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Approximation from the Exterior of Carathéodory Multifunctions *

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Abstract

We approximate a globally measurable multifunction $F(t,x)$ which takes compact values in an euclidean space by means of a decreasing sequence of globally measurable multifunctions $F_n(t,x)$ which are locally lipschitzian with respect to x, in the following cases: $F(t,\cdot)$ is upper semicontinuous and takes connected values, or $F(t,\cdot)$ is continuous.

Key words: Multifunction, approximation, lipschitzian.

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1 Introduction

The results on the approximation from the exterior of a globally measurable multifunction $F(t,x)$ which is upper semicontinuous with respect to x were mainly developed in the case in which F takes convex values [6,9,10,12–14], because of their applications in the qualitative study of differential inclusions [7,8-§7,11,12]. Indeed, if the approximating multifunctions are locally lipschitzian with respect to x, the solution set corresponding to the given orientor field can be shown to be acyclic, under suitable assumptions.

When the sets $F(t,x)$ are not convex, such a kind of approximation is not always possible, and it could be interesting to find some general conditions under which $F(t,x)$ can be approximated in the required way, i.e. by a decreasing

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sequence of measurable multifunctions $F_n(t, x)$ which are locally lipschitzian with respect to x. As is shown in [1], such an approximation can be useful again in the study of the topological properties of the solution set of a differential inclusion.

Our work starts from [4], where we considered the case in which $F(t, x) = F(x)$ and the sets $F(x)$ are compact, connected subsets of a euclidean space. In that context, we gave a positive answer to the above question, thanks to a suitable interpolation among subsets of $\mathbf{R}^p$ which applies, in particular, to those sets which are metric expansions of compact, connected sets.

In this paper we extend the quoted interpolation in two directions: first, in §2, we show that our technique can be fitted in order to deal with a wider class of sets, which includes the metric expansions of compact subsets of $\mathbf{R}^p$. Then, in §4, we prove that our interpolation preserves measurability in the case in which the given sets depend on a variable t which ranges over a measurable space T. Finally, in §5, we state and prove our main result, Theorem 5.1: the given multifunction $F(t, x)$ is defined on a measurable subset of $T \times X$, where X is a complete, separable metric space, and takes values in a euclidean space. Furthermore, three alternative assumptions are presented. In particular, in the case (i) we get the "parametrized" version of the main result of [4], which is exploited in [1] in order to show that the solution set of the corresponding differential inclusion $x' \in F(t, x)$ can be expressed as the (possibly empty) intersection of absolute retracts. Furthermore, in the cases (ii) or (iii), a new kind of result is given: if the sets $F(t, x)$ are simply supposed to be compact, but $F(t, x)$ is *continuous* with respect to x, the required approximation is again possible.

2 Interpolation among subsets of $\mathbf{R}^p$

Definition 2.1 If (Y, d) is a metric space, and C is a non-empty subset of Y, for every $x \in Y$ we denote by $\delta(x; C) = \inf_{y \in C} d(x, y)$ the *distance of* x from C. For every $\sigma > 0$, $B(x; \sigma)$ will stand for the closed ball with center at x and radius σ. If C, D are non-empty subsets of Y, the *excess of* C *over* D is defined by $e(C, D) = \sup_{x \in C} \delta(x; D)$, while $h(C, D) = \max(e(C, D), e(D, C))$ stands for the *Hausdorff distance* between C and D. We recall that h is actually a distance on the family of all non-empty, closed and bounded subsets of Y. If $C, D \subseteq Y$ and $\sigma > 0$, we also introduce the following notations:

$$C^\sigma = \cup\{B(x; \sigma)^\circ; \, x \in C\}, \quad C_\sigma = Y \setminus (Y \setminus C)^\sigma,$$
$$D \subset\subset C \Leftrightarrow D \subseteq C_\sigma \quad \text{for some } \sigma > 0. \tag{2.1}$$

We remark that, for every $\sigma > 0$, it is $\emptyset^\sigma = \emptyset$ and $Y_\sigma = Y$. Throughout this section, Y will be the space $\mathbf{R}^p$, endowed with its euclidean distance d, and B its closed unit ball. Now we are going to recall some notions from [3]. First of all we introduce the family $\mathcal{P}$ of all lipschitzian paths $\gamma : [0, 1] \to \mathbf{R}^p$: for every $\gamma \in \mathcal{P}$, we put $S(\gamma) = \gamma([0, 1])$, and denote by $l(\gamma)$ the *length* of γ. If $\gamma(0) = x$

and $\gamma(1) = y$, we say that γ *joins* x *and* y. If $E \subseteq \mathbf{R}^p$, we denote by $\mathcal{P}(E)$ the family of all paths $\gamma \in \mathcal{P}$ such that $S(\gamma) \subseteq E$.

Definition 2.2 The set E is said to be *Lipschitz-connected* [3, Def. 2.2] if for every $x, y \in E$ there exists a path $\gamma \in \mathcal{P}(E)$ which joins x and y; E is said to be *arcwise bounded* if there exists a constant $L > 0$ such that any two points x and $y \in E$ can be joined by a path $\gamma \in \mathcal{P}(E)$ with length $l(\gamma) \leq L$; the greatest lower bound of these constants L will be denoted by $\rho(E)$. A path $\eta \in \mathcal{P}(E)$ is said to be *minimal in* E if, for every path $\gamma \in \mathcal{P}(E)$ which joins $\eta(0)$ and $\eta(1)$, it is $l(\eta) \leq l(\gamma)$.

We denote by $\mathcal{A}$ the family of all closed, arcwise bounded subsets of $\mathbf{R}^p$. If $E \in \mathcal{A}$, for every $x, y \in E$ we put

$$d_E(x, y) = \min\{l(\gamma);\ \gamma \in \mathcal{P}(E),\ \gamma(0) = x,\ \gamma(1) = y\}.$$

This minimum surely exists, and d_E is a distance on E, which will be called the *intrinsic distance on* E (see Prop. 2.5 of [3]). Of course, it is always $d_E \geq d_{|E\times E}$, and the equality holds if and only if E is convex. Given a closed set $E \subseteq \mathbf{R}^p$, we denote by $\mathcal{K}(E)$ the family of all compact, non-empty subsets of E. We remark that $\mathcal{P}(E)$ is never empty (unless $E = \emptyset$), since it contains all constant paths $\gamma \equiv x$, with $x \in E$. Then, for all $C \in \mathcal{K}(E)$, $a \geq 0$, we put

$$Q^a(C; E) = \cup\{S(\gamma);\ \gamma \in \mathcal{P}(E),\ \gamma(0) \in C,\ l(\gamma) \leq a\}. \tag{2.2}$$

If $E \in \mathcal{A}$, we point out that $Q^a(C; E) = E$ as soon as $a \geq \rho(E)$. Furthermore, in any case, $C \subseteq Q^a(C; E) \subseteq C + aB$. If $C = \{x\}$ we shall write $Q^a(x; E)$. If $E \in \mathcal{A}$, it is easy to see that $Q^a(x; E) = \{y \in E : d_E(x, y) \leq a\}$. We also define in E an *intrinsic Hausdorff distance*, in the following way: for every C, $D \in \mathcal{K}(E)$ we put

$$h_E(C, D) = \min\{a \geq 0 : Q^a(C; E) \supseteq D,\ Q^a(D; E) \supseteq C\}.$$

Since $d_E \geq d_{|E\times E}$, it is easy to argue that the same relation holds between h_E and the restriction of h to $\mathcal{K}(E) \times \mathcal{K}(E)$. Furthermore, for all C, $D \in \mathcal{K}(E)$, $a, b \geq 0$,

$$h_E(Q^a(C; E),\ Q^b(D; E)) \leq h_E(C, D) + |a - b|. \tag{2.3}$$

Property (2.3) is shown in Theorem 2.10 of [4] in the case in which C and D are compact and connected and $E \in \mathcal{A}$: however, it is easy to see that the proof works even if C and D are only compact, and E is closed. Now we are going to recall a procedure given in [4], which relies on the transformation (2.2), and allows to "interpolate", under suitable conditions, connected subsets of $\mathbf{R}^p$. To this end we put forward some notions which were introduced in [4].

Definition 2.3 Let $\mathcal{F}$ be a family of non-empty subsets of $\mathbf{R}^p$, $\mathcal{Z}$ a set of indexes, $\mathcal{I}$ the family of all its finite subsets. For every $I \in \mathcal{I}$, let $\mathcal{F}_0(I)$ be a given set of mappings $I \to \mathcal{F}$ (let us say the *admissible families* of $\mathcal{F}^I$), $\Lambda(I)$ the set of all mappings $\lambda : I \to [0,1]$ such that $\sum_{i\in I} \lambda_i = 1$. A *wheighed operator* on $\mathcal{F}$ is a mapping $\Omega : \mathcal{D}(\mathcal{F}) \to \mathcal{F}$, where $\mathcal{D}(\mathcal{F}) = \cup_{I\in\mathcal{I}} (\mathcal{F}_0(I) \times \Lambda(I))$. We say that $\mathcal{D}(\mathcal{F})$ is the *domain* of Ω, or also the set of the *weighed families* Ψ which are *admissible* for Ω in $\mathcal{F}$.

When $\Psi \in \mathcal{D}(\mathcal{F})$ we shall write $\Psi = (D_i, \lambda_i)_{i\in I}$, $\Omega\Psi = \Omega_{i\in I}(D_i, \lambda_i)$. If $\lambda_i = 0$ on a set $J \subseteq I$ we shall also adopt, with an obvious meaning, the following notations: $(D_i, \lambda_i)_{i\notin J}$, $\Omega_{i\notin J}(D_i, \lambda_i)$.

Definition 2.4 A weighed operator $\Omega : \mathcal{D}(\mathcal{F}) \to \mathcal{F}$ is said to be a *stable mean operator* on $\mathcal{F}$ if, for every $I \in \mathcal{I}$ and every admissible family $(D_i)_{i\in I}$, the following properties hold:

(Ω_1) $\Omega_{i\in I}(D_i; \lambda_i) = \Omega_{i\neq j}(D_i; \lambda_i)$ *whenever* $\lambda_j = 0$;

(Ω_2) $\Omega_{i\in I}(D_i; \lambda_i) = D_j$ *whenever* $\lambda_j = 1$;

(Ω_3) $h(\Omega_{i\in I}(D_i; \lambda_i), \Omega_{i\in I}(D_i; \nu_i)) \leq L \max_{i\in I} |\lambda_i - \nu_i|$, *for any* $\lambda, \nu \in \Lambda(I)$, *where* L *is a positive constant which depends on the sets* D_i;

(Ω_4) *for every* $\lambda \in \Lambda(I)$ it is $\cap_{i\in I} D_i \subseteq \Omega_{i\in I}(D_i; \lambda_i) \subseteq \cup_{i\in I} D_i$.

If $\mathcal{F} = \mathcal{A}$, a family $(D_i)_{i\in I}$ will be called *admissible* if $D_i \cap D_j \neq \emptyset$, $i, j \in I$. Then $\mathcal{D}(\mathcal{A})$ is made up by all wheighed families $(D_i, \lambda_i)_{i\in I}$ such that the previous relation holds. In [4, Theorem 3.3] we showed the existence of a stable mean operator $\Omega : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$: the aim of this section is to extend its domain to more general families of sets.

Definition 2.5 Let $\emptyset \neq E \subseteq \mathbf{R}^p$: we denote by $\mathcal{H}(E)$ the family of all connected components of E; we also put $\mathcal{H}(\emptyset) = \emptyset$. If $C \subseteq D \subseteq \mathbf{R}^p$, the script $C \preceq D$ will mean that, for every $H \in \mathcal{H}(D)$, there exists $K \in \mathcal{H}(C)$ such that $K \subseteq H$. We shall write $C \div D$ if and only if, for every $K \in \mathcal{H}(C)$, $H \in \mathcal{H}(D)$, it is $K \cap D \neq \emptyset$, $C \cap H \neq \emptyset$. We also denote by $\mathcal{C}(E)$ the family of all compact subsets C of E such that $C \preceq E$.

We point out that the relation $\emptyset \preceq D$ can never hold, unless $D = \emptyset$. Furthermore, the following properties can be easily checked:

Proposition 2.6 *Let* $C, D, E \subseteq \mathbf{R}^p$. *If* $C \preceq D$ *and* $D \preceq E$, *then* $C \preceq E$. *If* $C \preceq D$ *and* $C \preceq E$, *then* $D \div E$.

We denote by $\mathcal{S}$ the family of all non-empty sets $E \subseteq \mathbf{R}^n$ such that $\mathcal{H}(E)$ is a finite subset of $\mathcal{A}$. If $E \in \mathcal{S}$, $\mathcal{H}(E) = \{K_1, \ldots, K_n\}$ and $C, D \preceq E$ we put $\rho(E) = \max_i \rho_i$ and $h_E(C, D) = \max_i h_i$, where, for every $i \in \{1, \ldots, n\}$, $\rho_i = \rho(K_i)$ and $h_i = h_{K_i}(C \cap K_i, D \cap K_i)$. It is easy to check that $h(C, D) \leq h_E(C, D)$ whenever $C, D \preceq E$, and

$$Q^a(C; E) \cap K_i = Q^a(C \cap K_i; K_i), \quad i = 1, \ldots, n, \tag{2.5}$$

$$C \preceq Q^a(C; E) \preceq E. \tag{2.6}$$

Theorem 2.7 *For every $C, D \in \mathcal{C}(E)$, $a, b \geq 0$, inequality (2.3) holds.*

Proof Let $C, D \preceq E$, $\mathcal{H}(E) = \{K_1, ..., K_n\}$ and, for every $i \in \{1, ..., n\}$, $h_i = h_{K_i}$, $\mu_i \doteq h_i(Q^a(C \cap K_i; K_i), Q^a(D \cap K_i; K_i)) = h_i(Q^a(C; E) \cap K_i, Q^b(D; E) \cap K_i)$, where the second equality follows from (2.5). We already know that (2.3) holds when E, C and D are respectively replaced by K_i, $C \cap K_i$, and $D \cap K_i$, so that $\mu_i \leq h_i(C \cap K_i, D \cap K_i) + |a - b|$. Now we only need to take the maximum over $i \in \{1, ..., n\}$, so as to get (2.3). □

Definition 2.8 We say that a family $(D_i)_{i \in I}$ in $\mathcal{S}$ is *admissible* if $D_i \div D_j$ for any $i, j \in I$. According to Def. 2.3, for every $I \in \mathcal{I}$ we denote by $\mathcal{S}_0(I)$ the set of all admissible families $(D_i)_{i \in I}$ in $\mathcal{S}$, and put $\mathcal{D}(\mathcal{S}) = \cup_{I \in \mathcal{I}} (\mathcal{S}_0(I) \times \Lambda(I))$.

Remark 2.9 Of course, in the particular case in which $D_i \in \mathcal{A}$ for every $i \in I$, Def. 2.8 is equivalent to the one we gave for the family $\mathcal{A}$. In any case, thanks to Prop. 2.6, a simple criterium in order that the given weighed family is admissible is the following one: There exists $C \subseteq \mathbf{R}^p$ such that $C \preceq D_i$ for every $i \in I$.

Theorem 2.10 *There exists a stable mean operator $\Omega : \mathcal{D}(\mathcal{S}) \to \mathcal{S}$.*

Proof We proceed by induction on the number $n = |I|$, as in the proof of Theorem 3.3 of [4]. If $n = 1$, $\mathcal{D}(\mathcal{S})$ is nothing but $\mathcal{S} \times \{1\}$, and Ω can be defined as the map $(D, 1) \mapsto D$. Now, let us suppose $n > 1$, take an admissible, wheighed family $\Psi = (D_i, \lambda_i)_{i \in I}$, with $|I| = n$, and put $E = \cup_{i \in I} D_i$. We are going to define $\Omega(\Psi)$ under the inductive assumption that Ω is already defined on families whose number of sets is less than n. Then properties (Ω_1)–(Ω_4) can be shown by induction. The inductive procedure, however, works with some additional properties: more precisely, we replace (Ω_3) and (Ω_4) with the following, stronger conditions:

$$
\begin{aligned}
& h_E(\Omega_{i \in I}(D_i; \lambda_i), \Omega_{i \in I}(D_i; \nu_i) \leq L \max_{i \in I} |\lambda_i - \nu_i|, \quad \lambda, \nu \in \Lambda(I), \\
& \cap_{i \in I} D_i \preceq \Omega_{i \in I}(D_i; \lambda_i) \preceq \cup_{i \in I} D_i, \quad \lambda \in \Lambda(I).
\end{aligned} \tag{2.9}
$$

So, let us put $\lambda_* = \min_{i \in I} \lambda_i$. If $2n\lambda_* \geq 1$, we define $\Omega_{i \in I}(D_i; \lambda_i) = E$. Otherwise, let J be the set of those indexes $i \in I$ such that $\lambda_i > \lambda_*$, and define $\mu = (\mu_i)_{i \in J}$ through the relations $(1 - n\lambda_*)\mu_i = \lambda_i - \lambda_*$, $i \in J$. Since $\mu \in \Lambda(J)$, and $|J| < n$, it is right to suppose that the set $C = \Omega_{i \in J}(D_i, \mu_i)$ is already defined, and satisfies the condition $\cap_{i \in J} D_i \preceq C \preceq \cup_{i \in J} D_i$. On the other hand, thanks to (2.9), we get $\cup_{i \in J} D_i \preceq E$, so that $C \preceq E$, and now we can define $\Omega_{i \in I}(D_i; \lambda_i) = Q^a(C; E)$, where $a = 2n\lambda_* \rho(E)$. Thanks to (2.6), property (2.9) holds by construction. As regards the other properties, we refer to the proof of Theorem 3.3 of [4], which can now easily be fitted to the present context. □

The next result puts in evidence two important classes of sets which are respectively contained in $\mathcal{S}$ and $\mathcal{A}$.

Proposition 2.11 *Let $C \subseteq \mathbf{R}^p$ compact, $\varepsilon > 0$, $K = C + \varepsilon B$. Then:*

(a) $K \in \mathcal{S}$; *(b) if C is also connected, $K \in \mathcal{A}$.*

Proof (a) Let us put $\sigma = \varepsilon/2$, and pick up points $x_1, \ldots, x_n \in K$ such that $\cup_{i=1}^n B(x_i; \sigma) \supseteq C$. For every $i \in \{1, \ldots, n\}$ let C_i be the connected component of $U \doteq C + \sigma B$ which contains x_i. Then $C_i \supseteq B(x_i; \sigma)$, so that $\cup_{i=1}^n C_i \supseteq U$, and there exists $J \subseteq \{1, \ldots, n\}$ such that $\mathcal{H}(U) = \{C_i;\, i \in J\}$. On the other hand, the sets $C_i + \sigma B$ $(i \in J)$ lie in $\mathcal{A}$ by virtue of (b), and their union gives $U + \sigma B = K$. Now it is enough to point out that the union of a finite number of sets of the family $\mathcal{A}$ (or more generally of $\mathcal{S}$) is in $\mathcal{S}$: to this end, we only need to consider Remark 2.7 of [4] and apply an easy inductive argument.

(b) See [3, Prop. 2.6]. □

Now we are going to show how the relation $\preceq$ we introduced before is preserved by continuous transformations. To this end, we recall some notions about multifunctions, which will be useful also in the next sections. Let X and Y two given sets, 2^Y the family of all subsets of Y: a mapping Φ from X to 2^Y will be called *a multifunction*, and simply denoted by $\Phi : X \rightrightarrows Y$. Whenever $A \subseteq Y$, we denote by $\Phi^{-1}(A)$ the subset of X where $\Phi(x) \cap A \neq \emptyset$.

Definition 2.12 If X and Y are topological spaces, a multifunction $\Phi : X \rightrightarrows Y$ is said to be *lower semicontinuous* if $\Phi^{-1}(A)$ is open whenever $A \subseteq Y$ is open, *upper semicontinuous* if $\Phi^{-1}(A)$ is closed whenever $A \subseteq Y$ is closed. If both properties hold together, we say that Φ is *continuous*.

Proposition 2.13 *Let X, Y be two topological spaces, $\Phi : X \rightrightarrows Y$ be a continuous multifunction, $U, V \subseteq X$. Let us adopt the same notations as in Def. 2.5, which make sense in any topological space, and suppose that $U \preceq V$. Then $\Phi(U) \preceq \Phi(V)$.*

Proof Let H be a connected component of $\Phi(V)$, and take an open set A and a closed set C which both contain H, but do not meet $\Phi(V) \setminus H$. Since the sets $\Phi^{-1}(H)$, $\Phi^{-1}(A)$, $\Phi^{-1}(C)$ have the same intersection with V and Φ is continuous, the set $\Phi^{-1}(H) \cap V$ is open and closed in V, so that it can be expressed as the union of some connected components of V. Let K be one of these components: since $U \preceq V$, we can find $Q \in \mathcal{H}(U)$ such that $Q \subseteq K$. Hence $\Phi(Q) \subseteq H$: since $\Phi(U) \subseteq \Phi(V)$, this actually means that the connected component of $\Phi(U)$ which contains the connected set $\Phi(Q)$ is contained in H. □

3 Preliminary results about measurable multifunctions

In the next section we are going to extend the interpolation given in Theorem 2.10 to the case in which the sets D_i depend on a parameter $t \in T$, where T is a measurable space. To this end, we need many "technical" results about measurable multifunctions, which will be useful also in §5. We refer to [5] for a lot of them, while the results at the end of the section which concern the behaviour of the connected components of a multifunction are less usual. From now on we suppose that $(T, \mathcal{L})$ is a measurable space, while (Y, d) is a metric space. We say that a multifunction $\Phi : T \rightrightarrows Y$ is *measurable* if $\Phi^{-1}(A) \in \mathcal{L}$ whenever $A \subseteq Y$ is open. The *domain* of Φ is the set of those $t \in T$ such that $\Phi(t) \neq \emptyset$, while its *graph* $\Gamma(\Phi)$ is made up by those pairs $(t, x) \in T \times Y$ such that $x \in \Phi(t)$. We also adopt the notations introduced in (2.1). From now on we suppose that the space Y is separable. We also recall the definition of a *complete* σ-field $\mathcal{L}$, given in [5]: in practice, $\mathcal{L}$ fulfils this condition whenever it is the domain of a complete, σ-additive measure μ.

Proposition 3.1 *Let $\Phi : T \rightrightarrows Y$ be measurable, $\sigma > 0$. Then the following multifunctions are measurable:*

(a) $\Phi' : t \mapsto X \setminus \Phi(t)$ (if Φ takes closed values or open values),

(b) $\bar{\Phi} : t \mapsto \overline{\Phi(t)}$, (c) $\Phi^\sigma : t \mapsto \Phi(t)^\sigma$, (d) $\Phi_\sigma : t \mapsto \Phi(t)_\sigma$.

Proof a) Let Φ take closed values, $A \subseteq Y$ be open: then $(\Phi')^{-1}(A) = T \setminus S$, where S is the subset of T where $\Phi(t) \supseteq A$. On the other hand, from Prop 3.2d (in which we put $\Psi(t) \equiv A$) we get $S \in \mathcal{L}$, so that Φ' is measurable. If Φ takes open values, its enough to express A as the union of a sequence of sets $A_n \subset\subset A$. Then Prop. 3.2e ensures that, for every $n \in \mathbf{Z}^+$, $(\Phi')^{-1}(A_n) \in \mathcal{L}$, so that the set $(\Phi')^{-1}(A) = \cup_n (\Phi')^{-1}(A_n)$ is measurable as well.

b)-c) It is enough to notice that, whenever A is open, $\bar{\Phi}^{-1}(A) = \Phi^{-1}(A)$ and $(\Phi^\sigma)^{-1}(A) = \Phi^{-1}(A^\sigma)$.

(d) If we apply (a), then property (c) to Φ', and again (a) to $(\Phi')^\sigma$, the assertion follows at once. □

Proposition 3.2 *Let Φ, $\Psi : T \rightrightarrows Y$ be measurable multifunctions. Then, with an obvious meaning of the notations, the following multifunctions are measurable*

(a) $\Phi \cup \Psi$; (b) $\Phi \cap \Psi$ (if $\mathcal{L}$ is complete);

(c) $\Phi \setminus \Psi$ (if Φ and Ψ take respectively open and closed values, or conversely).

Furthermore, let $L, M, N \subseteq T$ the sets where, respectively, $\Phi(t) \supseteq \Psi(t)$, $\Phi(t) \supset\supset \Psi(t)$, $\Phi(t) \cap \Psi(t) \neq \emptyset$. Then:

(d) if Φ takes closed values, $L \in \mathcal{L}$; (e) if Φ takes open values, $M \in \mathcal{L}$;

(f) if Φ takes open values, or Φ and Ψ take respectively closed and compact values, $N \in \mathcal{L}$.

Proof a) It is enough to notice that $(\Phi \cup \Psi)^{-1}(A) = \Phi^{-1}(A) \cup \Psi^{-1}(A)$.

b) Thanks to Theorem III.20 of [5], Φ and Ψ are measurable if and ony if $\Gamma(\Phi), \Gamma(\Psi) \in \mathcal{L} \otimes \mathcal{B}$. Now its enough to notice that $\Gamma(\Phi \cap \Psi) = \Gamma(\Phi) \cap \Gamma(\Psi)$ and apply again Theorem III.20 of [5].

c) Since $\Phi \setminus \Psi = (\Phi' \cup \Psi)'$, the assertion follows from Prop. 3.1a and 3.2a.

d) Let Λ be a countable, dense subset of Y, $\mathcal{V}$ be the family of all closed balls $V = B(x; \sigma)$, with $x \in \Lambda$, $\sigma \in \mathbf{Q}^+$. We claim that L agrees with the intersection

$\hat{L}$ of all sets of the kind $(T\setminus\Psi^{-1}(V))\cup\Phi^{-1}(V)$, where V ranges over $\mathcal{F}$. Indeed, let $t\in L$, $V\in\mathcal{V}$: if $\Psi(t)\cap V\neq\emptyset$, even more so it is $\Phi(t)\cap V\neq\emptyset$, because $\Psi(t)\subseteq\Phi(t)$: then $t\in\hat{L}$. Conversely, let us suppose that $t\in\hat{L}$, take $x\in\Psi(t)$ and a sequence of balls $V_n=B(\xi_n;\sigma_n)\in\mathcal{V}$, with $n\in\mathbf{Z}^+$, such that $\sigma_n\to 0$ as $n\to+\infty$ and, for every $n\in\mathbf{Z}^+$, $x\in V_n$: then, for every $n\in\mathbf{Z}^+$, $\Psi(t)\cap V_n\neq\emptyset$, so that $\Phi(t)\cap V_n\neq\emptyset$, and we can find a sequence of points $x_n\in\Phi(t)$ which converge to x as $n\to+\infty$. Since $\Phi(t)$ is closed, it must be $x\in\Phi(t)$, so that $t\in L$. Hence $L=\hat{L}\in\mathcal{L}$.

e) Let $\sigma_n\downarrow 0$ and, for every $n\in\mathbf{Z}^+$, consider the multifunction $\Phi_n: t\mapsto\Phi(t)_{\sigma_n}$: thanks to the previous point, the set $M_n\subseteq T$ where $\Phi_n(t)\supseteq\Psi(t)$ is measurable. Now it is enough to remark that $M=\cup_n M_n$.

f) Since $t\in T\setminus N$ if and only if $\Psi(t)\subseteq\Phi'(t)$, we only need to apply Prop. 3.1a and 3.2d. □

Now, let X be another metric space, denote by $\mathcal{B}$ its Borel σ-field and consider on $T\times X$ the product σ-field $\mathcal{E}=\mathcal{L}\otimes\mathcal{B}$. A multifunction $F:T\times X\rightrightarrows Y$ is said to be *globally measurable* if $F^{-1}(A)\in\mathcal{E}$ whenever A is open. We also say that $F(t,x)$ is *measurable with respect to* t if, for every $x\in X$, the multifunction $F(\cdot,x)$ is measurable, and that $F(t,x)$ is *continuous with respect to* x if, for every $t\in T$, the multifunction $F(t,\cdot)$ is continuous.

Proposition 3.3 *Let $F:T\times X\rightrightarrows Y$ be a multifunction which is measurable with respect to $t\in T$ and continuous with respect to $x\in X$. Then F is globally measurable.*

Proof Let $A\subseteq Y$ be open, $\Lambda=\{x_i;\ i\in\mathbf{Z}^+\}$ a dense subset of X. The assertion holds thanks to the following relation, which can be deduced from the continuity of $F(t,x)$ with respect to x: $F^{-1}(A)=\cap_j\cup_i\left(B(x_i;1/j)\times F(\cdot,x_i)^{-1}(A)\right)$. □

Remark 3.4 It is easy to check that, whenever $F:T\times X\rightrightarrows Y$ is globally measurable and $x:T\to Y$ is measurable, the multifunction $t\mapsto F(t,x(t))$ is measurable as well. The following result shows how this property can be generalized.

Proposition 3.5 *Let $\mathcal{L}$ and Y be complete, $E,W:T\rightrightarrows X$ two measurable multifunctions, and denote by the same scripts also their graphs. Let $W\subseteq E$, and consider a globally measurable multifunction $\Phi:E\rightrightarrows Y$. For every $t\in T$ let us denote respectively by $\Phi_W^+(t)$, $\Phi_W^-(t)$ the union and the intersection of all sets of the kind $\Phi(t,x)$, where x ranges over $W(t)$. Then:*

(a) Φ_W^+ is measurable; *(b) if Φ takes closed values, Φ_W^- is measurable.*

Proof a) Let $A\subseteq Y$ be open, $H\doteq(\Phi_W^+)^{-1}(A)$, π be the projection $(t,x)\mapsto x$. Then $H=\pi(W\cap\Phi^{-1}(A))$, and Theorem III.23 of [5] entails that $H\in\mathcal{L}$.

b) By De Morgan's laws, $\Phi_W^-=((\Phi')_W^+)'$. Now, thanks to Prop. 3.1a, Φ' is measurable, so that such is $\Psi=(\Phi'_W)^+$. On the other hand, Ψ takes open values, so that Prop 3.1a ensures again that $\Phi_W^-=\Psi'$ is measurable. □

Proposition 3.6 *Let $\mathcal{L}$ and Y be complete, $E \in \mathcal{L} \times \mathcal{B}$, Z a topological space. Let $F : E \rightrightarrows Y$, $\Gamma : T \times Z \rightrightarrows X$ be two globally measurable multifunctions. Let us suppose that, for every $t \in T$, $z \in Z$, $\Gamma(t,z) \subseteq E(t)$, and define $G : T \times Z \rightrightarrows Y$ by $G(t,z) = F(t, \Gamma(t,z))$. Then G is globally measurable in each one of the two following cases:*

(a) Γ takes open values.

(b) Γ takes compact values, $E(\cdot)$ takes closed values, and $F(t,x)$ is upper semicontinuous with respect to x.

Proof Let $\mathcal{T}$ be the Borel σ-field of Z, $\mathcal{D} = \mathcal{L} \otimes \mathcal{T}$. Thanks to Theorem III.2 of [5] it is enough to prove that $G^{-1}(U) \in \mathcal{D}$ whenever $U \subseteq Y$ is closed. So, let $U \subseteq Y$ be closed, and regard the set $F^{-1}(U)$ as the graph of a multifunction $\theta : T \rightrightarrows X$, with possibly empty values. Thanks to Theorem III.30 of [5], θ is measurable. Furthermore, $G^{-1}(U)$ can be expressed as the set of those pairs $(t,z) \in T \times Z$ such that $\theta(t) \cap \Gamma(t,z) \neq \emptyset$. We also remark that, in the case (b), θ takes closed values. Hence, in both cases, we can apply Prop. 3.2f, in which we replace $(T, \mathcal{L})$ by $(T \times Z, \mathcal{D})$ and put $\Phi(t,z) = \theta(t)$, $\Psi = \Gamma$, so as to conclude that $G^{-1}(U) \in \mathcal{D}$. □

From now on we denote by $\mathcal{U}$ a countable basis of open, connected subset for the topology of X, and by $\mathcal{F}$ the family of all connected subsets of Y which can be expressed as union of a finite number of elements of $\mathcal{U}$. Whenever $x \in C \subseteq Y$, the script $\Gamma(x;C)$ will stand for the connected component of C which contains x, while $\mathcal{F}(x;C)$ will be the family of those sets $K \in \mathcal{F}$ such that $x \in K \subset\subset C$.

Lemma 3.7 *Let $C \subseteq X$ be compact, $H \subseteq X$ be closed and, for every $n \in \mathbf{Z}^+$, K_n a closed set such that $K_{n+1} \subseteq K_n \subseteq C^{\sigma_n}$, where $\sigma_n \downarrow 0$. Then H meets all sets K_n if and only if it meets their intersection K.*

Proof Of course, whenever $H \cap K \neq \emptyset$, we get $H \cap K_n \neq \emptyset$ for all $n \in \mathbf{Z}^+$. Conversely, for every $n \in \mathbf{Z}^+$ let $x_n \in H \cap K_n$, and $y_n \in C$ such that $d(y_n, x_n) \leq \sigma_n$. Since C is compact, we can find a cluster point $y \in C$ for the sequence $(y_n)_n$, which enjoys this property also with respect to $(x_n)_n$, because $d(y_n, x_n) \to 0$ as $n \to +\infty$. Now it is easy to check that $x \in H \cap K$. □

Lemma 3.8 *Let $C \subseteq Y$, $x \in C$, and, for every $n \in \mathbf{Z}^+$, $C_n = C^{1/n}$.*

(a) If C is open, $\Gamma(x;C) = \cup \mathcal{F}(x;C)$.

(b) If C is compact $\Gamma(x;C) = \cap_n \Gamma(x;C_n)$.

Proof a) of course, the set $W = \cup \mathcal{F}(x;C)$ is contained in $\Gamma(x;C)$. In order to prove that $W = \Gamma(x;C)$, it will be enough to prove that W is open and closed in $\Gamma(x;C)$. To this end, let $y \in W$: then there exists $K \in \mathcal{F}(x;C)$ such that $y \in W$. Now, since C is open, $\Gamma(x;C)$ is open in turn, and we can find $V \in \mathcal{U}$ such that $y \in V \subset\subset \Gamma(x;C)$: now it is easy to check that $K \cup V \in \mathcal{F}(x;C)$, so that $V \subseteq W$, and actually W is open in $\Gamma(x;C)$. Now, let $(x_n)_n$ be a sequence in W which converges to a point $y \in \Gamma(x; C)$. Let $V \in \mathcal{U}$ be such that $y \in V \subset\subset \Gamma(x; C)$, and take $n \in \mathbf{Z}^+$ such that $x_n \in V$: since $x_n \in W$, there exists $K_n \in \mathcal{F}(x; C)$ such that $x_n \in K_n$. Hence $y \in K_n \cup V \doteq H_n \in \mathcal{F}(x; C)$, so that $y \in W$, and we can conclude that W is closed as well.

b) Let W be the right-hand side of the equality to be proved. Then, obviously, $\Gamma(x; C) \subseteq W$. Now, for every $n \in \mathbf{Z}^+$, let K_n be the closure of $\Gamma(x; C_n)$, $K = \cap_n K_n$. Then $W \subseteq K \subseteq \cap_n C_n = C$. Now, let us prove that K is connected. To this end, let U and V be two open disjoint sets which both meet K. Then, for every n, these sets cannot cover the connected set K_n, so that $K_n \cap H \neq \emptyset$, where $H = X \setminus (U \cup V)$: now Lemma 3.7 ensures that $K \cap H \neq \emptyset$. We just showed that, whenever two open, disjoint sets both meet the set K, they cannot cover it. Hence K is connected. But $x \in K \subseteq C$, so that actually $K \subseteq \Gamma(x; C)$, and even more so, $W \subseteq \Gamma(x; C)$. □

Proposition 3.9 *Let $E : T \rightrightarrows X$ be a measurable multifunction. Whenever $t \in T$, $x \in E(t)$ let us put $\Gamma(t, x) = \Gamma(x; E(t))$, otherwise $\Gamma(t, x) = \emptyset$. Then the multifunction $\Gamma : T \times X \rightrightarrows X$ is globally measurable whenever its values are of the following kind:*

(a) open; (b) compact.

Proof a) Let us order $\mathcal{F}$ in a sequence $(K_i)_i$. By virtue of Prop. 3.2e the set $S_i \subseteq T$ where $K_i \subset\subset E(t)$ is measurable. Now, let $x \in X$, $A \subseteq X$ be open, J the set of those indexes $i \in \mathbf{Z}^+$ such that $A \cap K_i \neq \emptyset$: then it is enough to remark that $\Gamma^{-1}(A) = \cup_{i \in J}(S_i \times K_i)$, by virtue of Lemma 3.8a.

b) Thanks to Theorem III.2 of [5] it is enough to show that $\Gamma^{-1}(U) \in \mathcal{L} \otimes \mathcal{B}$ whenever U is closed. For every $n \in \mathbf{Z}^+$, $t \in T$, $x \in X$ let $E_n(t) = E(t)^{1/n}$, $H_n(t, x)$ be the closure of $\Gamma(x; E_n(t))$. Now, thanks to the previous point and Prop. 3.1b, the multifunctions H_n are globally measurable, so that Theorem III.2 of [5] ensures that $H_n^{-1}(U) \in \mathcal{L} \otimes \mathcal{B}$. On the other hand, from Lemma 3.8b we argue that $\Gamma(t, x) = \cap_n H_n(t, x)$, so that Lemma 3.7 entails that $H^{-1}(U) = \cap_n H_n^{-1}(U) \in \mathcal{L} \otimes \mathcal{B}$. □

Proposition 3.10 *Let $\Phi, K : T \rightrightarrows Y$ be two measurable multifunctions which take respectively closed and compact values. Let us suppose that the values of K have non-empty interior and, for every $t \in T$, define $H(t)$ as the union of all the connected components of $K(t)$ which meet $\Phi(t)$. Then the multifunction H is measurable*

Proof Let $A \subseteq Y$ be open, $\Lambda = \{y_i; i \in \mathbf{Z}^+\}$ be a countable, dense subset of Y. For every $i \in \mathbf{Z}^+$ let us put $\Gamma_i(t) = \Gamma(y_i; K(t))$ and denote by S_i the set of those $t \in T$ such that $\Phi(t) \cap \Gamma_i(t) \neq \emptyset$. Thanks to Prop. 3.9b, Γ_i is measurable, so that Prop. 3.2f ensures that $S_i \in \mathcal{L}$. Now it is enough to notice that $H^{-1}(A) = \cup_i (S_i \cap \Gamma_i^{-1}(A))$. □

4 Interpolation among measurable multifunctions

The aim of this section is to prove that the interpolation given by Theorem 2.10 preserves measurability. To this end we recall Def. 2.3 and give the following one.

Definition 4.1 Let $\Omega : \mathcal{D}(\mathcal{F}) \to \mathcal{F}$ be a weighed operator on $\mathcal{F}$, $I \in \mathcal{I}$. We say that a family of multifunctions $(\Phi_i)_{i\in I}$ from a set T into $\mathcal{F}$ is *admissible* for Ω if, for every $t \in T$, the family $(\Phi_i(t))_{i\in I}$ is admissible. We say that Ω is *universally measurable* if, for every $I \in \mathcal{I}$, $\lambda \in \Lambda(I)$, and every admissible family $(\Phi_i)_{i\in I}$ of measurable multifunctions from a measurable space $(T, \mathcal{L})$ into $\mathcal{F}$, the multifunction $t \mapsto \Omega_{i\in I}(\Phi(t); \lambda_i)$ turns out to be measurable.

In order to prove that the operator Ω we built in §2 is universally measurable we put forward some technical results. The first one concerns the family of paths $\mathcal{P}$ which was introduced before Def. 2.2. Throughout this section, the script B_n will stand for $B(0; 1/n)^\circ$.

Lemma 4.2 *There exists a countable family $\mathcal{Q} \subseteq \mathcal{P}$ with the following property: for every $\eta \in \mathcal{P}$, $\varepsilon > 0$, there exists $\gamma \in \mathcal{Q}$ such that $l(\gamma) \leq l(\eta) + \varepsilon$ and, for every $r \in [0,1]$, $d(\gamma(r), \eta(r)) \leq \varepsilon$.*

Proof It is enough to take $\mathcal{Q}$ as the family of the poligonal paths γ which enjoy the following property: there exists a finite subset $\Delta = \{r_0, r_1, \ldots, r_n\}$ of $[0,1] \cap \boldsymbol{Q}$ (where $r_0 = 0$, $r_n = 1$, $r_0 < r_1 < \ldots < r_n$) such that $\gamma(\Delta) \subseteq \boldsymbol{Q}^p$ and γ' is constant on the interior of each interval of the corresponding subdivision. □

Lemma 4.3 *Let $E \subseteq \mathbf{R}^p$ be closed, C, $U \subseteq \mathbf{R}^p$ compact, $a \geq 0$, $\mathcal{Q}$ the family of the previous lemma. For every $n \in \mathbf{Z}^+$ let us put $E_n = E + B_n$. Then the three following conditions are equivalent:*

(a) $Q^a(C; E) \cap U \neq \emptyset$;

(b) for every $n \in \mathbf{Z}^+$ there exists a path $\gamma_n \in \mathcal{Q} \cap \mathcal{P}(E_n)$ such that

$$(\gamma(0) + B_n) \cap C \neq \emptyset, \quad (\gamma(1) + B_n) \cap U \neq \emptyset, \quad l(\gamma) \leq a + 1/n; \tag{4.1}$$

(c) for every $n \in \mathbf{Z}^+$ there exists a path $\gamma_n \in \mathcal{P}(E_n)$ such that (4.1) holds.

Proof $(a) \Rightarrow (b)$. There exists a path $\eta \in \mathcal{P}(E)$ such that $\eta(0) \in C$, $\eta(1) \in U$, $l(\eta) \leq a$: then it is enough to apply Lemma 4.2 with $\varepsilon = 1/n$, $n \in \mathbf{Z}^+$.

$(b) \Rightarrow (c)$: obvious.

$(c) \Rightarrow (a)$. Let $\gamma_n \in \mathcal{P}(E_n)$ be paths such that, for all $n \in \mathbf{Z}^+$, conditions (4.1) hold. Since C and U are compact, it is right to suppose, up to a subsequence, that $\gamma_n(0) \to x$ and $\gamma_n(1) \to y$ as $n \to +\infty$, where $x \in C$, $y \in U$. So, let $k \in \mathbf{Z}^+$ and apply Prop. 2.4 of [3] with $E = E_k$: then we find a path $\gamma \in \mathcal{P}(E_k)$ with length $l(\gamma) \leq a$ wich joins x and y. Hence, for all $k \in \mathbf{Z}^+$, $S(\gamma) \subseteq E_k$, so that actually $S(\gamma) \subseteq E$, because E is closed, and (a) is fulfilled. □

Proposition 4.4 *Let $C : T \to \mathcal{K}$, $E \to \mathcal{S}$ be two measurable multifunctions such that, for every $t \in T$, $C(t) \subseteq E(t)$. Then, for every $a \geq 0$, the multifunction $t \mapsto M_a(t) = Q^a(C(t); E(t))$ is measurable.*

Proof Let $a \geq 0$, $U \subseteq \mathbf{R}^p$ be compact. For every $n \in \mathbf{Z}^+$ let us put $E_n(t) = E(t) + B_n$. For every $\gamma \in \mathcal{P}$, let $A_n(\gamma)$ be the set of those $t \in T$ such that $E_n(t) \supseteq S(\gamma)$. Thanks to Prop. 3.2d, in which we put $\Phi(t) = E_n(t)$, $\Psi(t) \equiv S(\gamma)$, we get $A_n(\gamma) \in \mathcal{L}$. Since C is a measurable multifunction, the sets $T_n(\gamma) = A_n(\gamma) \cap C^{-1}(\gamma(0) + B_n^\circ)$ are measurable as well. Now it is enough to notice that, by virtue of the equivalence between (a) and (b) in Lemma 4.3, $M_a^{-1}(U) = \cap_n S_n$, where $S_n = \cup\{T_n(\gamma); \gamma \in \mathcal{Q}_n^a(U)\}$ and $\mathcal{Q}_n^a(U)$ is the countable family made up by those paths $\gamma \in \mathcal{Q}$ such that $(\gamma(1) + B_n) \cap U \neq \emptyset$, $l(\gamma) \leq a + 1/n$. Then $M_a^{-1}(U) \in \mathcal{L}$ whenever U is compact. On the other hand, if U is open, we get $M_a^{-1}(U) = \cup_n M_a^{-1}(U_n)$, where $(U_n)_n$ is any sequence of compact sets whose union gives U. Hence, M_a is a measurable multifunction. □

Lemma 4.5 *Let $E \subseteq \mathbf{R}^p$, $a \geq 0$, Λ be a dense subset of $\mathbf{R}^p$. For every $n \in \mathbf{Z}^+$ let us put $E_n = E + B_n$. Then the following conditions are equivalent:*

- *(a) $\rho(E) \leq a$.*
- *(b) there exists $\nu \in \mathbf{Z}^+$ such that, for all $n \geq \nu$ and $\xi \in \Lambda \cap E_n$ it is $Q^{a_n}(\xi; E_n) \supseteq \Gamma(\xi; E_n)$, where $a_n = a + 1/n$.*
- *(c) for every $x \in E$ it is $Q^a(x; E) \supseteq \Gamma(x; E)$.*

Proof *(a)* $\Rightarrow$ *(b)*. Since $E \in \mathcal{S}$, we can find $\nu \in \mathbf{Z}^+$ such that, whenever $n \geq \nu$ and K, $H \in \mathcal{H}(E)$, it is $(K + B_n) \cap (H + B_n) = \emptyset$. Now, let $n \geq \nu$, $\xi \in \Lambda \cap E_n$, $z \in \Gamma_n \doteq \Gamma(\xi; E_n)$. Then we can find x, $y \in E$ such that $d(x, \xi) \leq 1/2n$, $d(y, z) \leq 1/2n$. Since $n \geq \nu$, we get indeed $y \in \Gamma(x; E)$, so that, thanks to (a), we find a path $\eta \in \mathcal{P}(E)$, with length $l(\eta) \leq a$, which joins x and y. Now it is easy to build a path $\gamma \in \mathcal{P}(E_n)$ which joins ξ and z, whose length does not exceed a_n: to this end, it is enough to connect η with the two segments ξx, yz. Then $z \in Q^{a_n}(\xi; E_n)$.

(b) $\Rightarrow$ *(c)* Let $x \in E$, $y \in \Gamma(x; E)$. For every $n \geq \nu$ let us take $x_n \in \Lambda \cap (x + B_n)$, $y_n \in \Lambda \cap (y + B_n)$. Then there exists a path $\gamma_n \in \mathcal{P}(E_n)$ such that $l(\gamma_n) \leq a_n$, $\gamma_n(0) = x_n$, $\gamma_n(1) = y_n$. In particular, (4.1) holds, where we put $C = \{x\}$, $U = \{y\}$. Thanks to the implication *(b)* $\Rightarrow$ *(a)* of Lemma 4.3, we get $y \in Q^a(x; E)$.

(c) $\Rightarrow$ *(a)* Let $x \in E$, $y \in \Gamma(x; E)$: then $y \in Q^a(x; E)$, so that we can find a path $\gamma \in \mathcal{P}(E)$ which joins x and y, whose length does not exceed a. Since x and y are arbitrary, $\rho(A) \leq a$. □

Proposition 4.6 *Let $E : T \to \mathcal{S}$ be a measurable multifunction, and put, for every $t \in T$, $\beta(t) = \rho(E(t))$. Then β is measurable.*

Proof For every $n \in \mathbf{Z}^+$ let us put $E_n(t) = E(t) + B_n$. Let us also define Γ_n as Γ in Prop. 3.9, with E_n in place of E. Thanks to that result and Prop. 4.4, for every $\xi \in \Lambda \doteq \boldsymbol{Q}^p$, $a \geq 0$, the multifunctions $\Psi(t) = \Gamma_n(t, \xi)$, $\Phi(t) = Q^a(\xi, E_n(t))$ are measurable. Then Prop. 3.2d ensures that the set $A_n(\xi) \subseteq T$ where $Q^a(\xi, E_n(t)) \supseteq \Gamma_n(t, \xi)$ is measurable. Now it is enough to point out that, thanks to Lemma 4.5, $\beta^{-1}(]-\infty, a])$ can be expressed as $\cup_\nu \cap \{A_n(\xi);\, n \geq \nu, \xi \in \mathbf{Q}^p\}$. □

Theorem 4.7 *The mean operator $\Omega : \mathcal{D}(\mathcal{S}) \to \mathcal{S}$ given by Theorem 2.10 is universally measurable.*

Proof We proceed by induction on the number $n = |I|$, as in the proof of Theorem 2.10. If $n = 1$, an admissible family of multifunctions is nothing but a single multifunction Φ, while $\Lambda(I) = \{1\}$: then the mapping $t \mapsto \Omega(\Phi(t), 1) = \Phi(t)$ is obviously measurable. Now, let us suppose $n > 1$, take an admissible family of multifunctions $(\Phi_i)_{i \in I}$ and put, for shortness, $\Psi(t, \lambda) = \Omega_{i \in I}(\Phi_i(t); \lambda_i)$. Let $E(t) = \cup_{i \in I} \Phi_i(t)$, $\lambda \in \Lambda(I)$, $\lambda_* = \min_{i \in I} \lambda_i$. If $2n\lambda_* \geq 1$, we get $\Psi(t, \lambda) = E(t)$. Then, in this case, the mapping $\Psi(\cdot, \lambda)$ is surely mesurable, thanks to Prop. 3.2a. Otherwise, let us put $K(t) = \cup_{i \in J} \Phi_i(t)$ and $C(t) = \Omega_{i \in J}(\Phi_i(t), \mu_i)$, where J is the set of those indexes $i \in I$ such that $\lambda_i > \lambda_*$, while the coefficients μ_i are built as in the proof of Theorem 2.10. Since $|J| < n$, and we argue by induction, it is right to suppose that $C(\cdot)$ is measurable. Then we put, for every $a \geq 0$, $Z(t, a) = Q^a(C(t); E(t))$. Thanks to Prop. 4.4, the multifunction $t \mapsto Z(t, a)$ is mesurable. Furthermore, if in (2.3) we put $C = D = K(t)$, $E = E(t)$, and remember that $h_{|C(E) \times C(E)} \leq h_E$, we get the following Lipschitz property with respect to the Hausdorff distance: for all $t \in T$, $a, b \geq 0$, $h(Z(t, a), Z(t, b)) \leq |a - b|$. In particular, the multifunction $a \mapsto Z(t, a)$ is continuous. Since $Y = \mathbf{R}^p$ is separable, Prop. 3.3 ensures that Z is measurable with respect to the product of $\mathcal{L}$ with the Borel σ-field on $[0, +\infty[$. On the other hand, $\Omega_{i \in I}(\Phi_i(t), \lambda_i) = Z(t, \alpha(t))$, where the function $\alpha(t) = 2n\lambda_* \rho(E(t))$ is measurable, thanks to Prop. 4.6. Hence Remark 3.4 entails that $\Psi(\cdot, \lambda)$ is measurable also in the case $2n\lambda_* \leq 1$, and the proof is complete. □

Proposition 4.8 *Let I be a finite set of indices and, according to Def. 4.1, $(\Gamma_i)_{i \in I}$ an admissible family of measurable multifunctions from T to $\mathbf{R}^p$. Let $\lambda = (\lambda_i)_{i \in I}$ be a measurable function from T to $\Lambda(I)$ and put, for every $t \in T$, $\Phi(t) = \Omega_{i \in I}(\Gamma_i(t); \lambda_i(t))$. Then the multifunction $\Phi : T \rightrightarrows \mathbf{R}^p$ is measurable.*

Proof For every $t \in T$, $\lambda \in \Lambda(I)$ let us put $\Psi(t, \lambda) = \Omega_{i \in I}(\Gamma_i(t); \lambda_i)$. Thanks to Theorem 4.7 and property (Ω_3) in Def. 2.4, the multifunction $\Psi : T \times \Lambda(I) \rightrightarrows \mathbf{R}^p$ is measurable with respect to t and continuous with respect to λ, so that Prop. 3.3 entails that it is globally measurable. Since $\Phi(t) = \Psi(t, \lambda(t))$, from Remark 3.4 we easily argue that Φ is measurable. □

5 Approximation from the Exterior of Carathéodory Multifunctions.

We recall Def. 2.10 and some properties which are related to lower and upper semicontinuity. Let X, Y be metric spaces, $\Phi : X \rightrightarrows Y$ a multifunction. An easy consequence of the given definition is that Φ is upper semicontinuous if and only if, whenever $A \subseteq Y$ is open, the set of those points $x \in X$ such that $\Phi(x) \subseteq A$ is open. On the other hand, a necessary condition for the lower semicontinuity of Φ (which becomes also sufficient if Φ takes compact values) is the following one: for every compact subset of X and every $\sigma > 0$ the set of those points $x \in X$ such that $\Phi(x)^\sigma \supseteq K$ is open (see, for instance, Prop. 2.1 of [8]). From these arguments we easily infer that, whenever its values are compact (and non-empty), Φ is continuous if and only if it is continuous as a mapping from X into the family of all compact, non-empty subset of Y, in which we consider the Hausdorff distance of Def. 2.1: in other words, if and only if $h(\Phi(\xi), \Phi(x)) \to 0$ whenever $\xi \to x$. According to this setting, we say that Φ is *lipschitzian* if there exists a positive constant L such that, for every $\xi, x \in X$ it is $h(\Phi(\xi), \Phi(x)) \leq L\rho(\xi, x)$, where ρ is the distance on X. Then we say that F is *locally lipschitzian* if, for every point $\bar{x} \in X$, there exist a neighbourhood U of $\bar{x}$ and a positive constant $\bar{L}$ such that, for every $\xi, x \in U$ it is $h(\Phi(\xi), \Phi(x)) \leq \bar{L}\rho(\xi, x)$.

Now, let $(T, \mathcal{L})$ be a measurable space, $\mathcal{B}$ the Borel σ-field on X. From now on, whenever we consider a set $E \in \mathcal{L} \otimes \mathcal{B}$, we agree to identify it with the multifunction $E : T \rightrightarrows X$ whose graph is E. According to this convention, for every $t \in T$ we denote by $E(t)$ the set of those points $x \in X$ such that $(t, x) \in E$, while, for every $x \in X$, the script $E^{-1}(x)$ will stand for the set of those elements $t \in T$ such that $(t, x) \in E$. If $E \in \mathcal{L} \otimes \mathcal{B}$, we say that a multifunction $F : E \rightrightarrows Y$ is respectively *lower or upper semicontinuous, continuous or locally lipschitzian with respect to* x according to whether, for every $t \in T$, the multifunction $F(t, \cdot)$ enjoys the corresponding property on the set $E(t)$. Now we are going to present some alternative assumptions for our main result, which will be stated below.

(i) $F(t, x)$ is upper semicontinuous with respect to x, and takes connected values.

(ii) $F(t, x)$ is continuous with respect to x, and the sets $E(t)$ are open.

(iii) $F(t, x)$ is continuous with respect to x, X is locally compact and the sets $E(t)$ are closed and locally connected.

Theorem 5.1 *Let $(T, \mathcal{L})$ be a complete, measurable space, (X, ρ) a complete separable metric space, $E \in \mathcal{L} \otimes \mathcal{B}$, $F : E \rightrightarrows \mathbf{R}^p$ a globally measurable multifunction which takes compact, non-empty values. Let us suppose that (i), (ii) or (iii) holds. Then there exist globally measurable multifunctions $F_n : E \rightrightarrows \mathbf{R}^p$ ($n = 1, 2, \ldots$) which take compact values, are locally lipschitzean with respect to x and enjoy the following properties, for every pair $(t, x) \in E$:*

$$F(t, x) \subseteq F_{n+1}(t, x) \subseteq F_n(t, x), \quad n \in \mathbf{Z}^+,$$
$$h(F_n(t, x), F(t, x)) \to 0 \quad as \quad n \to +\infty.$$

Furthermore, in the case (i), the values of F_n are also connected.

The main tool for proving Theorem 5.1 will be Prop. 5.5, which can be seen as a parametrized version of Lemma 4.1 of [4], at least as far as condition (i) is concerned. We put forward some technical results about refinements and partitions of unity, because we need to build them in such a way to preserve measurability.

Lemma 5.2 *For every $i \in \mathbf{Z}^+$ let $A_i : T \rightrightarrows X$ be a measurable multifunction with open values. Then there exist measurable multifunctions $C_i : T \rightrightarrows X$ such that the family $(C_i(t))_i$ is a locally finite refinement of the previous one, that is to say: for every $t \in T$, $x \in X$ there exists a neighbourhood U of x which meets at most a finite number of sets $C_i(t)$, and for every $i \in \mathbf{Z}^+$, $t \in T$, it is $C_i(t) \subseteq A_i(t)$. Furthermore, for every $t \in T$, $\cup_i C_i(t) = \cup_i A_i(t)$.*

Proof Let $\sigma_i \downarrow 0$, $t \in T$. We are going to define by induction the sequence $(C_i(t))_i$ by putting $C_1(t) = A_1(t)$ and $C_i(t) = A_i(t) \setminus (\cup_{k=1}^{i-1} C_k(t))_{\sigma_i}$. Thanks to Propositions 3.2a,c and 3.1d, it is easy to prove by induction that the multifunctions C_i are measurable. Furthermore, for every $t \in T$, the sets $C_i(t)$ are obviously open, and cover the same set as the given ones: as regards the last statement, it is enough to remark that $C_i(t) \supseteq A_i(t) \setminus (\cup_{k=1}^{i-1} A_k(t))$. In order to prove that the family $(C_i(t))_i$ is locally finite, we consider the sets $V_{ij}(t) = (\cup_{k=1}^{i} C_k(t))_{\sigma_j}$, $i, j \in \mathbf{Z}^+$, and take $x \in X$: then we can find $i \in \mathbf{Z}^+$, $j \geq i$ such that $x \in C_i(t)$ and $B(x; 2\sigma_j) \subseteq C_i(t)$, so that $B(x;\sigma_j) \subseteq V_{ij}(t) \subseteq V_{jj}(t)$. Since $V_{jj}(t) \subseteq V_{hh}(t)$ for every $h \geq j$, and $V_{hh} \cap C_{h+1}(t) = \emptyset$, we actually get $B(x;\sigma_j) \cap C_h(t) = \emptyset$ for all $h > j$. □

Remark 5.3 It is easy to prove that, whenever $(C_i)_i$ is a locally finite sequence of open subsets of X and $\rho_i \downarrow 0$, the family $(C_i^{\rho_i})_i$ is again locally finite.

Remark 5.4 Let $(T, \mathcal{L})$ be a complete measurable space, and consider a sequence $(D_i)_i$ of measurable multifunctions from T to X which take open values. Let us suppose that, for every $t \in T$, the family $\mathcal{D}(t) \doteq (D_i(t))_i$ is locally finite, and denote by $D(t)$ its union. Then, for every $t \in T$, we can associate to $\mathcal{D}(t)$ a partition of unity is such a way to preserve measurability with respect to t. More precisely, if A is the graph of $D(\cdot)$, for every $i \in \mathbf{Z}^+$ we can define a globally measurable function $\lambda_i : A \to [0,1]$ which is locally lipschitzian with respect to x, and vanishes outside the graph of D_i, in such a way that, for every $(t,x) \in A$ it is $\sum_i \lambda_i(t,x) = 1$, where in the previous sum all but a finite number of terms are zero. To this end, it is enough to put $\lambda_i = \mu_i/\mu$, where $\mu_i(t,x)$ is the distance of x from $X \setminus D_i(t)$, while $\mu = \sum_i \mu_i > 0$ on A. Indeed, it is known that the functions λ_i are locally lipschitzian with respect to x (see, for instance, Theorem 0.2 of [2]). On the other hand, for every $i \in \mathbf{Z}^+$, $r > 0$, it is $\mu_i(t,x) \geq r$ if and only if (t,x) lies in the graph $D_{i,r}$ of the multifunction $t \mapsto D_i(t)_r$, which is measurable thanks to Prop. 3.1d. Since $\mathcal{L}$ is complete, Theorem III.30 of [5] ensures that $D_{i,r} \in \mathcal{L} \otimes \mathcal{B}$. Hence the functions μ_i are globally measurable, so that the functions λ_i enjoy the same property.

Proposition 5.5 *Let $E \in \mathcal{L} \times \mathcal{B}$, $F, H : E \rightrightarrows \mathbf{R}^p$ be two globally measurable multifunctions which take respectively compact and closed values. Let us suppose that (i), (ii) or (iii) holds, that H is lower semicontinuous with respect to x and, for every $(t,x) \in E$, $G(t,x) \subseteq H(t,x)$. Let $r > 0$ be given. Then there exists a globally measurable multifunction $\Phi : E \rightrightarrows \mathbf{R}^p$ which takes compact values, is locally lipschitzian with respect to x and fulfils the following inclusions, for every $(t,x) \in E$:*

$$F(t,x) \subseteq \Phi(t,x) \subseteq H(t,x) + rB, \tag{5.1}$$

$$\Phi(t,x) \subseteq F(t, B(x;r) \cap E(t)) + rB. \tag{5.2}$$

In the case (i), the values of Φ are also connected.

Proof For the sake of shortness, we shall denote by $(E, \mathcal{E})$ the measurable space in which $\mathcal{E}$ is the restriction to E of $\mathcal{L} \otimes \mathcal{B}$. Let $r > 0$ be given, $\sigma = r/2$: for every $(t,x) \in E$, let $F^\sigma(t,x)$ be the set $F(t,x) + \sigma B^\circ$, and denote by $\bar{F}^\sigma(t,x)$ its closure. Let $V(t,x)$ be the set of those points $\xi \in E(t)$ such that

$$F(t,\xi) \subseteq F^\sigma(t,x) \subseteq \bar{F}^\sigma(t,x) \subseteq H(t,\xi) + rB \tag{5.3}$$

and, for every $U \subseteq X$, $M(U)$ the set of those pairs $(t,x) \in E$ such that $U \cap E(t) \subseteq V(t,x)$. We claim that $M(U) \in \mathcal{L} \otimes \mathcal{B}$. To this end, let us denote by $M^+(U)$ and $M^-(U)$ the subsets of E where, respectively, the first and the third inclusion in (5.3) holds for every $\xi \in W(t) \doteq U \cap E(t)$. According to Prop. 3.5, let us consider the two measurable multifunctions $F^+ = F^+_W$, $H^- = H^-_W$. Let us replace in Prop. 3.2d the measure space $(T, \mathcal{L})$ with $(E, \mathcal{E})$, Y with X, and apply it to the multifunctions $\Phi(t,x) = F^\sigma(t,x)$ and $\Psi(t,x) = F^+(t)$: then we argue that $M^+(U) \in \mathcal{L} \times \mathcal{B}$. In a similar way, we can apply the same lemma to the multifunctions $\Psi(t,x) = \bar{F}^\sigma(t,x)$ and $\Phi(t,x) = H^-(t)$ and infer that $M^-(U) \in \mathcal{L} \times \mathcal{B}$ as well: now it is enough to remark that $M(U) = M^+(U) \cap M^-(U)$.

Now, let us consider a countable, dense subset Λ of X, and order $\Lambda \times \mathbf{Z}^+$ in a sequence (x_i, k_i), $i \in \mathbf{Z}^+$. Then, for every $i \in \mathbf{Z}^+$, let $0 < \rho_i \leq \sigma/3k_i$, $V_i = B(x_i, \rho_i)^\circ$, $U_i = B(x_i, 3\rho_i)^\circ$. For every $i \in \mathbf{Z}^+$, let us put $T_i = \pi(\Sigma_i)$, where $\Sigma_i = (T \times V_i) \cap M(U_i)$ and π is the projection $(t,x) \mapsto t$. Since $\Sigma_i \in \mathcal{L} \times \mathcal{B}$ and $\mathcal{L}$ is complete, Theorem III.23 of [5] ensures that $T_i \in \mathcal{L}$. Now, let $A_i : T \rightrightarrows X$ be the multifunction which agrees with V_i on $S_i \doteq T_i \setminus T_{i-1}$ (where we put $T_0 = \emptyset$). Of course, $A_i : T \rightrightarrows X$ is measurable and takes open values. Furthermore, for every $t \in T$, the family $(A_i(t))_i$ covers $E(t)$. Indeed, let $x \in E(t)$: since $x \in V(t,x)$ and the arguments at the beginning of this section show that $V(t,x)$ is open in $E(t)$, there exists $i \in \mathbf{Z}^+$ such that $x \in V_i$ and $U_i \cap E(t) \subseteq V(t,x)$. If i is the first of such indeces we get $t \in S_i$, so that $A_i(t) = V_i$, and actually $x \in A_i(t)$.

Now we can apply Lemma 5.2, and build new multifunctions C_i as there. Then we put $D_i(t) = C_i(t)^{\rho_i}$, $E_i(t) = D_i(t) \cap E(t)$. We recall that $C_i(t) \subseteq A_i(t) \subseteq V_i$, so that $\bar{D}_i(t) \subseteq \bar{V}_i^{\rho_i} \subseteq U_i$. Furthermore, thanks to Remark 5.3, the family $(D_i(t))_i$ is locally finite. Now, let us build the functions λ_i as in Remark 5.4 and, for every $i \in \mathbf{Z}^+$, $(t,x) \in E$, let us define $I(t,x)$ and $J(t,x)$ as the sets of those indeces $i \in \mathbf{Z}^+$ such that, respectively, $x \in D_i(t)$, $x \in \overline{C_i(t)}$. Now, for every $i \in I$, let us regard Σ_i as the graph of a multifunction $Z_i : T_i \rightrightarrows X$ with non-empty values. Thanks to Theorem III.22 of [5], Z_i admits a measurable selection $x_i : T_i \to X$. Then we put, for every $t \in S_i$, $K_i(t) = \bar{F}^\sigma(t, x_i(t))$ and point out that the following inclusions holds, whenever $t \in S_i$, $\xi \in E_i(t)$:

$$F(t,\xi) \subseteq K_i(t) \subseteq H(t,\xi) + rB. \tag{5.4}$$

Indeed, by construction, $(t, x_i(t)) \in \Sigma_i$, so that (5.3) holds with $x = x_i(t)$ and $\xi \in U_i$: since $E_i(t) \subseteq U_i$, even more so it holds on $E_i(t)$, and (5.4) follows at once. Now, if (i) holds let us put $H_i(t,x) = K_i(t)$. Otherwise, let us denote by $\Gamma_i(t,x)$ the set $\Gamma(x; E_i(t))$ or the set $\Gamma(x; \hat{E}_i(t))$ according to whether (ii) or (iii) holds, where $\hat{E}_i(t)$ is the intersection between $E(t)$ and the closure of $C_i(t)^{\rho_i/2}$. Then, in the cases (ii) and (iii), we define $H_i(t,x)$ as the union of all the connected component of $K_i(t)$ which contain the set $\Psi_i(t,x) = \bar{F}^\sigma(t, \Gamma_i(t,x))$. Now it is easy to check that, whatever condition holds among (i), (ii), and (iii),

$$F(t,\xi) \preceq H_i(t,\xi) \subseteq H(t,\xi) + rB, \tag{5.5}$$

$$H_i(t,\xi) \subseteq F(t, B(\xi; r) \cap E(t)) + rB, \tag{5.6}$$

where the script $\preceq$ is explained in Def. 2.5. Indeed, in the case (i) the set $H_i(t,\xi) = K_i(t)$ is connected, so that to the first relation in (5.5) is equivalent to an inclusion, namely the first inclusion in (5.4). On the other hand, when (ii) or (iii) holds, it is $\Psi_i(t,\xi) \preceq H_i(t,\xi)$ by construction. But $\Gamma_i(t,\xi)$ is a connected set which contains ξ, so that Prop 2.13, in which we put $\Phi = \bar{F}^\sigma(t,\cdot)$, $U = \{\xi\}$, $V = \Gamma_i(t,\xi)$, entails that $F(t,\xi) \preceq \Psi_i(t,\xi)$. Now, the first relation of (5.5) follows from Prop. 2.5. The second inclusion in (5.5) follows obviously from the second inclusion in (5.4), since $H_i(t,\xi) \subseteq K_i(t)$. As regards (5.6), it is enough to remark that, by construction, $H_i(t,\xi) \subseteq F(t, E_i(t)) + rB$, where $E_i(t) \subseteq U_i$, and the diameter of U_i does not exceed $2\sigma = r$. We also point out that, by virtue of Prop. 2.11, for every $(t,x) \in E$, $i \in I(t,x)$, it is $K_i(t) \in \mathcal{A}$ in the case (i), while $K_i(t) \in \mathcal{S}$ if (ii) or (iii) holds. In any case, $H_i(t,x) \in \mathcal{S}$. Now, thanks to Remark 2.9, it is easy to chek that, for every $(t,x) \in E$, the wheighed family

$$\Psi(t,x) = \{(H_i(t,x), \lambda_i(t,x));\ i \in I(t,x)\}, \tag{5.7}$$

is admissible. Now we are going to show that, for every $(t,x) \in E$, there exists a neighbourhood U of x such that, for every $\xi \in U \cap E(t)$,

$$\Omega_{i\in I(t,\xi)}(H_i(t,\xi), \lambda_i(t,\xi)) = \Omega_{i\in I(t,x)}(H_i(t,x), \lambda_i(t,\xi)). \tag{5.8}$$

To this end, we proceed in three different ways, according to whether (i), (ii) or (iii) is satisfied.

(i) We first remark that, whenever $J(t,x) \subseteq J \subseteq I(t,x)$, the family $\Psi_J(t,x)$ which turns out from (5.7) when J replaces $I(t,x)$ is again admissible, and such that $\Omega\,\Psi_J(t,x) = \Omega\,\Psi(t,x)$. Now, let $V \subseteq \cap\{D_i(t);\ i \in I(t,x)\}$ be an open neighbourhood of x which meets at most a finite number of sets $D_i(t)$, and let us withdraw from V all sets $\bar{C}_j(t)$ which meet it but do not contain x. Since the number of these closed sets is finite, the set U which is left is again an open neighbourhood of x, and for every $\xi \in U$ it is $J(t,\xi) \subseteq J(t,x)$, $I(t,x) \subseteq I(t,\xi)$. Now (5.8) follows easily, since in this case the sets $H_i(t,x)$ actually do not depend on x.

(ii) Let us call now W the neighbourhood U of x which was build in the previous case, and denote by U the connected component of $W \cap E(t)$ which contains x, so that, for every $\xi \in U$, $i \in I(t,\xi)$, it is $\Gamma_i(t,\xi) = \Gamma_i(t,x)$. Then, of course, $H_i(t,\xi) = H_i(t,x)$, and (5.8) holds for every $\xi \in U = U \cap E(t)$.

(iii) Let us build W as U was built in the case (i), but replacing $D_i(t)$ with $\hat{D}_i(t) = C_i(t)^{\rho_i/2}$. Now, let V be a neighbourhood of x such that $V \cap E(t)$ is connected, $\hat{W}$ the connected component of W which contains x, $U = V \cap \hat{W}$. Then $U \cap E(t)$ is connected, and contained in all sets $\hat{E}_i(t)$ $(i \in I(t,x))$. Then, for every $\xi \in U \cap E(t)$, we get again $\Gamma_i(t,\xi) = \Gamma_i(t,x)$, so that (5.8) holds.

Now we are going to show that the multifunction $\Phi(t,x) = \Omega\,\Psi(t,x)$ fulfils the required properties. In order to prove that Φ is globally measurable, we first claim that the multifunctions H_i are globally measurable. Indeed, thanks to Remark 3.4 and Prop. 3.1c,d, for every $i \in \mathbf{Z}^+$ K_i is measurable, so that, in the case (i), the assertion follows at once. If (ii) or (iii) holds, we point out that Γ_i is measurable, thanks to Prop. 3.9: in particular, as regards the case (iii), it is right to suppose that, for every $t \in T$, $\hat{E}_i(t)$ is compact, since it is a closed subset of U_i, and U_i can be chosen in such a way that $\bar{U}_i$ is compact. Hence Prop. 3.6 ensures that Ψ_i is measurable, and from Prop. 3.10 we argue that H_i is measurable as well. Now, for every $i \in \mathbf{Z}^+$, let M_i be the set of those pairs $(t,x) \in E$ such that $x \in D_i(t)$. From Prop. 3.2e, in which we consider the measure space $(E,\mathcal{E})$ and put $\Phi(t,x) = D_i(t)$, $\Psi(t,x) = \{x\}$, we get $M_i \in \mathcal{E}$. Then, whenever I is a finite subset of $\mathbf{Z}^+$, the set $M(I)$ of those pairs $(t,x) \in E$ such that $I(t,x) = I$ is measurable as well, since $M(I) = \cap_{i\in I} M_i \setminus \cup_{j\notin I} M_j$. On the other hand, if we apply Prop. 4.6, where we put the measure space $(E,\mathcal{E})$ in place of $(T,\mathcal{L})$, we infer that Φ is measurable on $M(I)$. Now it is enough to point out that $E = \cup\{M(I);\, I \in \mathcal{I}\}$, where $\mathcal{I}$ is the countable family of all finite subsets of $\mathbf{Z}^+$.

In order to prove that Φ is locally lipschitzian with respect to x, we can exploit property (5.8), and proceed exactly in the same way as in Lemma 4.1 of [4]. We refer to that paper for the details. Now we only need to prove (5.1) and (5.2): to this end, it is enough to consider (5.5) and (5.6), which hold for every $i \in I(t,\xi)$, and combine them with the following inclusions, which follow from property (Ω_4) of Def. 2.3:

$$\cap_{i\in I(t,x)} H_i(t,x) \subseteq \Phi(t,x) \subseteq \cup_{i\in I(t,x)} H_i(t,x). \qquad \square$$

Proof of Theorem 5.1. As Lemma 4.1 in [4], the previous result allows to build the approximating multifunctions, through an inductive procedure which starts from the case in which $H(t,x) \equiv \mathbf{R}^p$. Now the Proof of Theorem 5.1 follows, step by step, the one we gave in [4].

Remark 5.6 When $F(t,x)$ is subject to a given domination from the exterior, of the kind $F(t,x) \subseteq K(t)$, it can be useful to show that the approximating multifunctions preserve it. We point out that our technique allows us to do that, as soon as $K(\cdot)$ is measurable and takes closed values: to this end, it is enough to begin the inductive procedure of the previous proof with $H(t,x) = K(t)$.

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