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On Ideals in Hilbert Algebras

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Abstract

Connections between deductive systems of Hilbert algebras and ideals introduced by I. Chajda and R. Halaš are described.

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The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. A. Diego proved (cf. [4]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D. Busneag (cf. [1], [2]) and Y. B. Jun (cf. [5]) and some of their filters forming deductive systems were recognized. I. Chajda and R. Halaš introduced in [3] the concept of ideal in Hilbert algebra and described connections between such ideals and congruences. In this note we describe connections between such ideals and deductive systems.

Since there exist various modifications of the definition of Hilbert algebra, we use that of [1].

Definition 1 A *Hilbert algebra* is a triplet $\mathcal{H} = (H; \bullet, \mathbf{1})$, where H is a nonempty set, $\bullet$ is a binary operation and $\mathbf{1}$ is fixed element of H such that the following axioms hold for each $x, y, z \in H$:

(I) $x \bullet (y \bullet x) = \mathbf{1}$,

(II) $(x \bullet (y \bullet z)) \bullet ((x \bullet y) \bullet (x \bullet z)) = \mathbf{1}$,

(III) $x \bullet y = \mathbf{1}$ and $y \bullet x = \mathbf{1}$ imply $x = y$.

The following result was proved (cf. for example [4]):

Lemma 1 *Let $\mathcal{H} = (H; \bullet, \mathbf{1})$ be a Hilbert algebra and $x, y, z \in H$. Then*

(1) $x \bullet x = \mathbf{1}$,
(2) $\mathbf{1} \bullet x = x$,
(3) $x \bullet \mathbf{1} = \mathbf{1}$,
(4) $x \bullet (y \bullet z) = y \bullet (x \bullet z)$.

It is easily checked that in a Hilbert algebra $\mathcal{H}$ the relation $\leq$ defined by

$$x \leq y \iff x \bullet y = \mathbf{1}$$

is a partial order on H with $\mathbf{1}$ as the largest element.

Lemma 2 *Let $\mathcal{H} = (H; \bullet, \mathbf{1})$ be a Hilbert algebra and $x, y, z \in H$. Then*

(5) $(x \bullet y) \bullet y \geq x$,
(6) $(y \bullet z) \bullet (x \bullet z) \geq x \bullet y$,
(7) $(x \bullet y) \bullet (x \bullet z) \geq y \bullet z$.

Proof (5) is a consequence of (4) and (1). (6) is proved in [4]. (7) follows from (6) and (4). □

The following concept of ideal of Hilbert algebra was introduced in [3] by I. Chajda and R. Halaš.

Definition 2 A nonempty subset I of a Hilbert algebra $\mathcal{H} = (H; \bullet, \mathbf{1})$ is called an *ideal* of $\mathcal{H}$ if

(i) $\mathbf{1} \in I$,
(ii) $x \bullet y \in I$ for all $x \in H$, $y \in I$,
(iii) $(y_2 \bullet (y_1 \bullet x)) \bullet x \in I$ for all $x \in H$, $y_1, y_2 \in I$.

Observe that (i) follows from (ii) and (1). From (ii) it follows also that any ideal is a subalgebra. Moreover, putting in the above definition $y_1 = a$ and $y_2 = \mathbf{1}$, we obtain $(a \bullet x) \bullet x \in I$ for all $a \in I$ and $x \in H$. But if $a \leq y$ and $a \in I$, then by (2) we have $y = 1 \bullet y = (a \bullet y) \bullet y \in I$, which proves that the following lemma is true.

Lemma 3 *If I is an ideal of a Hilbert algebra $\mathcal{H}$ and $a \in I$, $x, y \in H$ and $a \leq x$, then $x \in I$ and $(a \bullet y) \bullet y \in I$.*

Definition 3 A nonempty subset D of a Hilbert algebra $\mathcal{H} = (H; \bullet, \mathbf{1})$ is called a *deductive system* of $\mathcal{H}$ if

(d_1) $\mathbf{1} \in D$,
(d_2) $x \in D$ and $x \bullet y \in D$ imply $y \in D$.

Directly from the above definition it follows

Lemma 4 *Let D be a deductive system of a Hilbert algebra $\mathcal{H}$. If $a \in D$ and $a \leq x$, then $x \in D$.*

Theorem 1 *A nonempty subset A of a Hilbert algebra $\mathcal{H}$ is an ideal if and only if it is a deductive system.*

Proof Let A be an ideal. Since $\mathbf{1} \in A$, then (d_1) is satisfied. To prove (d_2) assume $a \in A$ and $, a \bullet x = a_1 \in A$ for some $x \in H$. Then, by Lemma 3, $a_2 = (a \bullet x) \bullet x \in A$, which together with (1) and (2) gives

$$x = \mathbf{1} \bullet x = [((a \bullet x) \bullet x) \bullet ((a \bullet x) \bullet x)] \bullet x = [a_2 \bullet (a_1 \bullet x)] \bullet x \in A.$$

Thus $a \in A$ and $a \bullet x \in A$ imply $x \in A$, which proves (d_2). Hence A is a deductive system.

Conversely, if A is a deductive system, then $\mathbf{1} \in A$ and

$$a \bullet (x \bullet a) = x \bullet (a \bullet a) = a \bullet \mathbf{1} = \mathbf{1} \in A\,,$$

by Lemma 1. This for $a \in A$ implies $x \bullet a \in A$ for every $x \in H$. Hence (i) and (ii) are satisfied. We prove (iii). Let $a_1, a_2 \in A$. Then

$$a_2 \bullet [(a_2 \bullet (a_1 \bullet x)) \bullet x] = [a_2 \bullet (a_1 \bullet x)] \bullet (a_2 \bullet x) \geq (a_1 \bullet x) \bullet x \geq a_1$$

by (4), (7) and (5). But, by Lemma 4, for a deductive system A we have

$$a_2 \bullet [(a_2 \bullet (a_1 \bullet x)) \bullet x] \geq a_1$$

and $a_1, a_2 \in A$ imply $(a_2 \bullet (a_1 \bullet x)) \bullet x \in A$, which proves (iii). Thus A is an ideal. □

Let R be a binary relation defined on a set H and $\mathcal{H} = (H; \bullet, \mathbf{1})$ be a Hilbert algebra. R is said *compatible* if $\langle a, b\rangle \in R$ and $\langle c, d\rangle \in R$ imply $\langle a \bullet c,\ b \bullet d\rangle \in R$. The set

$$[\mathbf{1}]_R = \{a \in H : \langle a, \mathbf{1}\rangle \in R\}$$

is called a *kernel* of R.

Of course, a *congruence* on $\mathcal{H}$ is a compatible equivalence relation on H and *congruence kernel* is its class containing $\mathbf{1}$. From the results obtained in [3] it follows that the kernel of a reflexive and compatible relation defined on a Hilbert algebra is a deductive system. Moreover, if two congruences Θ and Φ on a Hilbert algebra have the same kernel, then $\Theta = \Phi$. This means that the kernel of every congruence on a Hilbert algebra is a deductive system and every congruence is uniquelly determined by its kernel. Hence, as a consequence of our Theorem 1 and the main result from [3], we obtain

Theorem 2 *Every congruence on a Hilbert algebra is uniquelly determined by a deductive system. Namely, if D is a deductive system of a Hilbert algebra $\mathcal{H} = (H; \bullet, \mathbf{1})$, then the relation Θ_D defined by*

$$\langle a, b\rangle \in \Theta_D \iff a \bullet b \in D \text{ and } b \bullet a \in D$$

is a congruence of $\mathcal{H}$ such that $[\mathbf{1}]_{\Theta_D} = D$.

References

[1] Busneag, D.: *A note on deductive systems of a Hilbert algebra.* Kobe. J. Math. **2** (1985), 29–35.

[2] Busneag, D.: *Hilbert algebras of fractions and maximal Hilbert algebras of quotients.* Kobe. J. Math. **5** (1988), 161–172.

[3] Chajda, I., Halaš, R.: *Congruences and ideals in Hilbert algebras.* Kyungpook Math. J. (in press).

[4] Diego, A.: *Sur les algébres de Hilbert.* Collection de Logique Math. Ser. A (Ed. Hermann, Paris) **21** (1966), 1–52.

[5] Jun, Y. B.: *Deductive systems of Hilbert algebras.* Math. Japon. **43** (1996), 51–54.