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# Some Notes to Certain Modification of the Oseen Problem *

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**Abstract**

The aim of this paper is to generalize the results of Farwig, Novotný and Pokorný presented in [3]. We consider certain class of modifications to the Oseen problem and show that the fundamental solution to the (modified) Oseen problem admits similar asymptotic properties as the fundamental solution to the (classical) Oseen problem.

**Key words:** Fundamental solution to the system of PDE's, Oseen problem, anisotropic asymptotic structure.

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## 1 Introduction

Let us consider the following problem

$$\left.\begin{array}{c} A(\mathbf{u}) + 2\lambda\frac{\partial \mathbf{u}}{\partial x_1} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{array}\right\} \text{ in } \Omega \subseteq \mathbb{R}^N, \qquad \mathbf{u} = \mathbf{u}_* \text{ at } \partial\Omega \ (\text{if } \Omega \neq \mathbb{R}^N), \tag{1.1}$$

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with $\Omega$ an unbounded domain in $\mathbb{R}^N$, $N = 2, 3$ and $A$ an elliptic operator given by

$$A = -\Delta + A_0 \,. \tag{1.2}$$

The operator $A_0$ is defined as follows

$$A_0 = \sum_{k,l=1\ k\le l}^{N} \alpha_{kl} \frac{\partial^2}{\partial x_k \partial x_l} \,, \tag{1.3}$$

$\alpha_{kl} \in \mathbb{R}$ for $k, l = 1, \dots, N$, $k \le l$; moreover, because of the ellipticity, $\alpha_{kk} < 1$ for $k = 1, \dots N$ (further conditions for $\alpha_{kl}$ will be stated later).

It is well known (cf. e.g. [1]) that many properties of solution to (1.1) are connected with the asymptotic properties of the fundamental solution to this system, see (1.4) below. Similar problem with $A(\mathbf{u}) = -\Delta\mathbf{u} + \alpha\frac{\partial^2 \mathbf{u}}{\partial x_1^2}$ was studied in [3] and in [6], where the authors took full advantage of the fact that the asymptotic behaviour of the derivative of the Oseen fundamental tensor with respect to the first variable is substantially better than with respect to other variables. These results were applied to studies of asymptotic properties of solutions describing steady flow of viscoelastic fluid in exterior plane and three-dimensional domains, see [4], [5].

Now we will be concerned with the study of the fundamental solution to system (1.1). As for notation, the symbols $\boldsymbol{\mathcal{O}}^\alpha$ and $\mathbf{e}$ with the components $\mathcal{O}^\alpha_{ij}$ and $e_j$, $i, j = 1, \dots, N$ denote the fundamental solution to (1.1). Let us consider in $\mathbb{R}^N$, $N = 2, 3$

$$\begin{aligned} A(\mathcal{O}^\alpha_{ij})(\mathbf{x}; 2\lambda) + 2\lambda \tfrac{\partial}{\partial x_1} \mathcal{O}^\alpha_{ij}(\mathbf{x}; 2\lambda) + \tfrac{\partial e_j}{\partial x_i}(\mathbf{x}) &= \delta_{ij}\delta(\mathbf{x}) \\ \tfrac{\partial \mathcal{O}^\alpha_{ij}}{\partial x_i}(\mathbf{x}; 2\lambda) &= 0 \end{aligned} \tag{1.4}$$

in the sense of $\mathcal{S}'$, i.e. the dual to the Schwartz class of functions, and with $\delta(\cdot)$ the Dirac delta distribution having support at $\mathbf{0}$. The basic idea how to construct the fundamental solution to (1.1), similarly as in [3], is to search the solution in the form

$$\boldsymbol{\mathcal{O}}^\alpha(\mathbf{x}; 2\lambda) = \boldsymbol{\mathcal{O}}(\mathbf{x}; 2\lambda) + \mathbf{E}^\alpha(\mathbf{x}; 2\lambda) \,, \tag{1.5}$$

where $\boldsymbol{\mathcal{O}}$ is the fundamental solution to the classical Oseen problem, i.e. system (1.1) with $\alpha_{kl} = 0$.

## 2 Notation, basic properties

Throughout this paper are vectors, vector– and tensor–valued functions printed boldfaced. Let $\Omega \subset \mathbb{R}^N$ be a domain. We use also the standard summation convention, i.e. we sum over twice repeated indices, from 1 to $N$. Further the Lebesgue spaces are denoted by $L^p(\Omega)$, the Sobolev spaces are denoted by $W^{k,p}(\Omega)$, both equipped with the standard norms $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, respectively.

Let $\mathcal{D}(\Omega)$ denote the space of compactly supported smooth functions which is equipped with the standard topology; its dual space called the space of distributions we denote by $\mathcal{D}'(\Omega)$. Let $T, G \in \mathcal{D}'(\Omega)$. The distributional derivative will be denoted by $D^\alpha T$, the direct product of distributions by $T \times G$ and the convolution of distributions by $T * G$, see e.g. [9] for further details.

Let us consider the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ defined by

$$\mathcal{S}(\mathbb{R}^N) = \{\varphi \in C^\infty(\mathbb{R}^N);\ \forall \alpha, \beta \in \mathbb{N}^N : \sup_{\mathbf{x} \in \mathbb{R}^N} |\mathbf{x}^\beta D^\alpha \varphi(\mathbf{x})| = C_{\alpha,\beta} < \infty\}.$$

The dual space called the space of tempered distributions is denoted by $\mathcal{S}'$. The derivative, the direct product and the convolution of tempered distributions are denoted identically as for distributions. We recall well known theorem about the convolution $f * g$ of the functions $f$ and $g$ belonging to certain $L^q(\mathbb{R}^N)$ spaces.

**Lemma 2.1** (Young)
*Let $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, $1 \le p \le \infty$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} \ge 1$. Then $f * g \in L^r(\mathbb{R}^N)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and*

$$||f * g||_r \le ||f||_p ||g||_q .$$

**Proof** See e.g [7]. □

Let us also recall some known facts about the fundamental solution to the classical Oseen problem, given by

$$\begin{aligned} (-\Delta + 2\lambda \tfrac{\partial}{\partial x_1}) \mathcal{O}_{ij}(\mathbf{x}; 2\lambda) + \tfrac{\partial e_j}{\partial x_i}(\mathbf{x}) &= \delta_{ij}\delta(\mathbf{x}) \\ \tfrac{\partial \mathcal{O}_{ij}}{\partial x_i}(\mathbf{x}; 2\lambda) = 0 . \end{aligned} \tag{2.1}$$

The fundamental pressure $\mathbf{e}$ can be taken in the form

$$e_j(\mathbf{x}) = \frac{\partial}{\partial x_j} \mathcal{E}(|\mathbf{x}|) \quad j = 1, \dots, N, \tag{2.2}$$

where $\mathcal{E}$ denotes the fundamental solution to the Laplace operator.
As for the explicit form of $\mathcal{O}$, see [1]. Let us only note that $\mathcal{O}$ is a symmetric tensor–valued function which is smooth outside the origin.

Denote

$$s(\mathbf{x}) = |\mathbf{x}| - x_1 \text{ and } \mathbf{x}' = (x_2, \dots x_N).$$

It can be proved that

$$\begin{aligned} &\text{(i) for } x_1 > 0 \text{ it holds } \tfrac{1}{2}\tfrac{|\mathbf{x}'|^2}{|\mathbf{x}|} \le s(\mathbf{x}) \le \tfrac{|\mathbf{x}'|^2}{|\mathbf{x}|} \\ &\text{(ii) for } x_1 \le 0 \text{ it holds } |\mathbf{x}| \le s(\mathbf{x}) \le 2|\mathbf{x}| . \end{aligned} \tag{2.3}$$

Finally we mention the homogeneity and asymptotic properties of the fundamental Oseen tensor. For the proofs we refer the reader to [1].

**Lemma 2.2** *Let $\mathcal{O}$ be the fundamental Oseen tensor in $\mathbb{R}^N$. Then*

$$\mathcal{O}_{ij}(\mathbf{x}; 2\lambda) = (2\lambda)^{N-2} \mathcal{O}_{ij}(2\lambda \mathbf{x}; 1) \quad i, j = 1, \dots, N .$$

**Lemma 2.3** *Let $N = 2$.*
*(i) Let $0 < |\mathbf{x}| \le \varepsilon$, $\varepsilon$ sufficiently small. Then there exists $C = C(\varepsilon)$ such that*

$$\begin{aligned} |\mathcal{O}_{12}(\mathbf{x};1)| &\le C \\ |\mathcal{O}_{11}(\mathbf{x};1)|,\ |\mathcal{O}_{22}(\mathbf{x};1)| &\le C\,|\ln|\mathbf{x}|| \\ |\nabla^k \mathcal{O}_{ij}(\mathbf{x};1)| &\le \tfrac{C}{|\mathbf{x}|^k} \quad k \ge 1,\ i,j = 1,2\,. \end{aligned}$$

*(ii) Let $|\mathbf{x}| > R$ with $R$ sufficiently large. Then there exists $C = C(R)$ such that*

$$\begin{aligned} \left|\tfrac{\partial^k}{\partial x_2^k}\mathcal{O}_{11}(\mathbf{x};1)\right| &\le \frac{C}{|\mathbf{x}|^{\frac{1+k}{2}}(1+s)^{\frac{1+k}{2}}} \quad k \ge 0 \\ |\widehat{\nabla}^k \mathcal{O}_{ij}(\mathbf{x};1)| &\le \frac{C}{|\mathbf{x}|^{\frac{2+k}{2}}(1+s)^{\frac{k}{2}}} \quad k \ge 0,\ i,j = 1,2\,, \end{aligned}$$

*where $\widehat{\nabla}^k \mathcal{O}_{ij}$ contains all derivatives of the $k$-th order of $\mathcal{O}$ except of $\frac{\partial^k \mathcal{O}_{11}}{\partial x_2^k}$.*

Let us note that in the case $(ii)$ we could get in some cases better uniform behaviour; nonetheless, in applications, it usually does not play any role and thus we skip it.

**Lemma 2.4** *Let $N = 3$.*
*(i) Let $0 < |\mathbf{x}| \le \varepsilon$, $\varepsilon$ sufficiently small. Then there exists $C = C(\varepsilon)$ such that*

$$|\nabla^k \mathcal{O}_{ij}(\mathbf{x};1)| \le \frac{C}{|\mathbf{x}|^{k+1}} \quad k \ge 0,\ i,j = 1,2,3\,.$$

*(ii) Let $|\mathbf{x}| > R$ with $R$ sufficiently large. Then there exists $C = C(R)$ such that*

$$\left|\frac{\partial^{k+l+m}}{\partial x_1^k \partial x_2^l \partial x_3^m}\mathcal{O}_{ij}(\mathbf{x};1)\right| \le \frac{C}{|\mathbf{x}|^{\frac{2+2k+l+m}{2}}(1+s)^{\frac{2+l+m}{2}}} \quad k,l,m \ge 0\,.$$

# 3 Main results

Using the definition of the fundamental solution to the Oseen problem (see (2.1)) and identity (1.5) we can rewrite (1.4) into

$$\begin{aligned} \Big[\Delta - A_0 - 2\lambda\tfrac{\partial}{\partial x_1}\Big] E^{\alpha}_{ij}(\mathbf{x};2\lambda) &= A_0 \mathcal{O}_{ij}(\mathbf{x};2\lambda) \\ \tfrac{\partial E^{\alpha}_{ij}(\mathbf{x};2\lambda)}{\partial x_i} &= 0\,. \end{aligned} \tag{3.1}$$

It is obvious that $e_j$ can be taken in the same form as for the classical Oseen problem, see (2.2). Our next goal is to determine the fundamental solution $E^*$ to $(3.1)_1$, it means the solution to the problem

$$\Big[\Delta - A_0 - 2\lambda\frac{\partial}{\partial x_1}\Big] E^*(\mathbf{x};2\lambda) = \delta(\mathbf{x})\,. \tag{3.2}$$

The construction of $E^*$ is based on the knowledge of the solution to another problem in $\mathbb{R}^N$

$$\Big[\Delta - 2\lambda\frac{\partial}{\partial x_1}\Big] F^*(\mathbf{x};2\lambda) = \delta(\mathbf{x})\,. \tag{3.3}$$

**Lemma 3.1** *The solution to (3.3) can be taken in the following form*

$$F^*(\mathbf{x}; 2\lambda) = -\frac{1}{N-1}\sum_{i=1}^{N} \mathcal{O}_{ii}(\mathbf{x}; 2\lambda), \tag{3.4}$$

*where $\mathcal{O}(\mathbf{x}; 2\lambda)$ is the fundamental solution to the Oseen problem.*

**Proof** It is a special case of Lemma 2.1 in [3]. □

Now we need to change the variables so that operator $\Delta - A_0 - 2\lambda\frac{\partial}{\partial x_1}$ acting on the $\mathbf{x}$-variables becomes the operator $\Delta - 2\lambda\frac{\partial}{\partial X_1}$ acting on new $\mathbf{X}$-variables. We choose linear change of variables and using the standard procedure we can get the desired relation.

To shorten notation, let us denote

$$\begin{gathered}
\beta_1 = 1-\alpha_{11} \quad \beta_2 = 1-\alpha_{22} \quad \beta_3 = 1-\alpha_{33} \\
A_1 = 4\beta_2\beta_3 - \alpha_{23}^2 \quad B_1 = 2\beta_3\alpha_{12} + \alpha_{13}\alpha_{23} \quad C_1 = 2\beta_2\alpha_{13} + \alpha_{12}\alpha_{23} \\
K_1 = 4\beta_1 - \frac{\alpha_{12}^2}{\beta_2} \quad K_3 = \beta_2 - \frac{\alpha_{23}^2}{4\beta_3} \\
K_2 = (\alpha_{23}^2 - 4\beta_2\beta_3)(\beta_3\alpha_{12}^2 + \beta_2\alpha_{13}^2 + \beta_1\alpha_{23}^2 + \alpha_{12}\alpha_{13}\alpha_{23} - 4\beta_1\beta_2\beta_3).
\end{gathered} \tag{3.5}$$

**Lemma 3.2** *Let the following equations hold*
*(i) for $N = 2$*

$$\begin{aligned}
X_1 &= \tfrac{2}{\sqrt{K_1}}x_1 + \tfrac{\alpha_{12}}{\beta_2\sqrt{K_1}}x_2 \\
X_2 &= \tfrac{x_2}{\sqrt{\beta_2}}
\end{aligned} \tag{3.6}$$

*(ii) for $N = 3$*

$$\begin{aligned}
X_1 &= \tfrac{A_1}{\sqrt{K_2}}x_1 + \tfrac{B_1}{\sqrt{K_2}}x_2 + \tfrac{C_1}{\sqrt{K_2}}x_3 \\
X_2 &= \tfrac{1}{\sqrt{K_3}}x_2 + \tfrac{\alpha_{23}}{2\beta_3\sqrt{K_3}}x_3 \\
X_3 &= \tfrac{x_3}{\sqrt{\beta_3}}.
\end{aligned} \tag{3.7}$$

*Then the operator $A$ given by (1.2) and (1.3) becomes the Laplace operator acting on the $\mathbf{X}$-variables.*

**Remark 3.1** Note that we obtain additional conditions for $\alpha_{kl}$, $k, l = 1, \dots, N$, $k \le l$, namely $K_j > 0$, $j = 1, 2, 3$ (see (3.6) and (3.7)). It is obvious that these conditions are fulfilled if and only if the operator $A$ is elliptic.

Let us denote for $N = 2$

$$\bar{\lambda} = \frac{2\lambda}{\sqrt{K_1}}$$

and for $N = 3$

$$\bar{\lambda} = \frac{A_1\lambda}{\sqrt{K_2}}.$$

**Lemma 3.3** *Let $N = 2, 3$. Then $E^*$ defined by*

$$E^*(\mathbf{x}; 2\lambda) = -\frac{C(N,\alpha)}{N-1} \sum_{i=1}^{N} \mathcal{O}_{ii}(\mathbf{X}; 2\bar{\lambda}) \tag{3.8}$$

*verifies equality (3.2), equality (3.8) is taken in the sense of $\mathcal{S}'$, variables $\mathbf{x}$ and $\mathbf{X}$ are connected by equations (3.6) or (3.7). The constants $C(2,\alpha)$ and $C(3,\alpha)$ are determined by the following equalities*

$$\begin{aligned} C(2,\alpha) &= \tfrac{2}{\sqrt{K_1}\sqrt{\beta_2}} \\ C(3,\alpha) &= \tfrac{A_1}{\sqrt{K_2}\sqrt{K_3}\sqrt{\beta_3}} \,. \end{aligned} \tag{3.9}$$

**Proof** Let $N = 2$. We put

$$E^*(\mathbf{x}; 2\lambda) = \frac{2}{\sqrt{K_1}\sqrt{\beta_2}} F^*(\mathbf{X}(\mathbf{x}); 2\bar{\lambda}) \,, \tag{3.10}$$

where $F^*$ is the solution to (3.3) and variables $\mathbf{x}$ and $\mathbf{X}$ are connected by (3.6). Then

$$\begin{aligned} \Big\langle (\Delta_{\mathbf{X}} - 2\bar{\lambda}\tfrac{\partial}{\partial X_1}) F^*(\mathbf{X}; 2\bar{\lambda}), \varphi(\mathbf{X}) \Big\rangle &= \Big\langle F^*(\mathbf{X}; 2\bar{\lambda}), (\Delta_{\mathbf{X}} + 2\bar{\lambda}\tfrac{\partial}{\partial X_1}) \varphi(\mathbf{X}) \Big\rangle \\ &= \tfrac{2}{\sqrt{K_1}\sqrt{\beta_2}} \Big\langle F^*(\mathbf{X}(\mathbf{x}); 2\bar{\lambda}), (\Delta_{\mathbf{x}} - A_0 + 2\lambda\tfrac{\partial}{\partial x_1}) \varphi(\mathbf{x}) \Big\rangle \\ &= \Big\langle E^*(\mathbf{x}; 2\lambda), (\Delta_{\mathbf{x}} - A_0 + 2\lambda\tfrac{\partial}{\partial x_1}) \varphi(\mathbf{x}) \Big\rangle \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$. In virtue of Lemma 3.1, it is seen that $E^*$ defined by (3.8) verifies (3.2). The proof for $N = 3$ is analogous. □

Let us mention two important consequences of change of variables stated above.

**Lemma 3.4** *Let (3.6) and (3.7) hold. Then there exist $L_i(N) > 0$, $i = 1, 2$ so that*

$$L_1(N)|\mathbf{X}| \le |\mathbf{x}| \le L_2(N)|\mathbf{X}| \,. \tag{3.11}$$

**Proof** The proof comes easily applying the Young inequality. □

**Lemma 3.5** *Let (3.6) and (3.7) hold. Then there exist constants $M_i(N) > 0$, $i = 1, \dots, 8$, so that the following estimates hold.*

(*i*) *Let $X_1 \ge 0$ and $x_1 \ge 0$. Then $M_1(N)s(\mathbf{X}) \le s(\mathbf{x}) \le M_2(N)s(\mathbf{X})$.*

(*ii*) *Let $X_1 < 0$ and $x_1 < 0$. Then $M_3(N)s(\mathbf{X}) \le s(\mathbf{x}) \le M_4(N)s(\mathbf{X})$.*

(*iii*) *Let $X_1 \ge 0$ and $x_1 < 0$. Then $M_5(N)s(\mathbf{X}) \le s(\mathbf{x}) \le M_6(N)s(\mathbf{X})$.*

(*iv*) *Let $X_1 < 0$ and $x_1 \ge 0$. Then $M_7(N)s(\mathbf{X}) \le s(\mathbf{x}) \le M_8(N)s(\mathbf{X})$,*

*where $s(\mathbf{x}) = |\mathbf{x}| - x_1$ (see also Section 2).*

**Proof** Let $N = 2$. We show for instance the case (iii).

Obviously ($|\mathbf{x}'| = |x_2|$) $|x_2| = |X_2|\sqrt{\beta_2}$. We have

$$s(\mathbf{X}) \leq |\mathbf{X}| \leq \frac{|\mathbf{x}|}{L_1} \leq \frac{s(\mathbf{x})}{L_1}.$$

For showing the inverse inequality we need to prove that there exists a constant $C > 0$ so that $|\mathbf{X}|^2 \leq C|X_2|^2$. The equations of change of variables (3.6) can be rewritten into

$$\begin{aligned} x_1 &= \tfrac{\sqrt{K_1}}{2}X_1 - \tfrac{\alpha_{12}}{2\sqrt{\beta_2}}X_2 \\ x_2 &= \sqrt{\beta_2}X_2\,. \end{aligned} \tag{3.12}$$

As $X_1 \geq 0$ and $x_1 < 0$, we see that

$$\begin{aligned} 0 \leq \tfrac{\sqrt{K_1}}{2}X_1 &< \tfrac{\alpha_{12}}{2\sqrt{\beta_2}}X_2 \\ X_1^2 &< \tfrac{\alpha_{12}^2}{K_1\beta_2}X_2^2\,. \end{aligned}$$

Hence

$$X_1^2 + X_2^2 \leq \frac{\alpha_{12}^2 + K_1\beta_2}{K_1\beta_2}X_2^2\,.$$

The rest of the proof is straightforward

$$s(\mathbf{x}) \leq 2|\mathbf{x}| \leq 2L_2|\mathbf{X}| \leq 2L_2C\frac{|X_2|^2}{|\mathbf{X}|} \leq 4L_2Cs(\mathbf{X})\,.$$

The proof for $N = 3$ is entirely analogous. □

**Remark 3.2** The importance of two preliminary lemmas is in the fact that the asymptotic properties estimated in the variable $\mathbf{X}$ hold in the variable $\mathbf{x}$ too.

Now we shall formulate the lemma connected with the fundamental solution to (3.2). For the proof we refer the reader to [1].

**Lemma 3.6** *(a) Let $N = 2$. Then*

(i) $\mathcal{O}_{11} \in L^p(\mathbb{R}^2)$ *for* $p \in (3, \infty)$

(ii) $\mathcal{O}_{ij} \in L^p(\mathbb{R}^2)$ *for* $p \in (2, \infty)$, $i + j \geq 3$

(iii) $\frac{\partial \mathcal{O}_{11}}{\partial x_2} \in L^p(\mathbb{R}^2)$ *for* $p \in (\frac{3}{2}, 2)$

(iv) ($\nabla \boldsymbol{\mathcal{O}}$ *except of* $\frac{\partial \mathcal{O}_{11}}{\partial x_2}$) $\in L^p(\mathbb{R}^2)$ *for* $p \in (1, 2)$.

*(b) Let $N = 3$. Then*

(i) $\mathcal{O}_{ij} \in L^p(\mathbb{R}^3)$ *for* $p \in (2, 3)$, $i, j = 1, 2, 3$

(ii) $\frac{\partial \mathcal{O}_{ij}}{\partial x_1} \in L^p(\mathbb{R}^3)$ *for* $p \in (1, \frac{3}{2})$, $i, j = 1, 2, 3$

(iii) $\frac{\partial \mathcal{O}_{ij}}{\partial x_k} \in L^p(\mathbb{R}^3)$ *for* $p \in (\frac{4}{3}, \frac{3}{2})$, $i, j = 1, 2, 3$ *and* $k = 2, 3$.

**Corollary 3.1** *(a) Let $N = 2$. Then*

$$\begin{aligned} E^* &\in L^p(\mathbb{R}^2) \text{ for } p \in (3, \infty) \\ \tfrac{\partial E^*}{\partial x_1} &\in L^p(\mathbb{R}^2) \text{ for } p \in (1, 2) \\ \tfrac{\partial E^*}{\partial x_2} &\in L^p(\mathbb{R}^2) \text{ for } p \in (\tfrac{3}{2}, 2). \end{aligned}$$

*(b) Let $N = 3$. Then*

$$\begin{aligned} E^* &\in L^p(\mathbb{R}^3) \text{ for } p \in (2, 3) \\ \tfrac{\partial E^*}{\partial x_1} &\in L^p(\mathbb{R}^3) \text{ for } p \in (1, \tfrac{3}{2}) \\ \tfrac{\partial E^*}{\partial x_k} &\in L^p(\mathbb{R}^3) \text{ for } p \in (\tfrac{4}{3}, \tfrac{3}{2}),\ k = 2, 3. \end{aligned}$$

**Proof** It is the consequence of Lemmas 3.3 and 3.6. □

We are now in a position to show the explicit formula for $E^\alpha_{ij}$.

**Theorem 3.1** *Let $N = 2, 3$. Then the solution to $(3.1)_1$, $\mathbf{E}^\alpha$, can be expressed in the form*

$$E^\alpha_{ij}(\mathbf{x}; 2\lambda) = \sum_{k,l=1\ k\leq l}^{N} \alpha_{kl} \int_{\mathbb{R}^N} \frac{\partial}{\partial x_k} E^*(\mathbf{x} - \mathbf{y}; 2\lambda) \frac{\partial}{\partial y_l} O_{ij}(\mathbf{y}; 2\lambda)\, d\mathbf{y}\,, \qquad (3.13)$$

*$i, j = 1, \ldots, N$, the convolutions on the right-hand side are taken in sense of convolution in $L^p$ spaces.*

**Proof** Combining the Young theorem (see Theorem 2.1) with Lemma 3.6 and with Corollary 3.1 we obtain that for $N = 2$, $\mathbf{E}^\alpha$ defined by (3.13) belongs to $L^p(\mathbb{R}^2)$ for $p \in (3, \infty)$ and for $N = 3$, $\mathbf{E}^\alpha$ belongs to $L^p(\mathbb{R}^3)$ for $p \in (2, 3)$, hence it is obvious that the convolutions in (3.13) are well defined in the sense of $\mathcal{S}'$. We proceed to show that $\mathbf{E}^\alpha$ defined by (3.13) satisfies $(3.1)_1$ in the sense of $\mathcal{S}'$. Let us fix arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and let $\eta_m \in \mathcal{D}(\mathbb{R}^N)$ be arbitrary sequence tending to 1 in the sense of the space $C^\infty(\mathbb{R}^N)$.[1] Throughout the proof, $A_{\mathbf{x}}$ and $A'_{\mathbf{x}}$ denote the operator $\Delta_{\mathbf{x}} - A_0 + 2\lambda\frac{\partial}{\partial x_1}$, and $\Delta_{\mathbf{x}} - A_0 - 2\lambda\frac{\partial}{\partial x_1}$, respectively,

[1] We say that $u_m \to 0$ in $C^\infty(\mathbb{R}^N)$ if $D^\alpha u_m \rightrightarrows 0$ $\forall \alpha \in \mathbb{N}^N$ on each compact subdomain of $\mathbb{R}^N$, cf. [9].

acting on the $\mathbf{x}$-variables. Then

$$\begin{aligned}
&\Big\langle A'_{\mathbf{x}} E^{\alpha}_{ij}(\mathbf{x}), \varphi(\mathbf{x})\Big\rangle = \Big\langle E^{\alpha}_{ij}(\mathbf{x}), A_{\mathbf{x}}\varphi(\mathbf{x})\Big\rangle \\
&= \lim_{m\to\infty} \sum_{k,l=1\ k\le l}^{N} \alpha_{kl}\Big\langle \tfrac{\partial}{\partial x_k} E^*(\mathbf{x};2\lambda) \times \tfrac{\partial}{\partial y_l} O_{ij}(\mathbf{y};2\lambda), \eta_m(\mathbf{x},\mathbf{y})\Big(A_{\mathbf{x}}\varphi(\mathbf{x}+\mathbf{y})\Big)\Big\rangle \\
&= \lim_{m\to\infty} \sum_{k,l=1\ k\le l}^{N} \alpha_{kl}\Big\langle E^*(\mathbf{x};2\lambda) \times O_{ij}(\mathbf{y};2\lambda), \tfrac{\partial^2}{\partial x_k \partial y_l}\Big[A_{\mathbf{x}}\Big(\eta_m(\mathbf{x},\mathbf{y})\varphi(\mathbf{x}+\mathbf{y})\Big)\Big]\Big\rangle \\
&- \lim_{m\to\infty} \sum_{k,l=1\ k\le l}^{N} \alpha_{kl}\Big\langle E^*(\mathbf{x};2\lambda) \times O_{ij}(\mathbf{y};2\lambda), \tfrac{\partial^2}{\partial x_k \partial y_l}\Big[\varphi(\mathbf{x}+\mathbf{y})(A_{\mathbf{x}}\eta_m(\mathbf{x},\mathbf{y}))\Big]\Big\rangle \\
&= \lim_{m\to\infty} \sum_{k,l=1\ k\le l}^{N} \Big(\alpha_{kl}\Big\langle A'_{\mathbf{x}} E^*(\mathbf{x};2\lambda) \times O_{ij}(\mathbf{y};2\lambda), \tfrac{\partial^2}{\partial x_k \partial y_l}\Big(\eta_m(\mathbf{x},\mathbf{y})\varphi(\mathbf{x}+\mathbf{y})\Big)\Big\rangle \\
&-\alpha_{kl}\Big\langle \tfrac{\partial}{\partial x_k} E^*(\mathbf{x};2\lambda) \times \tfrac{\partial}{\partial y_l} O_{ij}(\mathbf{y};2\lambda), \varphi(\mathbf{x}+\mathbf{y})\Big(\eta_m(\mathbf{x},\mathbf{y}) + A_{\mathbf{x}}\eta_m(\mathbf{x},\mathbf{y})\Big)\Big\rangle \\
&+\alpha_{kl}\Big\langle \tfrac{\partial}{\partial x_k} E^*(\mathbf{x};2\lambda) \times \tfrac{\partial}{\partial y_l} O_{ij}(\mathbf{y};2\lambda), \varphi(\mathbf{x}+\mathbf{y})\eta_m(\mathbf{x},\mathbf{y})\Big\rangle\Big) \\
&= \sum_{k,l=1\ k\le l}^{N} \alpha_{kl}\Big\langle \tfrac{\partial^2}{\partial y_k \partial y_l} O_{ij}(\mathbf{y};2\lambda), \varphi(\mathbf{y})\Big\rangle .
\end{aligned}$$

Thus $\mathbf{E}^{\alpha}$ defined by (3.13) satisfies $(3.1)_1$. The proof is complete. □

## 3.1 Behaviour of $\mathbf{E}^{\alpha}_{\mathbf{ij}}(\mathbf{x})$ for large $|\mathbf{x}|$

For simplicity we put $2\lambda = 1$; this is enabled due to the following lemma.

**Lemma 3.7** *Let $\mathbf{E}^{\alpha}$ be given by (3.13). Then*

$$\mathbf{E}^{\alpha}(\mathbf{x};2\lambda) = (2\lambda)^{N-2}\mathbf{E}^{\alpha}(2\lambda\mathbf{x};1)\,.$$

**Proof** It is a consequence of Lemma 2.2. □

**Theorem 3.2** *Let $N = 2$ and let $|\mathbf{x}| \gg 1$. Then for every $k \ge 0$, $E^{\alpha}_{ij}(\mathbf{x})$ satisfies the estimate*

$$\Big|\nabla^k E^{\alpha}_{ij}(\mathbf{x};1)\Big| \le \frac{C}{|\mathbf{x}|^{\frac{1+k}{2}}(1+s(\mathbf{x}))^{\frac{2+k}{2}}} \quad i,j = 1,2\,. \tag{3.14}$$

**Proof** Using the asymptotic structure of the Oseen fundamental tensor we get that for $|\mathbf{x}|$ large[2]

$$\nabla^k E^{\alpha}_{ij}(\mathbf{x};1) \sim \nabla^k\Big[\alpha_{22}\int_{\mathbb{R}^2} \frac{\partial}{\partial x_2} E^*(\mathbf{x}-\mathbf{y};1)\frac{\partial}{\partial y_2} O_{ij}(\mathbf{y};1)\, d\mathbf{y}\Big].$$

[2]Here and in what follows, $f \sim g$ for $|\mathbf{x}| \gg 1$ means that there exist $R, C_1, C_2 > 0$ such that for $|\mathbf{x}| > R$ we have $C_1|f(\mathbf{x})| \le |g(\mathbf{x})| \le C_2|f(\mathbf{x})|$.

Let $k=0$. Let $\mathbf{x}$ with $|\mathbf{x}| \gg 1$ be arbitrary but fixed point. Then

$$
\begin{aligned}
E^{\alpha}_{ij}(\mathbf{x};1) \sim \alpha_{22}\Big(&\int_{B_1(\mathbf{0})} \tfrac{\partial}{\partial x_2}E^*(\mathbf{x}-\mathbf{y};1)\tfrac{\partial}{\partial y_2}\mathcal{O}_{ij}(\mathbf{y};1)\,d\mathbf{y} \\
&+\int_{B_1(\mathbf{x})} \tfrac{\partial}{\partial x_2}E^*(\mathbf{x}-\mathbf{y};1)\tfrac{\partial}{\partial y_2}\mathcal{O}_{ij}(\mathbf{y};1)\,d\mathbf{y} \\
&+\int_{\mathbb{R}^2\setminus B_1(\mathbf{0})\setminus B_1(\mathbf{x})} \tfrac{\partial}{\partial x_2}E^*(\mathbf{x}-\mathbf{y};1)\tfrac{\partial}{\partial y_2}\mathcal{O}_{ij}(\mathbf{y};1)\,d\mathbf{y}\Big) \\
&\equiv I_1(\mathbf{x})+I_2(\mathbf{x})+I_3(\mathbf{x})\,.
\end{aligned}
$$

We start to estimate $I_1(\mathbf{x})$. Using Lemma 2.3 and Remark 3.2 we obtain

$$
\begin{aligned}
|I_1(\mathbf{x})| &\le C\int_{B_1(\mathbf{0})} \tfrac{1}{|\mathbf{X}-\mathbf{Y}|(1+s(\mathbf{X}-\mathbf{Y}))}\tfrac{1}{|\mathbf{y}|}\,d\mathbf{y} \\
\le C\int_{B_1(\mathbf{0})} \tfrac{1}{|\mathbf{x}-\mathbf{y}|(1+s(\mathbf{x}-\mathbf{y}))}\tfrac{1}{|\mathbf{y}|}\,d\mathbf{y} &\le \tfrac{C}{|\mathbf{x}|(1+s(\mathbf{x}))}\int_{B_1(\mathbf{0})}\tfrac{1}{|\mathbf{y}|}\,d\mathbf{y} \le \tfrac{C}{|\mathbf{x}|(1+s(\mathbf{x}))}\,.
\end{aligned}
$$

Analogously we can proceed with $I_2(\mathbf{x})$ to get the same estimate. The term $I_3(\mathbf{x})$ can be estimated by

$$
|I_3(\mathbf{x})| \le C\int_{\mathbb{R}^2} \frac{1}{|\mathbf{x}-\mathbf{y}|(1+s(\mathbf{x}-\mathbf{y}))}\,\frac{1}{|\mathbf{y}|(1+s(\mathbf{y}))}\,d\mathbf{y}\,.
$$

The estimates of convolutions of this type were studied by several authors in the past, recently these results have been extended to any dimension $N \ge 2$ by Kračmar, Novotný and Pokorný (see [2]). Since these calculations are quite technical, we can not deal with them here and we present only final estimate. For technical details see [2] and [8].

$$
|I_3(\mathbf{x})| \le \frac{C}{|\mathbf{x}|^{\frac12}(1+s(\mathbf{x}))}\,.
$$

Our next concern will be the behaviour of the derivative of $E^{\alpha}_{ij}$ for $|\mathbf{x}|$ sufficiently large.

We begin to give an estimate for $\frac{\partial}{\partial x_i}I_1(\mathbf{x})$, $i=1,2$. We have

$$
\begin{aligned}
\Big|\tfrac{\partial}{\partial x_i}I_1(\mathbf{x})\Big| &\le C\int_{B_1(\mathbf{0})} \big|\tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_2}E^*(\mathbf{x}-\mathbf{y};1)\big|\big|\tfrac{\partial}{\partial y_2}\mathcal{O}_{ij}(\mathbf{y};1)\big|\,d\mathbf{y} \\
&\le C\int_{B_1(\mathbf{0})} \tfrac{1}{|\mathbf{x}-\mathbf{y}|^{\frac32}(1+s(\mathbf{x}-\mathbf{y}))^{\frac32}}\tfrac{1}{|\mathbf{y}|}\,d\mathbf{y} \le \tfrac{C}{|\mathbf{x}|^{\frac32}(1+s(\mathbf{x}))^{\frac32}}\,.
\end{aligned}
$$

As for $I_2(\mathbf{x})$, first we change variables and we obtain

$$
\begin{aligned}
\Big|\tfrac{\partial}{\partial x_i}I_2(\mathbf{x})\Big| &\le \Big|\tfrac{\partial}{\partial x_i}\int_{B_1(\mathbf{0})} \tfrac{\partial}{\partial z_2}E^*(\mathbf{z};1)\tfrac{\partial}{\partial x_2}\mathcal{O}_{ij}(\mathbf{x}-\mathbf{z};1)\,d\mathbf{z}\Big| \\
&\le C\int_{B_1(\mathbf{0})} \tfrac{1}{|\mathbf{z}|}\tfrac{1}{|\mathbf{x}-\mathbf{z}|^{\frac32}(1+s(\mathbf{x}-\mathbf{z}))^{\frac32}}\,d\mathbf{z} \le \tfrac{C}{|\mathbf{x}|^{\frac32}(1+s(\mathbf{x}))^{\frac32}}\,.
\end{aligned}
$$

To estimate the last term $\frac{\partial}{\partial x_i}I_3(\mathbf{x})$ is more difficult. Similarly as above we will not deal with it and we present the final estimate, for technical details see [3], [6] and [8].

$$
\Big|\frac{\partial}{\partial x_i}I_3(\mathbf{x})\Big| \le \frac{C}{|\mathbf{x}|(1+s(\mathbf{x}))^{\frac32}}
$$

for $i=1,2$ and $|\mathbf{x}|$ sufficiently large.

For $k>1$ we can proceed analogously. The proof is complete. □

**Theorem 3.3** *Let $N = 3$ and let $|\mathbf{x}| \gg 1$. Then for every $k \geq 0$, $E^{\alpha}_{ij}(\mathbf{x})$ satisfies the estimate*

$$\left|\nabla^k E^{\alpha}_{ij}(\mathbf{x};1)\right| \leq \frac{C}{|\mathbf{x}|^{\frac{2+k}{2}}(1+s(\mathbf{x}))^{\frac{3+k}{2}}}\,. \tag{3.15}$$

**Proof** The proof can be done by analogy with the proof in two dimensions. □

## 3.2 Behaviour of $\mathbf{E}^{\alpha}_{\mathbf{ij}}(\mathbf{x})$ around zero

For $|\mathbf{x}|$ large is the behaviour of the derivative of the fundamental Oseen tensor with respect to the first variable different from the behaviour of the derivative with respect to the other variables and we used this fact in the previous part. For $|\mathbf{x}|$ small the situation is different, because the asymptotic behaviour of $\frac{\partial}{\partial x_1}\mathcal{O}_{ij}(\mathbf{x})$ is the same as the asymptotic behaviour of $\frac{\partial}{\partial x_k}\mathcal{O}_{ij}(\mathbf{x})$, $k = 2, 3$. The convolution $\frac{\partial}{\partial x_1}E^* * \frac{\partial}{\partial y_1}O_{ij}$ was studied in [3] and for this reason we have not to deal with this case. We only give the theorem, which was proved in [3].

**Theorem 3.4** *Let $|\mathbf{x}| \leq 2$. Then $E^{\alpha}_{ij}(\mathbf{x};1)$ satisfies the estimates*
*(a) for $N = 2$*

$$\begin{aligned} |E^{\alpha}_{ij}(\mathbf{x};1)| &\leq C\,|\ln|\mathbf{x}|| \\ |\nabla^k E^{\alpha}_{ij}(\mathbf{x};1)| &\leq C|\mathbf{x}|^{-k} \quad \textit{for } 0 < k \leq 2\,, \end{aligned} \tag{3.16}$$

*(b) for $N = 3$*

$$|\nabla^k E^{\alpha}_{ij}(\mathbf{x};1)| \leq C|\mathbf{x}|^{-k} \quad \textit{for } 0 \leq k \leq 2\,. \tag{3.17}$$

*Moreover,*

$$E^{\alpha}_{ij}(\mathbf{x},1) = I_1(\mathbf{x}) + I_2(\mathbf{x})\,,$$

*where $|\nabla^2 I_2(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|^{N-1}}$ for $|\mathbf{x}| \leq 2$ and $I_1(\mathbf{x})$, representing the singular part of the second gradient of $E^{\alpha}_{ij}(\mathbf{x};1)$, has the following property:*

$$\mathcal{F}\Big(\int_{\mathbb{R}^N} I_1(\cdot - \mathbf{y})D^{\beta}f(\mathbf{y})\;d\mathbf{y}\Big)(\xi) = m(\xi)\mathcal{F}(f)(\xi)\,,$$

*$|\beta| = 2$ and $m(\xi)$ represents the $L^p$-Fourier multiplier, $1 < p < \infty$. Therefore the integral operator $T$,*

$$Tf(\mathbf{x}) = \int_{\mathbb{R}^N} I_1(\mathbf{x} - \mathbf{y})D^{\beta}f(\mathbf{y})\;d\mathbf{y}$$

*maps for $|\beta| = 2$ the space $C^{\infty}_0(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, $1 < p < \infty$ and*

$$\begin{aligned} ||Tf||_{p,\mathbb{R}^N} &\leq C||f||_{p,\mathbb{R}^N} \\ ||Tf||_{p,(g),\mathbb{R}^N} &\leq C||f||_{p,(g),\mathbb{R}^N} \end{aligned}$$

*for all $g$, weights from the Muckenhoupt class*[3] $A_p$.

Now we are left with the task of verifying that $E^\alpha_{ij}(\mathbf{x})$ has zero divergence, it means that $\frac{\partial E^\alpha_{ij}(\mathbf{x};1)}{\partial x_i} = 0$ in $\mathcal{S}'$. It is not possible to verify this equality directly, since we are not able to verify the assumptions for interchanging of the derivative and the integral. We can not also follow the approach used in [6], because now in contradiction to the situation in [6], the assumptions for Hausdorff–Young inequality (see e.g. [7]) are not fulfilled as $E^\alpha_{ij} \notin L^p(\mathbb{R}^N)$ for $p \in (1,2)$. Hence, we have to employ another technique. Nevertheless, we can also here show[4]

**Theorem 3.5** *Let $N = 2,3$. Then*

$$\frac{\partial E^\alpha_{ij}(\mathbf{x};1)}{\partial x_i} = 0 \ \textit{in}\ \mathcal{S}'.$$

**Proof** We show the theorem only for $N = 2$, from the following technique will be seen that the same proof works also for $N = 3$.

Let $\mathbf{x} \neq 0$ be an arbitrary but fixed point in $\mathbb{R}^2$. The idea of the proof is to divide $\mathbb{R}^2$ into several subdomains and to modify each of terms from definition of $\mathbf{E}^\alpha$ in such a way that we will be able to verify the assumptions for interchanging of the derivative and the integral over each of subdomain for arbitrary but fixed point in $\mathbb{R}^2$, different from the origin.

We choose $\varepsilon > 0$ sufficiently small and $R > 0$ sufficiently large, so that

$$\begin{aligned}
&1.\ \mathbf{x} \in B^R_\varepsilon(\mathbf{0}) \text{ and } \mathbf{X} \in B^R_\varepsilon(\mathbf{0}) \\
&2.\ \forall \mathbf{z} \in \mathbb{R}^2, |\mathbf{z}| \le \varepsilon\,,\ (|\mathbf{x}-\mathbf{z}| \ge \delta \text{ and } |\mathbf{X}-\mathbf{Z}| \ge \delta) \quad \text{for some } \delta > 0 \qquad (3.18)\\
&3.\ \forall \mathbf{z} \in \mathbb{R}^2, |\mathbf{z}| \ge R\,,\ (|\mathbf{x}-\mathbf{z}| \gg 1 \text{ and } |\mathbf{X}-\mathbf{Z}| \gg 1).
\end{aligned}$$

We shall study $\frac{\partial}{\partial x_1} E^\alpha_{11}(\mathbf{x})$, for $\frac{\partial}{\partial x_i} E^\alpha_{ij}(\mathbf{x})$ and $i + j > 2$ we can proceed analogously. We have

$$\begin{aligned}
E^\alpha_{11}(\mathbf{x};1) &= \sum_{k,l=1\ k\le l}^{2} \alpha_{kl} \int_{\mathbb{R}^2} \tfrac{\partial}{\partial x_k} E^*(\mathbf{x}-\mathbf{y};1) \tfrac{\partial}{\partial y_l} \mathcal{O}_{11}(\mathbf{y};1) d\mathbf{y} \\
&\equiv J_1(\mathbf{x}) + J_2(\mathbf{x}) + J_3(\mathbf{x})\,.
\end{aligned}$$

Similarly as in the proof of Theorem 3.1, it is sufficient to concern with $J_2(\mathbf{x})$. First, we make the change of variables and divide the convolution into three

[3]The non-negative weight $g$ belongs to the Muckenhoupt class $A_p$, $1 \le p < +\infty$, if there is a constant $C$ such that

$$\sup_Q \left[ \left( \frac{1}{|Q|} \int_Q g(\mathbf{x}) d\mathbf{x} \right) \left( \frac{1}{|Q|} \int_Q g(\mathbf{x})^{-\frac{1}{p-1}} d\mathbf{x} \right)^{p-1} \right] \le C < \infty\,.$$

The supremum is taken over all cubes $Q$ in $\mathbb{R}^N$. Moreover, $||f||_{p,(g),\mathbb{R}^N} \equiv ||f g^{\frac{1}{p}}||_{p,\mathbb{R}^N}$.

[4]Note that, with some modifications, we could also employ the technique used in [3].

parts

$$\begin{aligned}J_2(\mathbf{x}) &= \alpha_{22} \int_{\mathbb{R}^2} \tfrac{\partial}{\partial x_2} E^*(\mathbf{x}-\mathbf{y};1) \tfrac{\partial}{\partial y_2} \mathcal{O}_{11}(\mathbf{y};1)\, d\mathbf{y} \\ &= -\alpha_{22}\Big( \int_{B_\varepsilon^R(\mathbf{0})} \tfrac{\partial}{\partial z_2} E^*(\mathbf{z};1) \tfrac{\partial}{\partial z_2} \mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z} \\ &\quad + \int_{B^R(\mathbf{0})} \tfrac{\partial}{\partial z_2} E^*(\mathbf{z};1) \tfrac{\partial}{\partial z_2} \mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z} \\ &\quad + \int_{B_\varepsilon(\mathbf{0})} \tfrac{\partial}{\partial z_2} E^*(\mathbf{z};1) \tfrac{\partial}{\partial z_2} \mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z}\Big).\end{aligned}$$

Now we apply the Gauss theorem, except of the integral over $B_\varepsilon(\mathbf{0})$. Let us note that due to the asymptotic properties of $E^*$ and $\mathcal{O}_{11}$ we have that

$$\lim_{r\to\infty} \int_{\partial B_r(\mathbf{0})} \frac{\partial}{\partial z_2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{z}) n_2 dS = 0\,.$$

Then

$$\begin{aligned}J_2(\mathbf{x}) &= \alpha_{22}\Big( \int_{B_\varepsilon^R(\mathbf{0})} \tfrac{\partial^2}{\partial z_2^2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z} \\ &\quad + \int_{B^R(\mathbf{0})} \tfrac{\partial^2}{\partial z_2^2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z} \\ &\quad - \int_{B_\varepsilon(\mathbf{0})} \tfrac{\partial}{\partial z_2} E^*(\mathbf{z};1) \tfrac{\partial}{\partial z_2} \mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z} \\ &\quad + \int_{\partial B_\varepsilon(\mathbf{0})} \tfrac{\partial}{\partial z_2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1) n_2(\mathbf{z})\, d_{\mathbf{z}}S\Big) \\ &\equiv \alpha_{22}(J_{21}(\mathbf{x}) + J_{22}(\mathbf{x}) - J_{23}(\mathbf{x}) + J_{24}(\mathbf{x}))\,,\end{aligned}$$

where $n_2$ denotes the second component of the vector of the unit outer normal to the ball with the radius $\varepsilon$.

Now we have to verify the assumptions for interchanging of the derivative and the integral separately for each $J_{2i}$, $i = 1,2,3,4$. We consider only $J_{21}(\mathbf{x})$, i.e. the most difficult term. Here we have to distinguish two cases, first $|\mathbf{x}|$ large and then $|\mathbf{x}|$ small.

(i) $|\mathbf{x}| \gg 1$

Due to the assumptions (3.18), the domain $B_\varepsilon^R(\mathbf{0})$ can be divided into 2 subdomains $B_\varepsilon^R(\mathbf{0}) = B_\varepsilon(\mathbf{x}) \cup (B_\varepsilon^R(\mathbf{0}) \setminus B_\varepsilon(\mathbf{x}))$. It is sufficient to verify the assumptions on $B_\varepsilon(\mathbf{x})$. But

$$\begin{gathered}\int_{B_\varepsilon(\mathbf{x})} \tfrac{\partial^2}{\partial z_2^2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1) d\mathbf{z} = \int_{B_\varepsilon(\mathbf{0})} \tfrac{\partial^2}{\partial x_2^2} E^*(\mathbf{x}-\mathbf{y};1)\mathcal{O}_{11}(\mathbf{y};1) d\mathbf{y} \\ \le C \int_{B_\varepsilon(\mathbf{0})} |\ln|\mathbf{y}||\, d\mathbf{y} \le C_1 \\ \left|\tfrac{\partial}{\partial x_1}\Big(\tfrac{\partial^2}{\partial z_2^2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\Big)\right| \le \tfrac{C_2}{|\mathbf{z}|^{\frac{3}{2}}(1+s(\mathbf{z}))^{\frac{3}{2}}} \tfrac{1}{|\mathbf{x}-\mathbf{z}|}\end{gathered}$$

and the function on the right hand-side is evidently integrable over $B_\varepsilon(\mathbf{x})$.

(ii) $|\mathbf{x}| < 2$

It suffices to study the $J_{23}(\mathbf{x})$ over $B_\sigma(\mathbf{x})$, where $\sigma < \min(\varepsilon, |\mathbf{x}| - \varepsilon)$.

$$\begin{gathered}\int_{B_\sigma(\mathbf{x})} \tfrac{\partial^2}{\partial z_2^2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\, d\mathbf{z} = \int_{B_\sigma(\mathbf{0})} \tfrac{\partial^2}{\partial x_2^2} E^*(\mathbf{x}-\mathbf{y};1)\mathcal{O}_{11}(\mathbf{y};1)\, d\mathbf{y} \\ \le C \int_{B_\sigma(\mathbf{0})} |\ln|\mathbf{y}|| \tfrac{1}{\delta^2}\, d\mathbf{y} \le C \\ \left|\tfrac{\partial}{\partial x_1}\Big(\tfrac{\partial^2}{\partial z_2^2} E^*(\mathbf{z};1)\mathcal{O}_{11}(\mathbf{x}-\mathbf{z};1)\Big)\right| \le C \tfrac{1}{|\mathbf{z}|^2} \tfrac{1}{|\mathbf{x}-\mathbf{z}|}\end{gathered}$$

Similarly we proceed with $J_{22}(\mathbf{x})$, $J_{23}(\mathbf{x})$ and $J_{24}(\mathbf{x})$ and finally we can conclude that we verify the assumptions for interchanging of the derivative and the integral for $\mathbf{x} \neq 0$; hence we have proved that $\frac{\partial E^\alpha_{ij}}{\partial x_i}(\mathbf{x};1) = 0$ a.e. in $\mathbb{R}^2$.

In order to show the equality in the sense of $\mathcal{S}'$, we fix arbitrarily $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and

$$\Big|\Big\langle \tfrac{\partial E^\alpha_{ij}(\mathbf{x};1)}{\partial x_i}, \varphi(\mathbf{x})\Big\rangle\Big| = \Big|\lim_{\varepsilon\to 0^+} \sum_{i=1}^{2} \int_{B^\varepsilon(\mathbf{0})} E^\alpha_{ij}(\mathbf{x};1) \tfrac{\partial \varphi(\mathbf{x})}{\partial x_i}\, d\mathbf{x}\Big|$$
$$= \Big|\lim_{\varepsilon\to 0^+} \Big(\int_{B^\varepsilon(\mathbf{0})} \tfrac{\partial E^\alpha_{ij}(\mathbf{x};1)}{\partial x_i}\varphi(\mathbf{x})\, d\mathbf{x} + \sum_{i=1}^{2} \int_{\partial B_\varepsilon(\mathbf{0})} E^\alpha_{ij}(\mathbf{x};1)\varphi(\mathbf{x}) n_i(\mathbf{x})\, d_{\mathbf{x}}S\Big)\Big|$$
$$\leq C \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(\mathbf{0})} |\ln|\mathbf{x}||\, d_{\mathbf{x}}S = 0\,,$$

where $\mathbf{n} = (n_1, n_2)$ denotes the vector of the unit outer normal to the ball with the radius $\varepsilon$. The theorem is shown. □

The results proved in this paper are summarized in the following theorem.

**Theorem 3.6** *Let $N = 2, 3$. Then the solution to (1.4) can be expressed in the form*

$$\mathcal{O}^\alpha(\mathbf{x}; 2\lambda) = \mathcal{O}(\mathbf{x}; 2\lambda) + \mathbf{E}^\alpha(\mathbf{x}; 2\lambda)\,,$$

*where $\mathcal{O}$ is the fundamental Oseen tensor and the remainder $\mathbf{E}^\alpha$ is given by (3.13) and has the asymptotic properties established in Theorems 3.2–3.4. Moreover,*

$$\mathbf{E}^\alpha(\mathbf{x}; 2\lambda) = (2\lambda)^{N-2}\mathbf{E}^\alpha(2\lambda\mathbf{x}; 1)\,.$$

# References

[1] Galdi, G. P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations I. *Springer Tracts in Natural Philosophy, Vol. 38, Springer, New York,* 1994.

[2] Kračmar, S., Novotný, A., Pokorný, M.: *Estimates of three-dimensional Oseen kernels in $L^p$-weighted spaces.* In: Sequiera A., Beirão da Veiga, H., Videman J.: Applied functional analysis. Kluwer Academic, Pluenum Publishers (1999), 281–316.

[3] Farwig, R., Novotný, A., Pokorný, M.: *The Fundamental Solution to a Modified Oseen Problem.* Zeitschrift für Analysis und ihre Anwendungen **19**, 3 (2000), 713–728.

[4] Novotný, A., Pokorný, M.: *Three-dimensional steady flow of viscoelastic fluid past an obstacle.* (accepted to Jour. of Math. Fluid Mech.).

[5] Novotný, A., Pokorný, M.: *Steady plane flow of viscoelastic fluid past an obstacle.* (submitted to Applications of Mathematics).

[6] Pokorný, M.: *Asymptotic behaviour of solutions to certain PDE's describing the flow of fluids in unbounded domains.* Ph.D. thesis, Charles University, Prague & Université de Toulon et du Var, 1999.

[7] Stein, E. M, Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. *Princeton University Press,* 1971.

[8] Trojek, P.: *Some mathematical problems connected with the flow of fluids in unbounded domains.* Master degree thesis, Palacký University, Olomouc, 1999.

[9] Vladimirov, V. S.: Generalized Functions in Mathematical Physics. *Nauka, Moscow,* 1979 (in Russian).