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MV-Algebras with Additive Closure Operators

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Abstract

In the paper, closure MV-algebras (i.e. MV-algebras with additive closure operators) as generalizations of topological Boolean algebras are introduced and studied. In particular, closure MV-algebras determined by idempotent elements, connections between closure MV-algebras and induced topological Boolean algebras and closed ideals in connection with congruences of MV-algebras are examined.

Key words: Closure MV-algebra, additive closure operator, topological Boolean algebra.

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The topological Boolean algebras (or closure algebras) have been introduced and studied (see e.g. [6]) as natural generalizations of the topological spaces defined by topological closure and interior, respectively, operators. The MV-algebras which have been introduced by C. C. Chang in [1] and [2] are algebraic counterparts of the Lukasiewicz infinite valued logic similarly as the Boolean algebras are for the classical two-valued logic. Every MV-algebra $\mathcal{A}$ contains the greatest Boolean subalgebra $B(\mathcal{A})$ which is formed by the additively idempotent elements. Moreover, the operations "$\oplus$" and "$\odot$" in $B(\mathcal{A})$ coincide with the lattice operations "$\vee$" and "$\wedge$", respectively. Hence the Boolean algebras can be considered as special cases of MV-algebras. Therefore, in the paper we introduce the additive closure and multiplicative interior, respectively, operators on MV-algebras that in the case $\mathcal{A} = B(\mathcal{A})$ are exactly the topological closure and interior, respectively, operators.

In the paper, the closure MV-algebras determined by the idempotent elements of MV-algebras are studied, connections between the additive closure operators of MV-algebras and the topological closure operators of the Boolean algebras of idempotent elements are shown and connections between the congruences and the closed ideals of MV-algebras are described. Recall the notion of an MV-algebra.

Definition 1 An algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of signature $\langle 2, 1, 0\rangle$ is called an *MV-algebra* if it satisfies the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(MV2) $x \oplus y = y \oplus x$;

(MV3) $x \oplus 0 = x$;

(MV4) $\neg\neg x = x$;

(MV5) $x \oplus \neg 0 = \neg 0$;

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$;

If $\mathcal{A}$ is an MV-algebra, set $x \odot y = \neg(\neg x \oplus \neg y)$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ for any $x, y \in A$, and $1 = \neg 0$. Then $(A, \odot, 1)$ is, among others, a commutative monoid, $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, and $(A, \oplus, 0, \vee, \wedge)$ and $(A, \odot, 1, \vee, \wedge)$ are lattice ordered monoids. For further necessary results concerning MV-algebras see [3] or [7].

The following definition of an additive closure operator on an MV-algebra generalizes that of a topological closure operator on a Boolean algebra.

Definition 2

a) Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra and $Cl\colon A \to A$ a mapping. Then Cl is called *an additive closure operator* on $\mathcal{A}$ if for each $a, b \in A$:

1. $Cl(a \oplus b) = Cl(a) \oplus Cl(b)$;
2. $a \leq Cl(a)$;
3. $Cl(Cl(a)) = Cl(a)$;
4. $Cl(0) = 0$.

b) If Cl is an additive closure operator on $\mathcal{A}$ then $\mathcal{A} = (A, \oplus, \neg, 0, Cl)$ is called *a closure MV-algebra* and $Cl(a)$ is called *the closure* of an element $a \in A$. An element a is said to be *closed* if $Cl(a) = a$.

Lemma 1 *Let $\mathcal{A}$ be a closure MV-algebra. Let $Int(a) = \neg Cl(\neg a)$ for each $a \in A$. Then for any $a, b \in A$ we have $Cl(a) = \neg Int(\neg a)$ and*

1'. $Int(a \odot b) = Int(a) \odot Int(b)$;

2'. $Int(a) \leq a$;

3'. $Int(Int(a)) = Int(a)$;

4'. $Int(1) = 1$.

($Int(a)$ will be called *the interior of* a and Int: $A \to A$ is *a multiplicative interior operator* on $\mathcal{A}$. An element $a \in A$ is called *open* if $Int(a) = a$.)

Proof

1'. $Int(a \odot b) = \neg Cl(\neg(a \odot b)) = \neg Cl(\neg a \oplus \neg b) = \neg(Cl(\neg a) \oplus Cl(\neg b)) = \neg Cl(\neg a) \odot \neg Cl(\neg b) = Int(a) \odot Int(b)$;

2'. $Int(a) = \neg Cl(\neg a) \leq \neg\neg a = a$;

3'. $Int(Int(a)) = Int(\neg Cl(\neg a)) = \neg(Cl(Cl(\neg a))) = \neg Cl(\neg a) = Int(a)$;

4'. $Int(1) = \neg Cl(\neg 1) = \neg Cl(0) = \neg 0 = 1$. □

Lemma 2 *For any* $a, b \in A$, $a \leq b$ *implies* $Cl(a) \leq Cl(b)$ *and* $Int(a) \leq Int(b)$.

Proof Let $a \leq b$. Then $Cl(a \vee b) = Cl(b)$, hence $Cl((b \odot \neg a) \oplus a) = Cl(b)$. Therefore $Cl(b \odot \neg a) \oplus Cl(a) = Cl(b)$, and so $Cl(a) \leq Cl(b)$.

Similarly, from $a \leq b$ we get $Int(a) = Int(a \oplus \neg b) \odot Int(b)$, hence $Int(a) \leq Int(b)$. □

Remark 1

a) It is known ([1]) that any Boolean algebra can be considered as a special case of an MV-algebra in which the operations "$\oplus$" and "$\odot$" coincide with the lattice operations "$\vee$" and "$\wedge$", respectively. It is then obvious that the closure Boolean algebras (by Definition 2) are exactly the topological Boolean algebras in the sense of the book [6], chapter III. Hence, closure MV-algebras are natural generalizations of topological Boolean algebras.

b) If $\mathcal{A}$ is any MV-algebra then the set $B(\mathcal{A}) = \{a \in A; a \oplus a = a\}$ of additive idempotents in $\mathcal{A}$ is a sublattice of the lattice $(A, \vee, \wedge)$ that is, moreover, the greatest Boolean sublattice ([1], Theorem 1.17). Note that $B(\mathcal{A})$ is, at the same time, the set of multiplicative idempotents in $\mathcal{A}$.

Lemma 3 *If* $\mathcal{A}$ *is an* MV*-algebra and* $a \in B(\mathcal{A})$ *then*

a) $y \odot a = y \wedge a$,

b) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,

for each $x, y \in A$.

Proof

a) Let $y \leq a$. Then $a \leq y \oplus a \leq a \oplus a = a$, thus $y \oplus a = a$, and hence, by [1], Theorem 1.15, $y \odot a = y = y \wedge a$.

Let now $y \in A$ be an arbitrary element. Obviously $y \odot a \leq y, a$. Let $z \in A, z \leq y, a$. Then, by the preceding, we have $z = z \odot a \leq y \odot a$, and hence $y \odot a = y \wedge a$.

b) Since $(a \wedge x) \oplus (a \wedge y) = (a \oplus a) \wedge (x \oplus a) \wedge (a \oplus y) \wedge (x \oplus y)$, by a), $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$. □

We will show that any idempotent element a in a closure MV-algebra $\mathcal{A}$ determines a closure MV-algebra on the interval $[0, a]$.

Theorem 4 *Let $\mathcal{A} = (A, \oplus, \neg, 0, Cl)$ be a closure MV-algebra and a be an idempotent element in $\mathcal{A}$. If we put*

$$x \oplus_a y = x \oplus y, \ \ 0_a = 0, \ \ \neg_a x = \neg(x \oplus \neg a) = \neg x \odot a, \ \ Cl_a(x) = a \odot Cl(x)$$

for any $x, y \in [0, a]$, then $I(a) = ([0, a], \oplus_a, \neg_a, 0_a, Cl_a)$ is also a closure MV-algebra. In $I(a), x \odot_a y = x \odot y$ and $Int_a(x) = a \odot Int(x \oplus \neg a)$ are satisfied for any $x, y \in [0, a]$.

Proof

a) Obviously, $([0, a], \oplus_a, 0)$ is a commutative monoid. We will verify the remaining axioms of an MV-algebra.

(MV4) $\neg_a \neg_a x = \neg(\neg_a x \oplus \neg a) = \neg(\neg(x \oplus \neg a) \oplus \neg a) = (x \oplus \neg a) \odot a = x \wedge a = x.$

(MV5) $x \oplus \neg_a 0 = x \oplus a = a.$

(MV6) $\neg_a(\neg_a x \oplus y) \oplus y = \neg(\neg_a x \oplus y \oplus \neg a) \oplus y = \neg(\neg(x \oplus \neg a) \oplus y \oplus \neg a) \oplus y = ((x \oplus \neg a) \odot a \odot \neg y) \oplus y = ((x \wedge a) \odot \neg y) \oplus y = (x \odot \neg y) \oplus y = x \vee y.$
Similarly $\neg_a(x \oplus \neg_a y) \oplus x = x \vee y.$

b) We will show that Cl_a is an additive closure operator on the MV-algebra $([0, a], \oplus_a, \neg_a, 0)$. Let $x, y \in [0, a]$.

1. By Lemma 3 we get
$Cl_a(x) \oplus Cl_a(y) = (a \odot Cl(x)) \oplus (a \odot Cl(y)) = a \odot (CL(x) \oplus Cl(y)) = a \odot Cl(x \oplus y) = Cl_a(x \oplus y).$
2. $x = x \wedge a \leq Cl(x) \odot a = Cl_a(x).$
3. $Cl_a(Cl_a(x)) = a \odot Cl(a \odot Cl(x)) \leq a \odot Cl(Cl(x)) = a \odot Cl(x) = Cl_a(x)$, hence by 2 we obtain $Cl_a(Cl_a(x)) = Cl_a(x)$.
4. $Cl_a(0) = a \odot Cl(0) = a \odot 0 = 0.$

Therefore $I(a) = ([0, a], \oplus_a, \neg_a, 0, Cl_a)$ is a closure MV-algebra.

At the same time, we have for any $x, y \in [0, a]$:

$x \odot_a y = \neg_a(\neg_a x \oplus_a \neg_a y) = \neg_a(\neg(x \oplus \neg a) \oplus \neg(y \oplus \neg a)) = \neg(\neg(x \oplus \neg a) \oplus \neg(y \oplus \neg a) \oplus \neg a) = (x \oplus \neg a) \odot (y \oplus \neg a) \odot a = (x \wedge a) \odot (y \wedge a) = x \odot y;$

$Int_a(x) = \neg_a Cl_a(\neg_a x) = \neg((a \odot Cl(\neg(x \oplus \neg a))) \oplus \neg a) = (\neg a \oplus \neg Cl(\neg(x \oplus \neg a))) \odot a = (\neg a \oplus Int(x \oplus \neg a)) \odot a = a \wedge Int(x \oplus \neg a) = a \odot Int(x \oplus \neg a).$ □

Definition 3 A subalgebra $\mathcal{C}$ of a closure MV-algebra $\mathcal{A}$ is called a *closure subalgebra* if $Cl(x) \in C$ for every $x \in C$.

Note Obviously, a subalgebra $\mathcal{C}$ is a closure subalgebra if and only if $Int(x) \in C$ for every $x \in C$.

Theorem 5 *The Boolean algebra $B(\mathcal{A})$ of additive idempotents of a closure MV-algebra $\mathcal{A}$ is a closure subalgebra of $\mathcal{A}$. That means, $B(\mathcal{A})$ endowed with the restriction of the operator Cl on $B(\mathcal{A})$ is a topological Boolean algebra.*

Proof Let $a \in B(\mathcal{A})$. Then $Cl(a) \oplus Cl(a) = Cl(a \oplus a) = Cl(a)$, hence $Cl(a) \in B(\mathcal{A})$. □

Let us now show that in the case of *complete MV-algebras* (i.e. such MV-algebras which are complete lattices with respect to the induced orders), every topological closure operator on the Boolean algebra of additively idempotent elements can be extended to a closure operator on the whole MV-algebra.

Theorem 6 *Let $\mathcal{A}$ be a closure MV-algebra and φ be a topological closure operator on the Boolean algebra $B(\mathcal{A})$. Then there is an additive closure operator Cl_φ on $\mathcal{A}$ such that its restriction on $B(\mathcal{A})$ is equal to φ.*

Proof Firstly we will show that $B = B(\mathcal{A})$ is a complete sublattice of $\mathcal{A}$. If $y_i \in B$, $i \in I$, and $y = \inf_A\{y_i; i \in I\}$, then $y \oplus y = \bigwedge_{i\in I} y_i \oplus \bigwedge_{i \in I} y_i$, hence $y \oplus y \leq y_j \oplus y_j$ for every $j \in I$, and thus $y \oplus y \leq y_j \oplus y_j = y_j$ for every $j \in I$. Therefore $y \oplus y \leq \bigwedge_{i\in I} y_i = y$, that means $y \in B$.

Dually for suprema.

Now, let $\varphi : B \to B$ be a topological closure operator on the Boolean algebra B. Let $\overline{\varphi}(x) = \varphi(\bigwedge(a; a \in B, x \leq a))$ for any $x \in A$. We will verify that $\overline{\varphi}$ is an additive closure operator on $\mathcal{A}$. Let $x, y \in A$.

1. $\overline{\varphi}(x \oplus y) = \varphi(\bigwedge(a; a \in B, x \oplus y \leq a))$,
   $\overline{\varphi}(x) \oplus \overline{\varphi}(y) = \varphi(\bigwedge(b; b \in B, x \leq b)) \oplus \varphi(\bigwedge(c; c \in B, y \leq c))$.

   It is clear that for any $a \in B$ satisfying $x \oplus y \leq a$, we have $\bigwedge(b; b \in B, x \leq b) \leq a$ and $\bigwedge(c; c \in B, y \leq c) \leq a$, hence $\bigwedge(b; b \in B, x \leq b) \oplus \bigwedge(c; c \in B, y \leq c) \leq a \oplus a = a$, therefore $\bigwedge(b; b \in B, x \leq b) \oplus \bigwedge(c; c \in B, y \leq c) \leq \bigwedge(a; a \in B, x \oplus y \leq a)$.

   Conversely, $x \oplus y \leq \bigwedge(b; b \in B, x \leq b) \oplus \bigwedge(c; c \in B, y \leq c)$, and thus $\bigwedge(a; a \in B, x \oplus y \leq a) \leq \bigwedge(b; b \in B, x \leq b) \oplus \bigwedge(c; c \in B, y \leq c)$.
   From this we get $\overline{\varphi}(x \oplus y) = \overline{\varphi}(x) \oplus \overline{\varphi}(y)$.

2. $x \leq \overline{\varphi}(x)$ by the definition of $\overline{\varphi}$.

3. $\overline{\varphi}(\overline{\varphi}(x)) = \overline{\varphi}(\varphi(\bigwedge(a; a \in B, x \leq a))) = \varphi(\varphi(\bigwedge(a; a \in B, x \leq a))) = \varphi(\bigwedge(a; a \in B, x \leq a)) = \overline{\varphi}(x)$.

4. $0 \in B$, hence $\overline{\varphi}(0) = \varphi(0) = 0$.

   Let us denote $Cl_\varphi = \overline{\varphi}$. Then Cl_φ is an additive closure operator on $\mathcal{A}$ and its restriction on B equals φ. □

Let us recall that if $\mathcal{A}$ is an MV-algebra and $\emptyset \neq I \subseteq A$ then I is called an *ideal* of $\mathcal{A}$, if

(i) $a \oplus b \in I$ for any $a, b \in I$ and

(ii) $x \leq a$ implies $x \in I$ for any $x \in A, a \in I$.

It is known ([1], Theorem 4.3, [3], Proposition 1.2.6) that ideals in MV-algebras are in a one-to-one correspondence with congruences. If I is an ideal in $\mathcal{A}$ then for the congruence θ_I corresponding to I, $(x, y) \in \theta_I$ if and only if $(x \odot \neg y) \oplus (\neg x \odot y) \in I$, for any $x, y \in A$. Denote by $\mathcal{A}/I = \mathcal{A}/\theta_I$ the factor MV-algebra of $\mathcal{A}$ by θ_I and let $\overline{x}$ denote the class of A/I containing x.

Definition 4 *Let $\mathcal{A}$ be a closure MV-algebra and I be an ideal of $\mathcal{A}$. Then I is called a c-ideal if $Cl(a) \in I$ for every $a \in I$.*

If $\mathcal{A}$ is a closure MV-algebra and I is an ideal of $\mathcal{A}$, set $Cl(\overline{x}) = \overline{Cl(x)}$ for every $x \in A$.

Theorem 7 *If $\mathcal{A}$ is a closure MV-algebra and I is a c-ideal of $\mathcal{A}$ then the factor MV-algebra $\mathcal{A}/I$ endowed with Cl is a closure MV-algebra.*

Proof Let $x, y \in A$, $\overline{x} = \overline{y}$. Then $(x, y) \in \theta_I$, hence $(x \odot \neg y) \oplus (\neg x \odot y) \in I$. Thus also $x \odot \neg y, \neg x \odot y \in I$, and therefore $Cl(x \odot \neg y), Cl(\neg x \odot y) \in I$. At the same time, $Cl(y) \oplus Cl(x \odot \neg y) = Cl(y \oplus (x \odot \neg y)) = Cl(x \vee y) \geq Cl(x)$, and since $\mathcal{A}$ is by [4], [5] a DRl-monoid, we obtain $Cl(x \odot \neg y) \geq Cl(x) - Cl(y) = Cl(x) \odot \neg Cl(y)$. And since $Cl(x \odot \neg y) \in I$, we also have $Cl(x) \odot \neg Cl(y) \in I$.

Similarly $\neg Cl(x) \odot Cl(y) \in I$, and hence $(Cl(x) \odot \neg Cl(y)) \oplus (\neg Cl(x) \odot Cl(y)) \in I$. That means $(Cl(x), Cl(y)) \in \theta_I$, and so the definition of the unary operation Cl on A/I is correct. (Therefore we have shown that θ_I is also a congruence of the closure MV-algebra.) Moreover, $Cl : A/I \to A/I$ satisfies all four conditions of additive closure operators. □

Corollary 8 *There is a one-to-one correspondence between the c-ideals and congruences of closure MV-algebras.*

Definition 5 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be closure MV-algebras and let $h : \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism of MV-algebras. Then h is called a *c-homomorphism* (or a *homomorphism of closure MV-algebras*) if

$$h(Cl(x)) = Cl(h(x))$$

for each $x \in A_1$.

Note It is obvious that a homomorphism h of MV-algebras a c-homomorphism if and only if

$$h(Int(x)) = Int(h(x))$$

for each $x \in A_1$.

Theorem 9 *Let $\mathcal{A}$ be a closure MV-algebra, a an open idempotent element in $\mathcal{A}$ and $h : A \to I(a)$ the mapping such that $h(x) = a \odot x$ for every $x \in A$. Then h is a surjective c-homomorphism of $\mathcal{A}$ onto $I(a)$.*

Proof Let $x, y \in A$. Then

$$\begin{aligned} h(x \odot y) &= a \odot (x \odot y) = (a \odot x) \odot (a \odot y) = h(x) \odot h(y) = h(x) \odot_a h(y), \\ \neg_a h(x) &= \neg(h(x) \oplus \neg a) = \neg((a \odot x) \oplus \neg a) = \neg(\neg a \vee x) = a \wedge \neg x, \end{aligned}$$

and hence by Lemma 3, $\neg_a h(x) = a \odot \neg x = h(\neg x)$. Moreover, $h(0) = a \odot 0 = 0$.

Hence h is a homomorphism of MV-algebras, and since $x = a \odot x = h(x)$ for each $x \in [0, a]$, h is surjective.

We will show that h is a c-homomorphism. Since a is open,

$$h(Int(x)) = a \odot Int(x) = Int(a) \odot Int(x) = Int(a \odot x) = Int(h(x)).$$

Let $y \leq a$. Then $Int(y) = Int(a \wedge y) = Int(a \odot (y \oplus \neg a)) = Int(a) \odot Int(y \oplus \neg a) = a \odot Int(y \oplus \neg a) = Int_a(y)$.

From this we get $Int(h(x)) = Int_a(h(x))$, and thus $h(Int(x)) = Int_a(h(x))$ for each $x \in A$. □

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