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Inexact Solution of Auxiliary Problems in Polyak Type Algorithms *

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Abstract

The Polyak algorithm is known to be an efficient algorithm for iterative solution of quadratic programming problems with inequality constraints, especially when applied to small problems or appropriately modified. It reduces the solution of the above problems to the conjugate gradient solution of a sequence of unconstrained minimization problems. The point of this note is to show that before reaching the solution of an auxiliary minimization problem, there is a feasible decrease direction that can be used in order to release some active constraints in such a way that the finite termination property of the original algorithm is preserved.

Key words: Quadratic programming, Polyak algorithm, inexact active set strategy.

1991 Mathematics Subject Classification: 90C20

1 Introduction

We shall be concerned with the problem to find

$$\min_{\xi \in \Omega} j(\xi) \tag{1}$$

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with $\Omega = \{\xi : c \leq \xi\}$, $j(\xi) = \frac{1}{2}\xi^T A\xi - \xi^T b$, b and c given column n-vectors, and A an $n \times n$ symmetric positive definite matrix.

One of suitable algorithms for the solution of (1) is the modified conjugate gradient method due to Polyak [6] that terminates at the solution of (1) in a finite number of iterations. The algorithm reduces the solution of (1) to minimization of $j(\xi)$ on a sequence of auxiliary sets called faces. The algorithm is known to achieve solution in a finite number of steps provided that the auxiliary problems are solved exactly. However, O'Leary [5] observed that it may be not necessary to solve intermediate problems to a high degree of accuracy, since the purpose of the minimization in the middle stage of computation is only to determine the next face. Though she did not give any theory, O'Leary reduced the number of iterations to about a half with an algorithm in which the accuracy of the conjugate gradient minimization was refined during the course of iterations. Later, the present author [2] gave a condition in terms of the norm of violation of the Kuhn-Tucker conditions that grants the existence of a feasible direction that enables us to find $y \in \Omega$ so that $j(y)$ is less then the minimum of $j(\xi)$ on the face that is just explored and developed theoretically supported algorithms that preserve the finite termination property of the original Polyak algorithm and except inexact solutions of auxiliary problems. This result was then used to give some additional insight into the finite termination property of related purely iterative algorithms by Friedlander and Martínez [4] and the present author [3].

The presentation of the basic idea in [2] turned out to be not suitable for later applications and its proof a bit more complicated than necessary. Here we give a simplified condition on leaving the face so that the resulting algorithm preserves the finite termination property of the original algorithm and prevents repeated examination of any face.

2 Notations and preliminaries

It is well known that the solution to the problem (1) always exists, and it is necessarily unique. The solution is fully determined by the Kuhn–Tucker optimality conditions [1]. For arbitrary $x \in \Omega$, let us define the residual $r = r(x)$ by

$$r = Ax - b. \tag{2}$$

Then x is the unique solution of (1) if for $i = 1, \ldots, n$,

$$x_i = c_i \text{ implies } r_i \geq 0 \quad \text{and} \quad x_i > 0 \text{ implies } r_i = 0. \tag{3}$$

Let N denote the set of all indices so that

$$\mathcal{N} = \{1, 2, \ldots, n\}.$$

The set of all indices for which the variables x_i are at their bounds is called an *active set* of x. We shall denote it by $\mathcal{A}(x)$ so that

$$\mathcal{A}(x) = \{i \in \mathcal{N} : x_i = c_i\}.$$

Its complement

$$\mathcal{F}(x) = \{i \in \mathcal{N} : x_i > c_i\}$$

and subset

$$\mathcal{B}(x) = \{i \in \mathcal{N} : x_i = c_i \text{ and } r_i > 0\}$$

are called a *free set* and a *binding set*, respectively.

To enable an alternative reference to the Kuhn–Tucker conditions, we shall introduce a notation for the parts of $r(x)$ that are defined by

$$\begin{aligned} \varphi_i(x) &= r_i(x) \quad \text{for } i \in \mathcal{F}(x), & \varphi_i(x) &= 0 \quad \text{for } i \in \mathcal{A}(x), \\ \beta_i(x) &= 0 \quad \text{for } i \in \mathcal{F}(x), & \beta_i(x) &= r_i^-(x) \quad \text{for } i \in \mathcal{A}(x), \end{aligned}$$

where we have used the notation $r_i^- = \min\{r_i, 0\}$. Thus the Kuhn–Tucker conditions (3) are satisfied iff the *projected gradient* $\nu(x) = \varphi(x) + \beta(x)$ is equal to zero.

The Euclidean norm of x will be denoted by $\|x\|$. Analogous notation will be used for the induced matrix norm, so that the spectral condition number $\kappa(A)$ of the matrix A is defined by

$$\kappa(A) = \|A\| \, \|A^{-1}\| .$$

For any vector x and a decomposition $\mathcal{I}$, $\mathcal{J}$ of the set of indices $\mathcal{N} = \{1, \ldots, n\}$, let us denote by $x_{\mathcal{I}}$ and $x_{\mathcal{J}}$ the parts of x whose indices belong to $\mathcal{I}$ and $\mathcal{J}$, respectively. Corresponding to this decomposition of the set of indices, we partition and rearrange also the vectors r, b and c and the matrix A. With this notation, (2) is equivalent to

$$\begin{pmatrix} r_{\mathcal{I}} \\ r_{\mathcal{J}} \end{pmatrix} = \begin{pmatrix} A_{\mathcal{I}\mathcal{I}} & A_{\mathcal{I}\mathcal{J}} \\ A_{\mathcal{J}\mathcal{I}} & A_{\mathcal{J}\mathcal{J}} \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}} \\ x_{\mathcal{J}} \end{pmatrix} - \begin{pmatrix} b_{\mathcal{I}} \\ b_{\mathcal{J}} \end{pmatrix}, \tag{4}$$

and the minimization of $j(\xi)$ on the face

$$\mathcal{W}_{\mathcal{I}} = \{\xi : \xi_i = c_i \text{ for } i \in \mathcal{I}\}$$

amounts to unconstrained minimization of

$$j_{\mathcal{J}}(\xi_{\mathcal{J}}) = \frac{1}{2}\xi_{\mathcal{J}}^T A_{\mathcal{J}\mathcal{J}}\, \xi_{\mathcal{J}} - \xi_{\mathcal{J}}^T (b_{\mathcal{J}} - A_{\mathcal{J}\mathcal{I}}\, c_{\mathcal{I}}). \tag{5}$$

3 Modification of Polyak algorithm

Let us briefly describe the original Polyak algorithm without attempting to go into details that are available elsewhere [5, 6]. Starting from feasible x^0 and $\mathcal{I} = \mathcal{B}(x^0)$, the conjugate gradient iterations $x^1, x^2, \ldots$ are generated in order to minimize $j(\xi)$ on the face $\mathcal{W}_{\mathcal{I}}$ until either $x^k \notin \Omega$ or the minimum of $j(\xi)$ on $\mathcal{W}_{\mathcal{I}}$ is reached at $x^k \in \Omega$. In the first case, x^k is modified so that $x^k \in \Omega$, $j(x^k) < j(x^{k-1})$ and $\mathcal{I} \subset \mathcal{A}(x^k)$, and then the process continues from x^k with $\mathcal{I} = \mathcal{A}(x^k)$. The symbol $\subset$ denotes a proper subset in this section. In the other case, we check whether x^k satisfies $\beta(x^k) = o$. If so, x^k is the solution, otherwise we set $\mathcal{I} = \mathcal{B}(x^k)$ and restart from x^k.

A part of the proof of the finite termination property of the Polyak algorithm that is essential for us is based on the observations that the number of different faces which are generated during the solution is finite and that the faces generated by the minimizers of auxiliary problems can never repeat as far as the sequence $\{j(x^k)\}$ is decreasing.

Our development of the algorithm which accepts the approximate solutions of auxiliary minimization problems is based on an observation that we can preserve the finite termination property if we stop the minimization on the face $\mathcal{W}_\mathcal{I}$ at some x^k when we are able to generate x^{k+1} so that $\mathcal{I} \supset \mathcal{A}(x^{k+1})$ and $j(x^{k+1}) < \min\{j(\xi) : \xi \in \mathcal{W}_\mathcal{I}\}$. We shall look for x^{k+1} in the form $x^{k+1} = x^k - \alpha d$ with a suitable decrease direction $-d$ with the assistance of the following theorem.

Theorem *Let $x \in \Omega$ and $\kappa(A)^{1/2} \le \Gamma$. Denote $r = r(x)$, $\mathcal{I} = \mathcal{A}(x)$, and suppose that d satisfies*

$$r^T d \ge \|d\|^2 \quad \text{and} \quad \|d\| > \Gamma\|\varphi(x)\|. \tag{6}$$

Then the vector $y = x - \|A\|^{-1} d$ satisfies

$$j(y) < \min\{j(\xi) : \xi \in \mathcal{W}_\mathcal{I}\}. \tag{7}$$

Proof Let x, Γ and d satisfy the assumption of Theorem and (6). Let us denote $\mathcal{J} = \mathcal{F}(x)$ and notice that $r^T d \ge \|d\|^2$ implies

$$j(y) - j(x) = \frac{1}{2}\|A\|^{-2} d^T A d - \|A\|^{-1} d^T r \le -\frac{1}{2}\|A\|^{-1}\|d\|^2. \tag{8}$$

Since $\|r_\mathcal{J}\| = \|\varphi(x)\|$, it follows by assumptions and (6) that

$$\|d\|^2 > \kappa(A)\|r_\mathcal{J}\|^2. \tag{9}$$

Substituing (9) to (8) yields

$$j(y) - j(x) < -\frac{1}{2}\,\|A^{-1}\|\,\|r_\mathcal{J}\|^2. \tag{10}$$

Now let $x \in \Omega$ and let us denote by $\bar{x}$ and $\bar{r}$ the minimizer of $j(\xi)$ on $\mathcal{W}_\mathcal{I}$ and the corresponding residual vector, respectively. Direct computations yield

$$j(x) - j(\bar{x}) = j\big(\bar{x} + (x - \bar{x})\big) - j(\bar{x}) = \frac{1}{2}(x - \bar{x})^T A (x - \bar{x}) + \bar{r}^T (x - \bar{x}). \tag{11}$$

If we now rearrange the indices and take into account that $\bar{r}_\mathcal{J} = o$ and $x_\mathcal{I} = \bar{x}_\mathcal{I}$, we can further simplify the right hand side of (11) to get

$$j(x) - j(\bar{x}) = \frac{1}{2}(x_\mathcal{J} - \bar{x}_\mathcal{J})^T A_{\mathcal{J}\mathcal{J}} (x_\mathcal{J} - \bar{x}_\mathcal{J}). \tag{12}$$

To express $(x_{\mathcal{J}} - \bar{x}_{\mathcal{J}})$ in terms of $r_{\mathcal{J}}$, let us write the formula for r and $\bar{r}$ in the form (4) and subtract them so that

$$\begin{pmatrix} r_{\mathcal{I}} - \bar{r}_{\mathcal{J}} \\ r_{\mathcal{J}} \end{pmatrix} = \begin{pmatrix} A_{\mathcal{I}\mathcal{I}} & A_{\mathcal{I}\mathcal{J}} \\ A_{\mathcal{J}\mathcal{I}} & A_{\mathcal{J}\mathcal{J}} \end{pmatrix} \begin{pmatrix} o \\ x_{\mathcal{J}} - \bar{x}_{\mathcal{J}} \end{pmatrix}. \tag{13}$$

In particular, since $A_{\mathcal{J}\mathcal{J}}$ is also positive definite, it follows that $x_{\mathcal{J}} - \bar{x}_{\mathcal{J}} = A_{\mathcal{J}\mathcal{J}}^{-1} r_{\mathcal{J}}$ and by (12)

$$j(x) - j(\bar{x}) = \frac{1}{2} r_{\mathcal{J}}^T A_{\mathcal{J}\mathcal{J}}^{-1} r_{\mathcal{J}}. \tag{14}$$

Taking into account the interlacing properties of the spectra of principal submatrices of symmetric matrices, we get

$$\frac{1}{2} r_{\mathcal{J}}^T A_{\mathcal{J}\mathcal{J}}^{-1} r_{\mathcal{J}} \leq \frac{1}{2} \|A_{\mathcal{J}\mathcal{J}}^{-1}\| \|r_{\mathcal{J}}\|^2 \leq \frac{1}{2} \|A^{-1}\| \|r_{\mathcal{J}}\|^2, \tag{15}$$

so that by (10)

$$j(y) - j(\bar{x}) = \big(j(y) - j(x)\big) + \big(j(x) - j(\bar{x})\big) < -\frac{1}{2} \|A^{-1}\| \, \|r_{\mathcal{J}}\|^2 + \frac{1}{2} \|A^{-1}\| \, \|r_{\mathcal{J}}\|^2 = 0$$

$\square$

Now let us define $d(x) = \beta(x)$ for any x and assume that

$$\|d(x^k)\| > \Gamma \|\varphi(x^k)\|$$

and $x^k \in \Omega$. Then $d = \beta(x^k)$ satisfies the assumptions of Theorem so that for $x^{k+1} = x^k - \alpha_{cg} d$ with $\alpha_{cg} = r^T d / d^T A d$ which minimizes $j(x^k - \alpha d)$ with respect to α we get

$$j(x^{k+1}) \leq j(x^k - \|A\|^{-1} d) < j(x^k).$$

It is easy to see that $\mathcal{A}(x^k) \supset \mathcal{A}(x^{k+1})$ and $x^{k+1} \in \Omega$. It is also easy to check that we can carry out the same reasoning with $d(x) = r^-(x)$, where the entries r_i^- of r^- are defined by $r_i^- = \min\{r_i, 0\}$.

4 Comments and conclusions

We have proved a simple theoretical result that has been used to develop a variant of the Polyak algorithm which accepts approximate solutions of auxiliary linear problems and which preserves the finite termination property even for the dual degenerate problems that are characterized by $r_i(\bar{x}) = 0$ for some $i \in \mathcal{A}(\bar{x})$. The condition (6) can be used to replace the fixed bound on $\|\varphi(x^k)\|$ that is usually used without any theory to control the precision of solution of auxiliary problems.

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