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# Convolution Product of Periodic Distributions and the Dirichlet Problem on the Unit Disc

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**Abstract**

A short description of a convolution product of periodic distributions is presented together with its applications to the Dirichlet problem for the Laplace equation on the unit disc with distributional data.

**Key words:** Periodic distribution, convolution product, Poisson integral, Dirichlet problem.

**1991 Mathematics Subject Classification:** 34A12, 34A45

## 1 Periodic distributions

Let $\mathcal{P}$ denote the set of all smooth complex $2\pi$-periodic functions defined on $\mathbb{R}$. Similarly, the symbol $L^2_{2\pi}$ will be denoted the space of $2\pi$-periodic locally square integrable functions on $\mathbb{R}$. Denote by

$$(\varphi,\psi) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\overline{\psi}(t)dt$$

the usual inner product of $\varphi$ and $\psi$ in $L^2_{2\pi}$. Put by definition

$$(\varphi,\psi)_k := \sum_{j=0}^{k}\binom{k}{j}(\varphi^{(j)},\psi^{(j)}) \tag{1}$$

for $\varphi, \psi \in \mathcal{P}$. For every $k \in \mathbb{N}_0$ formula (1) gives an inner product in $\mathcal{P}$. Of course, $\mathcal{P}$ is not complete with any of them. But we can consider the complection $\mathcal{P}_k$ of $\mathcal{P}$ under the norm $\|\cdot\|_k := (\cdot,\cdot)_k^{\frac{1}{2}}$. $\mathcal{P}_k$ is a Hilbert space with respect to the norm $\|\cdot\|_k$. It is easy to see that

$$\|\varphi\|_{L^2_{2\pi}} = \|\varphi\|_0 \leq \|\varphi\|_1 \leq \ldots \leq \|\varphi\|_k \leq \cdots \tag{2}$$

for $\varphi \in \mathcal{P}$. $\mathcal{P}_k$ may be regarded as a subspace of $L^2_{2\pi}$ ([1]). It is easy to check that $\varphi$ is in $\mathcal{P}_k$ if and only if its distributional derivatives $D^\alpha\varphi \in L^2_{2\pi}$ for $\alpha = 0, 1, \ldots k$. From above, it follows that $\varphi$ has weak derivatives $\varphi^{(\alpha)} = D^\alpha\varphi$, $\alpha = 0, 1, \ldots k$ and $\varphi^{(k-1)}$ is an absolutely continuous function on $\mathbb{R}$. It may be shown that (1) is true for $\varphi, \psi \in \mathcal{P}_k$, too.

**Theorem 1** *A function $\varphi$ from $L^2_{2\pi}$ is in $\mathcal{P}_k$ if and only if*

$$\sum_{\nu\in\mathbb{Z}} (\nu^2+1)^k |c_\nu(\varphi)|^2 < \infty, \quad c_\nu(\varphi) := (\varphi, e_\nu), \tag{3}$$

*where $e_\nu := e^{i\nu(\cdot)}$. Moreover,*

$$\|\varphi\|_k^2 = \sum_{\nu\in\mathbb{Z}} (\nu^2+1)^k |c_\nu(\varphi)|^2. \tag{4}$$

**Proof** See [2]. □

Let us equip the vector space $\mathcal{P}$ with the family of the norms (2). The dual space of $\mathcal{P}$ will be denoted by $\mathcal{P}'$. If $\Lambda \in \mathcal{P}'$ then it will be called a periodic distribution. Each periodic distribution will be extended by continuity on some space $\mathcal{P}_k$. This extension of $\Lambda$ may be regarded as an element of $\mathcal{P}_{-k}$, where $\mathcal{P}_{-k}$ is the dual space of $\mathcal{P}_k$. A function $\varphi \in L^2_{2\pi}$ is identified with the linear form $(\varphi, \cdot)$. It may be shown that

$$\mathcal{P}' \supset \ldots \supset \mathcal{P}_{-k} \supset \ldots \supset \mathcal{P}_{-1} \supset \mathcal{P}_0 = L^2_{2\pi} \supset \mathcal{P}_1 \supset \ldots \supset \mathcal{P}_k \supset \ldots \supset \mathcal{P} \tag{5}$$

(see [1]).

As an immediate corollary from the Riesz theorem we obtain

**Theorem 2** *A linear continuous form $\Lambda$ in $\mathcal{P}_{-k}$ may be written as follows*

$$\Lambda(\varphi) = (f_\Lambda, \varphi)_k, \tag{6}$$

*where $f_\Lambda$ is a fixed element of $\mathcal{P}_k$ and $\varphi$ runs trough $\mathcal{P}_k$.*

## 2 Convolution product

If $\varphi$ and $\psi$ are in $L^2_{2\pi}$ then we take

$$(\varphi * \psi)(x) := 2\pi(\varphi(x - \cdot), \overline{\psi}). \tag{7}$$

Since $\varphi(x - \cdot)$ is in $L^2_{2\pi}$ for each $x \in \mathbb{R}$ therefore formula (7) is sensible and $\varphi * \psi \in L^2_{2\pi}$.

**Theorem 3** *For $\varphi$, $\psi \in L^2_{2\pi}$ we have*

$$c_\nu(\varphi * \psi) = 2\pi c_\nu(\varphi) c_\nu(\psi), \quad \nu \in \mathbb{Z}.$$

**Proof** See [4, p. 168]. □

We are now in a position to define a convolution product of a linear form $\Lambda \in \mathcal{P}_{-k}$ and $\varphi \in \mathcal{P}_k$.

**Definition 1** If $\Lambda(\cdot) = (f_\Lambda, \cdot)_k$ then the function

$$(\Lambda * \varphi)(x) := 2\pi (f_\Lambda(x - \cdot), \overline{\varphi})_k \tag{8}$$

is called the convolution product of $\Lambda$ and $\varphi$.

It is easy to verify using integrating by parts that if $\Lambda \in \mathcal{P}_{-k_1}$ and $\varphi \in \mathcal{P}_{k_2}$ $k_1, k_2 \in \mathbb{N}_0$, $k_1 \le k_2$, then

$$(f^1_\Lambda(x - \cdot), \varphi)_{k_1} = (f^2_\Lambda(x - \cdot), \varphi)_{k_2}$$

provided $f^1_\Lambda$, $f^2_\Lambda$ are different representations of $\Lambda$ given by Theorem 2.

**Theorem 4** *If $\Lambda \in \mathcal{P}_{-k}$ and $\varphi \in \mathcal{P}_k$ then*

$$c_\nu(\Lambda * \varphi) = 2\pi c_\nu(\Lambda) c_\nu(\varphi), \tag{9}$$

*where $c_\nu(\Lambda) := \Lambda(e_\nu)$.*

**Proof** Note that

$$\begin{aligned} c_\nu(\Lambda) &= \Lambda(e_\nu) = (f_\Lambda, e_\nu)_k = \sum_{j=0}^{k} \binom{k}{j} (f_\Lambda^{(j)}, e_\nu^{(j)}) \\ &= \frac{1}{2\pi} \sum_{j=0}^{k} \binom{k}{j} \int_{-\pi}^{\pi} f_\Lambda^{(j)}(t) (e^{-i\nu t})^{(j)} dt \\ &= \frac{1}{2\pi} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_{-\pi}^{\pi} f_\Lambda(t) (e^{-i\nu t})^{(2j)} dt \\ &= \frac{1}{2\pi} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (-i\nu)^{2j} \int_{-\pi}^{\pi} f_\Lambda(t) e^{-i\nu t} dt \\ &= (1 + \nu^2)^k c_\nu(f_\Lambda). \end{aligned}$$

In accordance with Definition 1 we have

$$\begin{aligned} (\Lambda * \varphi)(x) &= 2\pi (f_\Lambda(x - \cdot), \overline{\varphi})_k \\ &= 2\pi \sum_{j=0}^{k} \binom{k}{j} ((f_\Lambda(x - \cdot))^{(j)}, \overline{\varphi}^{(j)}) \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (f_\Lambda^{(j)} * \varphi^{(j)})(x). \end{aligned}$$

Hence, by Theorem 4 we obtain

$$\begin{aligned} c_\nu(\Lambda * \varphi) &= 2\pi \sum_{j=0}^{k} (-1)^j \binom{k}{j} c_\nu(f_\Lambda^{(j)}) c_\nu(\varphi^{(j)}) \\ &= 2\pi \sum_{j=0}^{k} (-1)^j \binom{k}{j} (f_\Lambda^{(j)}, e_\nu)(\varphi^{(j)}, e_\nu) \\ &= 2\pi c_\nu(f_\Lambda) c_\nu(\varphi) \sum_{j=0}^{k} (-1)^j \binom{k}{j} (-i\nu)^j (-i\nu)^j \\ &= 2\pi(1+\nu^2) c_\nu(f_\Lambda) c_\nu(\varphi) = 2\pi c_\nu(\Lambda) c_\nu(\varphi). \end{aligned}$$

This finishes the proof. □

**Theorem 5** *A linear form $\Lambda$ is in $\mathcal{P}_{-k}$, $k \geq 1$ if and only if*

$$\sum_{\nu\in\mathbb{Z}} \frac{|c_\nu(\Lambda)|^2}{(\nu^2+1)^k} < \infty \tag{10}$$

**Proof** See [2]. □

As an immediate consequence of Theorem 3 and Theorem 4 we have the following

**Theorem 6** *If $\psi \in \mathcal{P}_l$, $\varphi \in \mathcal{P}_k$, $\Lambda \in \mathcal{P}_{-k}$, $k, l \in \mathbb{N}_0$ and $l \geq k$ then*

$$(\Lambda * \varphi) * \psi = \Lambda * (\varphi * \psi). \tag{11}$$

## 3 The Poisson integral

In this sequel we shall need the function

$$P_r(t) = \sum_{\nu\in\mathbb{Z}} r^{|\nu|} e^{i\nu t}, \ 0 \leq r < 1, \ t \in \mathbb{R} \tag{12}$$

(see [3]). It is easy to show that $P_r(t) = P_r(-t)$ and $P_r(t) \in \mathbb{R}$.

Let $c_\nu \in \mathbb{C}$ for $\nu \in \mathbb{Z}$ and

$$\limsup_{|\nu|\to\infty} |c_\nu|^{\frac{1}{n}} \leq 1. \tag{13}$$

**Theorem 7** *For every sequence $(c_\nu)_{\nu\in\mathbb{Z}}$ fulfilling* (13) *the function*

$$v(x,y) := \sum_{\nu=0}^{\infty} c_\nu z^\nu + \sum_{\nu=1}^{\infty} c_{-\nu} (\overline{z})^\nu, \quad z = x + iy \tag{14}$$

*of real arguments $x$ and $y$ is harmonic in $B_1(0) = \{(x,y) : x^2 + y^2 < 1\}$. Moreover, the function*

$$u(r,t) := \sum_{\nu\in\mathbb{Z}} c_\nu r^{|\nu|} e^{i\nu t}, \quad 0 \le r < 1,\ t \in \mathbb{R}$$

*is a polar representation of $v$.*

**Proof** From (13) it follows that the series in (14) are almost uniformly convergent on $B_1(0)$. Therefore $v$ is a harmonic function. For $z = re^{it}$ we obtain

$$u(r,t) = \sum_{\nu=0}^{\infty} c_\nu r^{|\nu|} e^{i\nu t} + \sum_{\nu=1}^{\infty} c_{-\nu} r^{|\nu|} e^{-i\nu t} = \sum_{\nu=0}^{\infty} c_\nu z^\nu + \sum_{\nu=1}^{\infty} c_{-\nu} (\overline{z})^\nu = v(x,y).$$

Of course, $u(r,\cdot) \in \mathcal{P}$. This finishes the proof. □

**Theorem 8** *For every distribution $\Lambda \in \mathcal{P}'$, $P_r * \Lambda$ is a polar representation of a harmonic function. Moreover, if $\Lambda \in \mathcal{P}_k$ then*

$$\lim_{r\to 1} \frac{1}{2\pi}(P_r * \Lambda) = \Lambda \tag{15}$$

*in $\mathcal{P}_k$ for every $k \ge 0$, but if $k < 0$ then the* (15) *holds in the sense of weak topology in $\mathcal{P}_k$.*

**Proof** Let $\Lambda = \sum_{\nu\in\mathbb{Z}} c_\nu e_\nu$. From Theorem 1 and Theorem 5 it follows that there exists $k \in \mathbb{Z}$ such that $\sum_{\nu\in\mathbb{Z}} (\nu^2+1)^k |c_\nu|^2 < \infty$. Therefore condition (13) is fulfilled. According to Theorems 7, 3 and 4 the function

$$P_r(t) * \Lambda = 2\pi \sum_{\nu\in\mathbb{Z}} c_\nu r^{|\nu|} e^{i\nu t}, \quad 0 \le r < 1,\ t \in \mathbb{R} \tag{16}$$

is the polar representation of the harmonic function given by (14).

Now, we shall prove (15) for $k \in \mathbb{N}_0$. According to (4), we have

$$\left\| \frac{1}{2\pi}(P_r * \Lambda) - \Lambda \right\|_k^2 = \sum_{\nu\in\mathbb{Z}} (1+\nu^2)^k |r^{|\nu|} c_\nu - c_\nu|^2 = \sum_{\nu\in\mathbb{Z}} (1+\nu^2)^k |c_\nu|^2 (1 - r^{|\nu|})^2.$$

Let $\epsilon > 0$ and $\nu_0 \in \mathbb{N}$ such that

$$\sum_{|\nu|>\nu_0} (1+\nu^2)^k |c_\nu|^2 (1 - r^{|\nu|})^2 < \epsilon.$$

Put $M := \max_{|\nu|\le\nu_0} |c_\nu|(1+\nu^2)^k$. There exists $r_0$, $0 < r_0 < 1$ such that $(1 - r^{|\nu|})^2 < \epsilon$ for $r \ge r_0$ and $|\nu| \le \nu_0$. Therefore

$$\left\| \frac{1}{2\pi}(P_r * \Lambda) - \Lambda \right\|_k^2 < \epsilon(M+1)$$

for $r \ge r_0$. This ends the proof of (15) if $k \ge 0$.

In case when $k \leq -1$ we have to show that

$$\lim_{t \to 1} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} (P_r * \Lambda)(t) \varphi(t) dt = \Lambda(\varphi) \tag{17}$$

for $\varphi \in \mathcal{P}_{|k|}$. Put $\tilde{\varphi}(t) := \varphi(-t)$. In accordance with Theorem 6 we have

$$\int_{-\pi}^{\pi} (\Lambda * P_r)(t) \varphi(t)\, dt = \int_{-\pi}^{\pi} (\Lambda * P_r)(0 - t) \tilde{\varphi}(t)\, dt$$

$$= [(\Lambda * P_r) * \overline{\tilde{\varphi}}](0) = [(\Lambda * P_r) * \tilde{\overline{\varphi}}](0) = [\Lambda * (P_r * \tilde{\overline{\varphi}})](0)$$

$$= [\Lambda * (\overset{\sim}{\overline{P_r * \varphi}})](0) = [\Lambda * (\overline{P_r \tilde{*} \varphi})](0)$$

$$= 2\pi (f_\Lambda(0 - \cdot), P_r \tilde{*} \varphi)_k = 2\pi (f_\Lambda, P_r * \varphi)_k = 2\pi \Lambda(P_r * \varphi).$$

It was shown in the first part of this proof that $P_r * \varphi \to 2\pi\varphi$ in $\mathcal{P}_{|k|}$. Therefore, by continuity of $\Lambda$ we have (17). This finishes the proof. □

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