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Span in Incidence Structures of Independent Sets Defined on Projective Space *

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Abstract

As in [2], to every incidence structure we can construct an incidence structure of independent sets. In this paper an incidence structure defined by means of points and hyperplanes of a projective space is investigated. In the corresponding incidence structure of independent sets there is a span (i.e. the maximal distance of two p-element independent sets of points) determined for some $p > 2$.

Key words: Incidence structure, independent set.

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Definition 1 Let G and M be sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called an *incidence structure*[1].

Let $A \subseteq G$, $B \subseteq M$ be non-empty sets. Then we denote

$$A^{\uparrow} = \{m \in M \mid gIm \ \ \forall g \in A\}, \qquad B^{\downarrow} = \{g \in G \mid gIm \ \ \forall m \in B\}.$$

For the empty set we put $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$. And moreover, we denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$, $g^{\uparrow} := \{g\}^{\uparrow}$, $m^{\downarrow} := \{m\}^{\downarrow}$ for $A \subseteq G$, $B \subseteq M$ and $g \in G$, $m \in M$.

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[1] It is called *kontext* more frequently (Wille, [1]). The name incidence structure is used with regards to consecutive geometric applications.

Definition 2 Let $\mathcal{J} = (G, M, I)$ be an incidence structure. A sequence

$$(g_0, m_0, g_1, m_1, \dots, g_{r-1}, m_{r-1}, g_r),$$

where $g_i \in G$ for $i \in \{0, \dots, r\}$, $m_j \in M$ for $j \in \{0, \dots, r-1\}$ and $g_j I m_j, g_{j+1} I m_j$ for all $j \in \{0, \dots, r-1\}$, is called a *join* of elements g_0, g_r.

A positive integer r is said to be a *length of a join* of elements g_0, g_r. We suppose that the join (g, m, g) has a length 0. If a join of two elements of G exists, then we say that they are *joinable*. The minimal length of all joins of elements $g, h \in G$ we call a *distance* of these elements and denote by $v(g, h)$. The maximal distance of any two elements of G is said to be a *span* of G and denoted by $d(G)$. If $|g^{\uparrow}| = |m^{\downarrow}| = 1$ for all $g \in G$, $m \in M$, then we put $d(G) = 0$.

In what follows we denote $A_a := A - \{a\}$, $B_m := B - \{m\}$ for $A \subseteq G$, $B \subseteq M$, respectively.

Definition 3 The set $A \subseteq G$ is said to be *independent* in G if $a \notin A_a^{\uparrow\downarrow}$ for all $a \in A$.

Consider a subset $A \subseteq G$. For $a \in A$ let us put $X^A(a) := A_a^{\uparrow} - a^{\uparrow}$. Then $X^A(a) = \emptyset$ if and only if $a \in A_a^{\uparrow\downarrow}$. A is independent in G if and only if $X^A(a) \neq \emptyset$ for all $a \in A$.

Prof. Machala has defined ([2], [3]) a norming mapping in incidence structures and incidence structures of independent sets.

Definition 4 Let a non-empty set $A \subseteq G$ be independent in G. If $\mathcal{X} = \{X^A(a) \mid a \in A\}$, then for every choice $Q^A = \{m_a \in X^A(a) \mid X^A(a) \in \mathcal{X}\}$ we define a *norming mapping* $\alpha : A \to Q^A$ by the formula $\alpha(a) = m_a$ for all $a \in A$.

In a similar way we define: A set B is independent in M if $m \notin B_m^{\downarrow\uparrow}$ for all $m \in B$. Let us put $Y^B(m) := B_m^{\downarrow} - m^{\downarrow}$ for each $m \in B$. B is independent in M if and only if $Y^B(m) \neq \emptyset$ for all $m \in B$. Let a non-empty set $B \subseteq M$ be independent in M. We put $\mathcal{Y} = \{Y^B(m) \mid m \in B\}$ and $Q^B = \{g_m \in Y^B(m) \mid Y^B(m) \in \mathcal{Y}\}$. The mapping $\beta : B \to Q^B : m \mapsto g_m$ is a mapping norming the set B.

Theorem 1 *Let $\mathcal{J} = (G, M, I)$ be an incidence structure and $A \subseteq G$ be independent. Then each norming mapping $\alpha : A \to Q^A$ is injective and Q^A is independent in M.*

The dual statement also holds:

Theorem 2 *Let $\mathcal{J} = (G, M, I)$ be an incidence structure and $B \subseteq M$ be independent. Then each norming mapping $\beta : B \to Q^B$ is injective and Q^B is independent in G.*

For the proofs of Theorems 1 and 2 see [3].

Definition 5 Let us consider an incidence structure $\mathcal{J} = (G, M, I)$ and a positive integer $p \geq 2$. Let G^p and M^p be the sets of all independent sets of G and M of cardinality p, respectively. Then $\mathcal{J}^p = (G^p, M^p, I^p)$ is called an *incidence structure of independent sets* of $\mathcal{J}$ where AI^pB if and only if there exists a norming mapping $\alpha : A \to B$ for $A \in G^p$, $B \in M^p$.

Let us consider a projective space $\mathcal{P}^n$ of finite dimension $n > 2$ over a field K which can be uderstood as a set of all subspaces of a vector space V over K of dimension $n + 1$. Projective dimension of subspaces in $\mathcal{P}^n$ is defined with a help of dimension of subspaces in V by the formula $\dim_{\mathcal{P}} U = \dim_V U - 1$ for any subspace U of V. Then the projective space $\mathcal{P}^n$ has projective dimension n. The subspaces of projective dimension 0 $(1, 2, n-1)$ are points (lines, planes, hyperplanes). The empty set is a subspace of $\mathcal{P}^n$ and $\dim_{\mathcal{P}} \emptyset = -1$. In what follows we will consider the notion of dimension of a subspace in the projective sense. However, we put $\dim_{\mathcal{P}} U := \dim U$, i.e. the index $\mathcal{P}$ will be omitted. A subspace of $\mathcal{P}^n$ generated by a point-set A will be denoted by $[A]$.

As in [4], we remind the following well-known formula:

Proposition 1 *If U and V are subspaces of $\mathcal{P}^n$, then*

$$\dim U + \dim V = \dim(U + V) + \dim(U \cap V).$$

Proposition 2 *Let $U_1, \ldots, U_k$, $1 \leq k \leq n + 1$, be hyperplanes in $\mathcal{P}^n$ and let $n_k = \{1, \ldots, k\}$. Then the following conditions are equivalent:*

$$\forall i \in n_k : \Big(\bigcap_{j \in n_k - \{i\}} U_j \Big) \not\subseteq U_i, \tag{1}$$

$$\dim \Big(\bigcap_{j \in n_k} U_j \Big) = n - k. \tag{2}$$

For the proof see [4].

Let us suppose that an incidence structure $\mathcal{J} = (G, M, I)$ on the projective space $\mathcal{P}^n$ is defined as follows: G is a set of all points of $\mathcal{P}^n$, M is a set of all hyperplanes of $\mathcal{P}^n$ and I is an incidence relation: xIU if and only if the point x lies in the hyperplane U.

For elements of M we will use symbols $U, V, W, \ldots$ Then we suppose that $U^{\downarrow} := U$ and so on.

Let us consider an incidence structure of independent sets $\mathcal{J}^p = (G^p, M^p, I^p)$ corresponding to $\mathcal{J}$ where $2 < p \leq n + 1$.

Let $A = \{a_1, \ldots a_p\} \in G^p$, $B = \{b_1, \ldots b_p\} \in G^p$. We denote $U = [A]$, $V = [B]$ and for all $i \in \{1, \ldots, p\}$ we put $A_i := A - \{a_i\}, B_i := B - \{b_i\}$, $U_i := [A_i]$, $V_i := [B_i]$. Obviously $\dim U = \dim V = p - 1$, $\dim U_i = \dim V_i = p - 2$. In what follows we suppose that $a_i \neq b_i$ for all i and a line passing through the points a_i, b_i will be denoted by $c_i = a_i b_i$, i.e. $c_i = \{a_i\} + \{b_i\}$.

Theorem 3 *The following statements are equivalent for two distinct independent sets $A, B \in G^p$, $2 < p \leq n$:*

(1) $v(A,B) = 1$.

(2) There exists a subspace W of dimension $n-p$ which intersects all the lines c_i and $W \cap U = W \cap V = \emptyset$.

Proof (1) $\Longrightarrow$ (2) From $v(A,B) = 1$ the existence of norming mappings α, β follows with the property $\beta\alpha(A) = B$. Let us put $\alpha(A) = R$ and $\alpha(a_i) = Z_i$. Then $R = \{Z_1, \dots, Z_p\} \in M^p$. We choose such a denotation that $\beta\alpha(a_i) = \beta(Z_i) = b_i$. Since R is independent in M it follows from Proposition 2 that $\dim R^{\downarrow} = n-p$ and $\dim R_i^{\downarrow} = n-p+1$ for all $i \in \{1, \dots, p\}$. If we put $W = R^{\downarrow} = \bigcap_{1 \leq i \leq p} Z_i$, then $W \subset R_i^{\downarrow}$ and W is a hyperplane in R_i for each i. By the assumption $\alpha(a_i) = Z_i$ where $Z_i \in A_i^{\uparrow} - a_i^{\uparrow}$. Hence $a_i \notin Z_i$ and $a_i \notin W$. We also obtain $a_i \in R_i^{\downarrow}$. Moreover, $\beta(Z_i) = b_i$ where $b_i \in R_i^{\downarrow} - Z_i$. The line c_i is contained in $R_i^{\downarrow}$ and is not contained in W. Hence it intersects W in one point for each i.

Let $r \in W \cap U$. Then r is not contained in all subspaces U_i. Let for instance $r \notin U_1$. Then $U = r + U_1$. It is clear that $r \in Z_1$, $U_1 \subset Z_1$ which implies $U \subset Z_1$. Thus $a_1 \in Z_1$ and that is a contradiction. Therefore $W \cap U = \emptyset$ and similarly $W \cap V = \emptyset$.

(2) $\Longrightarrow$ (1) For each i we put $Z_i = U_i + W$. Since $W \cap U = \emptyset$ and $\dim W = n-p$ it is clear that Z_i is a hyperplane and $a_i \notin Z_i$. Let us denote $R = \{Z_1, \dots, Z_p\}$. From $U_i \subseteq Z_i$ we get $Z_i \in A_i^{\uparrow}$ and $a_i \notin Z_i$ implies that $Z_i \notin a_i^{\uparrow}$. Thus $Z_i \in A_i^{\uparrow} - a_i^{\uparrow}$, the mappingí $\alpha : a_i \mapsto Z_i$ is norming and R is independent in M. Since the lines c_i intersect W we get in proper denotation that $V_i \subset Z_i$. Obviously $b_i \notin Z_i$ and $b_i \in R_i^{\downarrow}$, that is $b_i \in R_i^{\downarrow} - Z_i$. Hence $\beta : Z_i \to b_i$ is a norming mapping and $\beta\alpha(A) = B$. □

We put

$$Q = \sum_{1 \leq i \leq p} c_i, \qquad Q_j = \sum_{i \neq j} c_i.$$

Then $U + V = Q$. If we denote $\dim(U+V) = l$ and $\dim(U \cap V) = r$, then $\dim U + \dim V = 2(p-1) = l + r$.

Definition 6 One says that the sets A, B are in a *basic position* if $v(A,B) = 1$ and the subspace $U \cap V$ has minimal dimension.

Proposition 3 *If the sets A, B are in the basic position, then $p \geq \frac{n+1}{2}$ if and only if* $\dim Q = n$.

Proof Let $p \geq \frac{n+1}{2}$. Then $2p - 2 \geq n-1$ and $l + r \geq n-1$. Since r is minimal admissible and hence l is maximal admissible number, we get $l = n$. Assume that $\dim Q = n$. This yields $2(p-1) = n + r$ and $p = \frac{n+1}{2} + \frac{r+1}{2}$. From $r \geq -1$ we obtain $\frac{r+1}{2} \geq 0$ and $p \geq \frac{n+1}{2}$. □

Proposition 4 *Let the sets A, B be in the basic position. Then $p \leq \frac{n+1}{2}$ if and only if $U \cap V = \emptyset$.*

Proof Let $p \leq \frac{n+1}{2}$. Then $2p-2 \leq n-1$ and $l+r \leq n-1$. For $r=-1$ we have $l \leq n$, which is always fulfilled. From the requirement of minimality of r it follows that $U \cap V = \emptyset$. Assume $U \cap V = \emptyset$, that is $r=-1$. Then $l-1 = 2p-2$ and $2p = l+1$. Since $l \leq n$ we obtain $2p \leq n+1$ and $p \leq \frac{n+1}{2}$. □

Proposition 5 *Let the sets A, B be in the basic position and $p = \frac{n+1}{2}$. Then $\dim Q_j = n-2$ for each j.*

Proof Since $p = \frac{n+1}{2}$ it is clear that n is odd and $n \geq 5$. Let us put $n = 2q+1$. In Q_j there exist $\frac{n+1}{2} - 1 = \frac{n-1}{2} = q$ lines c_i. If R is a subspace and m is a line in $\mathcal{P}^n$, then $\dim(R+m) \leq \dim R + 2$. It follows that for lines $m_1, \ldots, m_l$ from $\mathcal{P}^n$ we get $\dim(\sum_{1 \leq i \leq n} p_i) \leq 2l-1$. Thus $\dim Q_j \leq 2q-1$ and from $2q-1 = n-2$ we have $\dim Q_j \leq n-2$. If $\dim Q_j < n-2$, for some j, then $\dim Q < n$ and that is a contradiction to $\dim Q = n$. Therefore $\dim Q_j = n-2$. □

Proposition 6 *Let the sets A, B be in the basic position and $p > \frac{n+1}{2}$. Then $\dim Q_j = n-1$ or $\dim Q_j = n-2$ and there always exists such i that $\dim Q_i = n-1$.*

Proof We know that $\dim U = \dim V = p-1$, $\dim U_i = \dim V_i = p-2$, $Q = U+V$, $Q_i = U_i + V_i$. Moreover $\dim Q = n$ by Proposition 3 and hence $\dim(U \cap V) = 2p-n-2$. Let us show that $\dim Q_i = n-2$ iff $U \cap V = U_i \cap V_i$: Assume $\dim Q_i = n-2$. Then $\dim U_i + \dim V_i = 2p-4 = \dim Q_i + \dim(U_i \cap V_i) = n-2+\dim(U_i \cap V_i)$. This yields $\dim(U_i \cap V_i) = 2p-n-2 = \dim(U \cap V)$. Let $U \cap V = U_i \cap V_i$. Then $2p-4 = \dim Q_i + 2p-n-2$ and $\dim Q_i = n-2$. It follows that $\dim Q_i = n-1$ iff $\dim(U_i \cap V_i) < \dim(U \cap V)$. Since there always exists such i that $U_i \cap V_i \neq U \cap V$ we obtain that always exists such Q_i that $\dim Q_i = n-1$. □

Remark 1 If $\dim Q_j = n-1$ for certain $j \in \{1, \ldots, p\}$, then $p > \frac{n+1}{2}$: The subspace Q_j has maximal dimension $2p-3$. In case of $p \leq \frac{n+1}{2}$ we get $2p-3 \leq n-2$ and $\dim Q_j \leq n-2$. That is a contradiction.

Further, let us put $x_i = c_i \cap W$ and $X = \sum_{1 \leq i \leq p} x_i$, $X_j = \sum_{i \neq j} x_i$. Then $X, X_j \subseteq W$, $Q = X+U$ and $Q_i = X_i + U_i$.

Proposition 7 *Let the sets A, B be in the basic position. Then $\dim Q_i = n-2$ if and only if $\dim X_i = n-p-1$ and $\dim Q_i = n-1$ if and only if $X_i = W$.*

Proof It is obvious that $\dim X_i = \dim Q_i + \dim(X_i \cap U_i) - \dim U_i$. If $\dim Q_i = n-2$, then $\dim X_i = n-2-1-p+2 = n-p-1$. Similarly, for $\dim Q_i = n-1$ we have $\dim X_i = n-p$. If $\dim X_i = n-p-1$, then $\dim Q_i = n-p-1+p-2+1 = n-2$ and from $\dim X_i = n-p$ we get $\dim Q_i = n-1$. □

Example 1

1. Let $p = n$. Then $\dim Q = n$ and W is a point. Obviously $X = X_i = W$ for each i and thus $\dim Q_i = n - 1$ for each i.

2. Consider $n = 6$, $p = 4$. It means that W is a plane. All points x_i cannot lie on a line. If any three points of x_i do not lie on a line, then $\dim Q_i = n-1 = 5$ for each i. All the lines c_i are pairwise disjoint. Let for instance x_1, x_2, x_3 be pairwise distinct points lying on a line h in W. Then $x_4 \notin h$. We get $\dim Q_4 = n - 2 = 4$ and $\dim Q_j = 5$ for all $j \neq 4$. The lines c_i are pairwise distinct again. Let $x_1 = x_2$. Then $\dim Q_3 = \dim Q_4 = 4$ and $\dim Q_1 = \dim Q_2 = 5$.

Theorem 4 *Let $c_1, \ldots, c_p$ be lines for $2 < p \leq n$ and $Q = \mathcal{P}^n$. Then the following statements are equivalent:*

1. *Q_j is a hyperplane for each $j \in \{1, \ldots, p\}$.*
2. *There exists precisely one subspace W of dimension $k = n - p$ which does not contain any of lines c_i and intersects all of them.*

Proof (1) $\Longrightarrow$ (2) Assume that $\bigcap_{i \neq j} Q_i \subseteq Q_j$ for certain j. From $c_j \subset \bigcap_{i \neq j} Q_i$ we have $c_j \subseteq Q_j$ and $Q \subseteq Q_j$. That is a contradiction. The set $\{Q_i \mid i \in \{1, \ldots, p\}\}$ of hyperplanes is independent. If we put $W = \bigcap_{1 \leq i \leq p} Q_i$, then $\dim W = n - p = k$ by Proposition 2. Since $c_j \not\subseteq Q_j$ and Q_j is a hyperplane in $\mathcal{P}^n$ we obtain that $x_j = c_j \cap Q_j$ is a point and $c_j \not\subset W$. From $c_j \subset \bigcap_{i \neq j} Q_i$ it follows that $x_j \in \bigcap_{1 \leq i \leq p} Q_i$ and $x_j \in W$. Thus W intersects all the lines c_i.

Let Z be a subspace of dimension k which intersects all the lines c_i and does not contain any of them. If we denote $z_i = c_i \cap Z$, then $Z' = \sum_{1 \leq i \leq p} z_i \subseteq Z$. Let us put $Z_j = \sum_{i \neq j} z_i$ for each $j \in \{1, \ldots, p\}$. Then $Z_j \subseteq Z'$ and $\dim Z_j \leq k$. On the lines c_i we select points a_i distinct from z_i and x_i. Let us denote $A = \{a_1, \ldots, a_p\}$, $A_i = A - \{a_i\}$ and $U = [A]$, $U_i = [A_i]$. Then $Q = U + Z$, $Q_i = U_i + Z_i$. The set A is independent: Let $a_i \in A_i^{\uparrow\downarrow} = U_i$. Then $a_i \in Q_i$. Since $c_i \not\subseteq Q_i$ and Q_i is a hyperplane we get $a_i = c_i \cap Q_i = x_i$. That is a contradiction. Thus $\dim U = p - 1$ and $\dim U_i = p - 2$. For given i we obtain $\dim U_i + \dim Z_i = p - 2 + \dim Z_i = \dim Q_i + \dim(Z_i \cap U_i) = n - 1 + \dim(Z_i \cap U_i)$ and $\dim Z_i = n - p + 1 + \dim(Z_i \cap U_i)$. Since $\dim Z_i \leq k$ it is obvious that $Z_i \cap U_i = \emptyset$ and $\dim Z_i = k$. Hence $Z_i = Z' = Z$ and $z_i \in Z_i$. Then $z_i \in Q_i$ and $z_i \in c_i$, that is $z_i \in \bigcap_{1 \leq i \leq p} Q_i = W$. This yields $Z \subseteq W$ and since $\dim Z = k$ we get $Z = W$.

(2) $\Longrightarrow$ (1) By the assumption $x_i = c_i \cap W$ are points. Obviously $B = \sum_{1 \leq i \leq p} x_i \subseteq Q$ and $B_j = \sum_{i \neq j} x_i \subseteq Q_j$. If $\dim Q_i < n - 2$, then $\dim Q < n$ and this is a contradiction to $Q = \mathcal{P}^n$. Thus $\dim Q_i \geq n - 2$. Let $\dim Q_i = n - 2$ for certain i. If $\dim B_i = k$, then $B_i = W$ and $x_i \in B_i \subseteq Q_i$, that is $\dim(Q_i \cap c_i) \geq 0$. We know that $\dim Q_i + \dim c_i = n - 1 = \dim(Q_i + c_i) + \dim(Q_i \cap c_i)$ which implies $\dim(Q_i + c_i) = n - 1 - \dim(Q_i \cap c_i)$ and hence $\dim Q \leq n - 1$. That is a contradiction. Thus $\dim B_i = k - 1$. If we select a point $y_i \neq x_i$ on the line c_i, then $y_i \notin W$ and for $W' = y_i + B_i$ we get $\dim W' = k$. Thus W' intersects all the lines c_i and this is a contradiction.

Let $\dim Q_i = n$. We select points $a_i \in c_i$ distinct from x_i and we put $A = \{a_1, \dots, a_p\}$, $A_i = A - \{a_i\}$ and $U_i = [A_i]$ again. Then $\dim U_i \leq p - 2$, $\dim B_i \leq n-p$ and $Q_i = U_i + B_i$. From $n + \dim(U_i \cap B_i) = \dim U_i + \dim B_i \leq n-2$ we get $\dim(U_i \cap B_i) \leq -2$ which is a contradiction. Thus $\dim Q_i = n - 1$. □

Remark 2 Let $p \leq \frac{n+1}{2}$. Then $\dim Q_j < n - 1$ foar each $j \in \{1, \dots, p\}$ by Remark 1. If $k = n - p$, then $k \geq p - 1$. If we select points $x_i \in c_i$ for $i \in \{1, \dots, p\}$, then $\dim(\sum_{1 \leq i \leq p} x_i) \leq p - 1 \leq k$. Thus there exist such subspaces of dimension k that they intersect all the lines c_i.

In the following propositions 8–13 we assume that $Q = \mathcal{P}^n$ and $\dim Q_i = n - 1$ for all $i \in \{1, \dots, p\}$. By Theorem 4 there exists a uniquely determined subspace W of dimension $n - p$ for which $W \subseteq Q_i$. Recall that $x_i = c_i \cap W$ for all i.

Proposition 8 *$U \cap W = \emptyset \Leftrightarrow a_i \notin W$ for each $i \in \{1, \dots, p\}$.*

Proof If $U \cap W = \emptyset$, then obviously $a_i \notin W$. Let $a_i \notin W$ for each $i \in \{1, \dots, p\}$ and assume that $x \in U \cap W$. There exists $i \in \{1, \dots, p\}$ such that $x \notin U_i$. Since U_i is a hyperplane in U we get $U = U_i + \{x\}$. Moreover, $U_i \subseteq Q_i$, $W \subseteq Q_i$ and $x \in Q_i$, that is $U \subseteq Q_i$. This implies $a_i \in Q_i$. Since $a_i \notin W$ we have $a_i \neq x_i$ and $c_i = a_i + x_i$. Now from $x_i \in Q_i$ it follows that $c_i \subset Q_i$ and $Q \subseteq Q_i$. That is a contradiction to $Q = \mathcal{P}^n$. □

Proposition 9 *$V \cap W = \emptyset \Leftrightarrow b_i \notin W$ for each $i \in \{1, \dots, p\}$.*

Proposition 10 *$U_i + V = \mathcal{P}^n \Leftrightarrow b_i \neq x_i$.*

Proof Let $b_i \neq x_i$. Then $c_i = b_i + x_i$. Obviously $Q_i \subseteq U_i + V$. Let $b_i \in Q_i$. Since $x_i \in Q_i$ we have $c_i \subseteq Q_i$ and $Q_i + c_i = Q \subseteq Q_i$ which is a contradiction. Thus $b_i \notin Q_i$ and $Q_i + \{b_i\} = \mathcal{P}^n$. However, $Q_i + \{b_i\} \subseteq U_i + V$ yields $\mathcal{P}^n = U_i + V$. □

Proposition 11 *$V_i + U = \mathcal{P}^n \Leftrightarrow a_i \neq x_i$.*

Remark 3 If $b_i = x_i$, then $U_i + V = Q_i$ and hence $\dim(U_i + V) = n - 1$.

Proposition 12 *$U \cap V \nsubseteq U_i \Leftrightarrow b_i \neq x_i$.*

Proof Since $U + V = Q = \mathcal{P}^n$ we get $\dim(U \cap V) = 2p - n - 2$. Let $b_i \neq x_i$ which means $\mathcal{P}^n = U_i + V$. Assume that $U \cap V \subseteq U_i$. Then $U_i \cap V = U \cap V$. However, $\dim(U_i \cap V) = 2p - n - 3 = \dim(U \cap V)$ and this is a contradiction. Let $b_i = x_i$. From $U_i \cap V \subseteq U \cap V$ and Remark 3 we get $\dim(U_i \cap V) = 2p - n - 2 = \dim(U \cap V)$. It follows that $U_i \cap V = U \cap V$ and $U \cap V \subseteq U_i$. □

Proposition 13 *$U \cap V \nsubseteq V_i \Leftrightarrow a_i \neq x_i$.*

Remark 4 If $U \cap W = \emptyset$, then $a_i \neq x_i$ for each $i \in \{1, \ldots, p\}$ and thus $U \cap V \not\subseteq V_i$ for each $i \in \{1, \ldots, p\}$. Similarly for $V \cap W = \emptyset$. If $U \cap V \not\subseteq V_i$ for each $i \in \{1, \ldots, p\}$, then $a_i \neq x_i$ for each $i \in \{1, \ldots, p\}$ and $U \cap W = \emptyset$ by Proposition 8. Similarly for $U \cap V \not\subseteq U_i$.

Corollary 1 *Let $Q = \mathcal{P}^n$, $\dim Q_i = n - 1$ and $(U \cap V) \not\subseteq U_i$, $(U \cap V) \not\subseteq V_i$ for each $i \in \{1, \ldots, p\}$. Then the sets A, B are in the basic position.*

In order to determine a span of G^p one has to find such sets $A, B \in G^p$, $A = \{a_1, \ldots a_p\}$, $B = \{b_1, \ldots b_p\}$, that $v(A, B)$ is maximal. For brevity we suppose that $a_i \neq b_j$ for all $i, j \in \{1, \ldots, p\}$. It follows from the definition of a norming mapping that the renumbering of elements from A, B does not make any difference. If any element of A is equal to any element of B, then obviously $v(A, B)$ is not greater than by the converse assumption.

Theorem 5 *In an incidence structure $\mathcal{J}^{n+1}$ there is $d(G^{n+1}) = 0$.*

Proof Let $A \in G^{n+1}$. Then $X^A(a_i) = Z_i \in M$ for $i \in \{1, \ldots, n + 1\}$. There exists a unique choice $Q^A = \{Z_1, \ldots, Z_{n+1}\}$ from the set $\mathcal{X}$ and thus a unique norming mapping of A. Hence in $\mathcal{J}^{n+1}$ we have $|A^{\uparrow}| = 1$, similarly $|B^{\downarrow}| = 1$ for all $B \in M^{n+1}$. Therefore $d(G^{n+1}) = 0$. □

Theorem 6 *If $2(p - 1) < n$, then $d(G^p) = 2$.*

Proof 1. Let $U \cap V = \emptyset$. This is equivalent to $\dim(U + V) = \dim Q = 2p - 1$. Consider the lines $c_1, \ldots, c_p$. According to Remark 2 there exist infinitely many subspaces of dimension $n - p$ intersecting all the lines c_i and not containing any of them. Obviously $2p - 1 = 1 + 2(p - 1) = 1 + (2 + \ldots + 2)$. It means that the subspace Q generated by lines $c_1, \ldots, c_p$ has maximal dimension and thus for $Q_j = \sum_{i \neq j} c_i$ we get $c_j \cap Q_j = \emptyset$. If we select points $x_i \in c_i$, $x_i \neq a_i, b_i$, $i \in \{1, \ldots, p\}$, then $X = \{x_1, \ldots, x_p\}$ is an independent set: Let $x_j \in X_j^{\uparrow\downarrow}$ where $X_j = X - \{x_j\}$. Since $X_j^{\uparrow\downarrow} \subseteq Q_j$ we get $x_j \in c_j \cap Q_j$ and it is a contradiction. Hence the set X generates a subspace R of dimension $p - 1$. To Q there exists a complementary subspace S, i. e. $Q + S = \mathcal{P}^n$, $Q \cap S = \emptyset$. Then $2p - 1 + \dim S = n - 1$ and $\dim S = n - 2p$. We get $R \cap S = \emptyset$ and $\dim(R + S) = p - 1 + n - 2p + 1 = n - p$. Let us put $W = R + S$. Since $U + R = Q$ we have $\dim(U \cap R) = \dim U + \dim R - \dim Q = p - 1 + p - 1 - 2p + 1 = -1$ which yields $U \cap R = \emptyset$. Similarly $V \cap R = \emptyset$. Since $W \cap Q = R$ we also obtain $U \cap W = V \cap W = \emptyset$. Therefore $v(A, B) = 1$ by Theorem 3.

2. Let $U \cap V \neq \emptyset$. Then $\dim(U + V) \leq 2p - 2 < n$. It is easy to see that there exists a subspace T of dimension $p - 1$ with the property $T \cap U = T \cap V = \emptyset$. We select independent points $a'_1, \ldots, a'_p$ in T and denote $A' = \{a'_1, \ldots, a'_p\}$. Then from 1. it follows that $v(A, A') = v(A', B) = 1$ and hence $v(A, B) \leq 2$. It is not difficult to find an example of $v(A, B) = 2$ $(U = V)$. □

In what follows we assume that $p = n$.

Remark 5 It follows immediately from Theorem 3 that a distance of sets $A, B \in G^n$ is equal to 1 if and only if all the lines $c_1, \dots, c_p$ pass through a point w which is contained neither in U nor in V.

Definition 7 The sets $A, B \in G^n$ are said to be in a *general position* if the following conditions are valid:
1. $U \neq V$,
2. $b_j \notin U_i, a_j \notin V_i$ for all $i, j \in \{1, \dots, n\}$.

Remark 6 Let $a_i, b_i \notin U \cap V$ for all $i \in \{1, \dots, n\}$. Then the sets $A, B \in G^n$ are in the general position.

Theorem 7 *If $A, B \in G^n$ are in the general position, then $v(A, B) \leq n - 1$.*

Proof 1. Let $n = 3$. Then (by assumption) $b_2, b_3 \notin U_1$, $a_2, a_3 \notin V_1$ and $U_1 \neq V_1$. If the lines c_2, c_3 have a point $w_1 \in c_1$ in common, then $w_1 \neq a_1, b_1$ and $v(A, B) = 1$. Let $v(A, B) \neq 1$. The definition of the general position implies that at least one of the lines c_i (under a proper denotation) is contained neither in U nor in V. Let c_1 be that line. Then on c_1 there exists a point $w_1 \neq a_1, b_1$ such that $V_1 \not\subseteq R$ where $R = w_1 + U_1$ and for a point of intersection $g = V_1 \cap R$ we get $g \neq b_2, b_3, a_2, a_3$. In the plane R we select a line q passing through g which is not contained in V and does not contain w_1; we denote by a'_2, a'_3 its points of intersection with the lines w_1a_2, w_1a_3. The lines a'_2b_2, a'_3b_3 are distinct, contained in a plane $S = q + V_1$ and thus they have a point w_2 in common. Then there exist norming mappings $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$\{a_1, a_2, a_3\} \stackrel{\beta_1\alpha_1}{\longrightarrow} \{a'_1 = b_1, a'_2, a'_3\} \stackrel{\beta_2\alpha_2}{\longrightarrow} \{b_1, b_2, b_3\}.$$

Thus $v(A, B) = 2$.

2. Let $n \geq 4$ and suppose that in every projective space $\mathcal{P}^{n-1}$ of dimension $n-1$ there is $v(A', B') \leq n-2$ for independent sets A', B' of $\mathcal{P}^{n-1}$ in the general position. We show that $v(A, B) \leq n - 1$ for independent sets A, B of $\mathcal{P}^n$ in the general position.

Let $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ be independent sets of $\mathcal{P}^n$ in the general position. Then for instance $a_i \notin V_1$ for all $i \in \{1, \dots, n\}$, and hence $U_1 \neq V_1$. On c_1 we can select a point w_1 such that:

a) $w_1 \neq a_1, b_1$,

b) $R = w_1 + U_1$ is a hyperplane in $\mathcal{P}^n$,

c) $V_1 \not\subseteq R$ and then for $P = V_1 \cap R$ we get $\dim P = n - 3$,

d) $b_j \notin P$ for all $j \in \{2, \dots, n\}$,

e) P does not intersect any of lines w_1a_i.

Let us select a subspace Q of R containing P and not containing w_1, $\dim Q = n-2$. Then Q is a hyperplane in R and thus it intersects all lines w_1a_i at points a'_i, $i \in \{2, \dots, n\}$. Obviously $a'_i \notin V_1$, $b_i \notin Q$ for $i \in \{2, \dots, n\}$. If we put $S = Q + V_1$, then S is a hyperplane in $\mathcal{P}^n$, and hence it is a projective space of dimension $n - 1$. The sets $A' = \{a'_2, \dots, a'_n\}$, $B' = \{b_2, \dots, b_n\}$ are independent in S and they are in the general position.

By assumption $v(A',B') \leq n-2$, hence there exist norming mappings $\alpha'_2,\dots,\alpha'_{n-1}$ and $\beta'_2,\dots,\beta'_{n-1}$ such that $B' = \beta'_{n-1}\alpha'_{n-1}\dots\beta'_2\alpha'_2(A')$. If we put

$$X_i^j = [w_j, {}^{j-1}a_2,\dots,{}^{j-1}a_{i-1},{}^{j-1}a_{i+1},\dots,{}^{j-1}a_n]$$

for $j \in \{2,\dots,n-1\}$, $i \in \{2,\dots,n\}$ where w_j are properly selected points, then $({}^1a_2,\dots,{}^1a_n) \overset{\alpha'_2}{\to} (X_2^2,\dots,X_n^2) \overset{\beta'_2}{\to} ({}^2a_2,\dots,{}^2a_n) \overset{\alpha'_3}{\to} (X_2^3,\dots,X_n^3) \to \dots \overset{\beta'_{n-1}}{\to} (b_2,\dots,b_n)$.

The sets $A_j = \{b_1,{}^ja_2,\dots,{}^ja_n\}$, $j \in \{1,\dots,n-1\}$, are independent in $\mathcal{P}^n$. Let us put

$$Y_i^1 = [w_1,a_1,\dots,a_{i-1},a_{i+1},\dots,a_n],$$

$$Y_i^j = [w_j,{}^{j-1}a_1,\dots,{}^{j-1}a_{i-1},{}^{j-1}a_{i+1},\dots,{}^{j-1}a_n]$$

where $b_1 := {}^{j-1}a_1$ for $i \in \{1,\dots,n\}$, $j \in \{2,\dots,n-1\}$.

Then $(a_1,\dots,a_n) \overset{\alpha_1}{\to} (Y_1^1,\dots,Y_n^1) \overset{\beta_1}{\to} (b_1,{}^1a_2,\dots,{}^1a_n) \overset{\alpha_2}{\to} (Y_1^2,\dots,Y_n^2) \overset{\beta_2}{\to} (b_1,{}^2a_2,\dots,{}^2a_n) \to \dots \overset{\beta_{n-1}}{\to} (b_1,\dots,b_n)$. This yields $v(A,B) \leq n-1$ in $\mathcal{P}^n$. □

Proposition 14 *If $A,B \in G^n$, then there exists a set $A' \in G^n$ such that $v(A,A') = 1$ and A',B are in the general position.*

Proof Let us select an arbitrary point $w_1 \notin U,V$. In the hyperplane V we select a subspace R of dimension $n-2$ such that it does not contain any of points b_i and any of intersections $a_iw_1 \cap V$, $i \in \{1,\dots,n\}$. Then consider an arbitrary hyperplane U' contaning R and not containing w_1. We put $a'_i = a_iw_1 \cap U'$, $i \in \{1,\dots,n\}$. It is obvious that $a'_i, b_i \notin U' \cap V$ for all $i \in \{1,\dots,n\}$, $v(A,A') = 1$ and the sets $A' = \{a'_1,\dots,a'_n\}$, B are in the general position by Remark 6. □

Theorem 8 *If $p = n$, then $d(G^p) \leq n$.*

Proof If A, B are in the general position, then $d(G^p) \leq n-1$ by Theorem 7. If they are not in the general position, then we select a set A' according to Proposition 14. Hence $v(A,A') = 1$, $v(A',B) \leq n-1$ anh this yields $v(A,B) \leq n$. □

Theorem 9 *Let $n = 3$ and $U = V$. Then $v(A,B) = 2$ if and only if the triangles A,B are perspective (i.e. lines c_1,c_2,c_3 have one point in common).*

Proof 1. Let the triangles A,B be perspective. Then there exists a point $r \in U$, $r = c_1 \cap c_2 \cap c_3$. At least one of lines c_i, $i \in \{1,2,3\}$, must fulfil a condition $r \neq a_i, b_i$. Let c_1 be such a line. Select an arbitrary point $w_1 \notin U$ and a point a'_1 on the line a_1w_1 such that $a'_1 \neq a_1, w_1$. Lines a'_1w_1 and b_1r have a_1 in common and hence the lines a'_1b_1, w_1r have a point denoted by w_2 in common. It is obvious that there exist intersections $a'_2 = a_2w_1 \cap b_2w_2$ and

$a'_3 = a_3w_1 \cap b_3w_2$. For $A' = \{a'_1, a'_2, a'_3\}$ we get $v(A, A') = 1 = v(A', B)$ by Remark 5. Thus $v(A, B) = 2$.

2. Let $v(A, B) = 2$. Then there exist points $w_1, w_2 \notin U$, $w_1 \neq w_2$, and an independent set $A' = \{a'_1, a'_2, a'_3\} \in G^3$ with a property

$$\{a_1, a_2, a_3\} \to \{a'_1, a'_2, a'_3\} \to \{b_1, b_2, b_3\}.$$

For the points a'_i we obtain $a'_i \in a_iw_1$, $a'_i \in b_iw_2$ for all $i \in \{1, 2, 3\}$. That implies $a'_i = a_iw_1 \cap b_iw_2$, $i \in \{1, 2, 3\}$. If the lines a_iw_1 and b_iw_2 have a point in common, then also the lines a_ib_i and w_1w_2 for $i \in \{1, 2, 3\}$ have a point in common. Denote $r = w_1w_2 \cap U$ and we get $r \in c_i$ for all $i \in \{1, 2, 3\}$. Thus the triangles A, B are perspective. □

Proposition 15 *Let $n = 3$ and $U = V$. If the triangles A, B are not perspective, then $v(A, B) = 3$.*

Proof From $U = V$ we get $v(A, B) > 1$ by Remark 5. Moreover, $v(A, B) \neq 2$ by Theorem 9 and $v(A, B) \leq 3$ by Theorem 8. □

Corollary 2 *If $p = n = 3$, then $d(G^3) = 3$.*

An open problem is to determine a span for n and p fulfilling an equality $\frac{n+1}{2} \leq p < n$. The solution of that requires an analysis of rather complicated incidence relations in $\mathcal{P}^n$. As an illustration we present a particular case for $n = 4, p = 3$.

Proposition 16 *Let $n = 4$, $p = 3$. If the intersection of planes U, V is a point q and $q \notin a_ia_j$, $q \notin b_ib_j$ for all distinct $i, j \in \{1, 2, 3\}$, then the sets A, B are in the basic position.*

Proof If $U \cap V = \{q\}$, then $U + V = Q = \mathcal{P}^n$. Suppose for instance $\dim Q_3 = 2$. Then the lines a_1a_2, b_1b_2 have a point x in common. Since $x \in U \cap V$ we get $x = q$ and $q \in a_1a_2$ which is a contradiction. Thus all Q_i are hyperplanes in $\mathcal{P}^n$. According to Theorem 4 there exists a unique line intersecting all c_i. Moreover, $U \cap V \not\subseteq V_i, U_i$ for all $i \in \{1, 2, 3\}$ and from Remark 4 we get $U \cap W = V \cap W = \emptyset$. It follows from Theorem 3 that the sets A, B are in the basic position. □

Theorem 10 *If $n = 4$, then $d(G^3) = 2$.*

Proof 1. Let $U \neq V$. We select points $r \in U$, $s \in V$ such that $r, s \notin U \cap V$ and $r \notin a_ia_j, s \notin b_ib_j$ for all distinct i, j. Now let us select a line t intersecting the line rs such that $t \cap U = t \cap V = \emptyset$ and consider a plane $T = rs + t$. Then $T \cap U = \{r\}$, $T \cap V = \{s\}$. In T we select an independent set $A' = \{a'_1, a'_2, a'_3\}$ such that $r \notin a'_ia'_j$ and $s \notin a'_ia'_j$. The sets A, A' and A', B are in the basic position by Proposition 16. Thus $v(A, A') = v(A', B) = 1$ and $v(A, B) \leq 2$.

2. Let $U = V$. Then each line W intersecting all lines c_i is contained in the plane U. It follows from Theorem 3 that $v(A, B) > 1$. In U we select a point $r \notin a_ia_j$, $r \notin b_ib_j$. Now let us consider a plane T containing r such that $T \cap U = \emptyset$ and proceed analogously to 1. We have obtained that $v(A, B) \leq 2$ and thus $v(A, B) = 2$. □

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