\psi : X \to U$ be a compact w-map from any metric space into U. Then for every $\varepsilon > 0$ there exists a finite dimensional subspace $E^{n(\varepsilon)}$ of E and a compact w-map $\psi_\varepsilon : X \to U$ such that:*

[1] Let $\psi : X \to Y$ be a map such that $\psi(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the contraction of ψ to the pair (A, B) we understand a map $\psi' : A \to B$ with the same values as ψ. A contraction of ψ to the pair (A, Y) is simply the restriction $\psi|_A$ of ψ to A. Hence, by the contraction of w-map φ we mean a w-map $\varphi' = (\sigma_{\varphi'}, w_{\varphi'})$, where $\sigma_{\varphi'}$ and $w_{\varphi'}$ are the contractions of σ_φ and w_φ, respectively.

[2] See introduction of Section 3.

(5.1.1) *the w-maps ψ_ε, ψ are w-homotopic,*
(5.1.2) $\psi_\varepsilon(X) \subset E^{n(\varepsilon)}$,
(5.1.3) $d_H(\varphi(x), \varphi_\varepsilon(x)) < \varepsilon$ *for every* $x \in X$.[3]

In Step 3 we shall distinguish two cases:

Special case ($X = U$, where U is an open subset of a normed space E.)

Since $\varphi : U \to U$ is a compact w-map, in view of Lema 1 for every n we get a finite dimensional subspace $E^{m_n} \subset E$ and a compact w-map $\varphi_n : U \to U$ such that:

a) $\varphi \sim_w \varphi_n$,
b) $\varphi_n(U) \subset E^{m_n}$,
c) $d_H(\varphi(x), \varphi_n(x)) < \frac{1}{n}$ for every $x \in U$.

We let $U_n = U \cap E^{m_n}$. Now, for every n, we consider the following commutative diagram in which all the arrows represent either the inclusions or the contractions of the compact w-map φ_n:

$$\begin{array}{ccc} U_n & \xrightarrow{i} & U \\ {\scriptstyle \widetilde{\varphi}_n}\downarrow & \overset{\hat{\varphi}_n}{\swarrow} & \downarrow{\scriptstyle \varphi_n} \\ U_n & \xrightarrow{i} & U\,. \end{array}$$

Reasoning as in Step 2 we deduce that $\widetilde{\varphi}_n$ and φ_n are Lefschetz w-maps and $\Lambda(\varphi_n) = \Lambda(\widetilde{\varphi}_n)$. Moreover, we get that φ is a Lefschetz w-map and $\Lambda(\varphi) = \Lambda(\varphi_n)$, because $\varphi \sim_w \varphi_n$. Let assume now that $\Lambda(\varphi) \neq 0$. Then $\Lambda(\widetilde{\varphi}_n) \neq 0$ for every n. Now by applying Step 1, for every n, we get that $\widetilde{\varphi}_n$ has a fixed point and hence φ_n has also a fixed point. So, we have a sequence $\{x_n\}$ such that:

$(*)$ $x_n \in \varphi_n(x_n)$,
$(**)$ $d_H(\varphi(x_n), \varphi_n(x_n)) < \frac{1}{n}$.

From $(*)$ and $(**)$ we infer that there exists the sequence $\{y_n\}$ such that:

(i) $y_n \in \varphi(x_n)$,
(ii) $d(x_n, y_n) = \|x_n - y_n\| \leq d_H(\varphi_n(x_n), \varphi(x_n)) < \frac{1}{n}$.

Since φ is a compact w-map, then we may assume without loss of generality that:

(iii) $\lim_n y_n = x_0 \in U$.

Consequently, from *(ii)* we deduce that:

$(***)$ $\lim_n x_n = x_0$.

Hence, in view of *(i)*, *(iii)*, $(***)$ and the upper semicontinuity of φ, it is not difficult to see that $x_0 \in \varphi(x_0)$.

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[3]Recall that if (X, d) is a metric space, $\varepsilon > 0$ and $B \subset X$, nonempty, bounded and closed, then the Hausdorff distance between A and B is defined by:

$$d_H(A, B) := \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\},$$

where dist is the distance of the point from the set.

General case ($X \in ANR$ and $\varphi \in \mathbb{K}_w(X)$)

Since $X \in ANR$, then there exist the continuous maps $r : U \to X$ and $s : X \to U$ such that $r \circ s = id_X$, where U is some open subset of some normed space E. Let us consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{s} & U \\ {\scriptstyle \varphi}\Big\downarrow & \overset{\varphi \circ r}{\swarrow} & \Big\downarrow{\scriptstyle s \circ \varphi \circ r} \\ X & \xrightarrow{s} & U . \end{array}$$

Now, reasoning as in the second step, we obtain the desired conclusion. This completes proof. □

Remark In the above proof we have adopted to the case of w-maps the method due to A. Granas for single valued maps (see [8]).

Now, we are going to generalize the above result. First we recall the necessary notions and facts. Let (X, A) be pair of spaces and let $\mathcal{H}$ be the Darbo homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$. For a pair (X, A) let us consider the graded vector space $\mathcal{H}_*(X, A) = \{\mathcal{H}_n(X, A)\}_{n \geq 0}$. A w-map $\varphi : (X, A) \to (X, A)$ is called a Lefschetz w-map provided $\varphi_* = \{\varphi_n : \mathcal{H}_n(X, A) \to \mathcal{H}_n(X, A)\}$ is a Leray endomorphism. For such φ we can define the Lefschetz number $\Lambda(\varphi)$ of φ by putting $\Lambda(\varphi) = \Lambda(\varphi_*)$. In the following definition we define a class of w-maps, which is definitely larger than a class of compact w-maps, moreover, for which the Lefschetz Fixed Point Theorem remain true (see [5] or [6]).

Definition 5.6 A w-map $\varphi : X \to X$ is said to be a *compact absorbing contraction* if there exists an open subset U of X such that $\overline{\varphi(U)}$ is a compact subset of U and $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U)$.

We shall use notation $\varphi \in \mathbb{CAC}_w(X)$.

Note 5.1 If $\varphi \in \mathbb{CAC}_w(X)$ and U satisfies the above conditions and K is a compact subset of X, then there exists $n \in N$ such that $\varphi^n(K) \subset U$.

Given a w-map $\varphi : (X, A) \to (X, A)$. Let us denote by $\varphi_X : X \to X$ and $\varphi_A : A \to A$ the evident contractions of φ. From Property 4.6 we obtain:

Proposition 5.3 *Let $\varphi : (X, A) \to (X, A)$ be a w-map. If any two of φ, φ_A and φ_X are Lefschetz w-map, then so is the third and, moreover,*

$$\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_A).$$

From Note 5.1 and the fact of applying the Darbo Homology functor $\mathcal{H}$ with compact carriers and coefficients in $\mathbb{Q}$ we obtain:

Proposition 5.4 *If $\varphi : (X, A) \to (X, A)$ is a w-map and a subset A of X satisfies conditions of Definition 5.6, then φ_* is weakly nilpotent.*

After these preliminaries we are able to formulate and prove the following (see [5]):

Theorem 2 (The Lefschetz Fixed Point Theorem) *Let $X \in ANR$ and $\varphi \in \mathbb{CAC}_w(X)$. Then:*

1. φ is a Lefschetz w-map;

2. $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.

Proof Let U be an open subset of X satisfying all properties of Definition 5.6 and let $\overline{\varphi} : U \to U$ be the contraction $\varphi|_U$ of φ to U. Let us consider also $\widetilde{\varphi} : (X, U) \to (X, U)$, where $\widetilde{\varphi}(x) = \varphi(x)$ for every $x \in X$. Then in view of Proposition 5.4 and Property 4.8 we obtain that $\widetilde{\varphi}$ is a Lefschetz w-map and $\Lambda(\widetilde{\varphi}) = 0$. Since $\overline{\varphi}$ is a compact w-map and $U \in ANR$ (by Proposition 5.2), then from Theorem 1 we get that $\overline{\varphi}$ is a Lefschetz w-map. Hence, from Proposition 5.3 we deduce that φ is a Lefschetz w-map and $\Lambda(\varphi) = \Lambda(\overline{\varphi})$. Let us assume now that $\Lambda(\varphi) \neq 0$. Since $\Lambda(\overline{\varphi}) \neq 0$, then by applying once again Theorem 1 we have that $\overline{\varphi}$ has a fixed point and hence we get that φ has a fixed point. □

Let us observe that if $\varphi : X \to Y$ is a w-map and X is connected then $\sum_{y \in Y} w_\varphi(x, y)$ does not depend on $x \in X$ (see [10]). So, for X connected it makes sense to speak of the index of the w-map φ, $I(\varphi) = \sum_{y \in Y} w_\varphi(x, y)$, which is well defined.

Corollary *Let X be an acyclic ANR (i.e. $\mathcal{H}_0(X) \approx \mathbb{Q}$ and $\mathcal{H}_n(X) = 0$ for every $n \geq 1$) or, in particular, a convex subset of a normed space and let $\varphi : X \to X$ be a w-map with $I(\varphi) \neq 0$. Then:*

1. if $\varphi \in \mathbb{K}_w(X)$, then φ has a fixed point;

2. if $\varphi \in \mathbb{CAC}_w(X)$, then φ has a fixed point.

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