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Some Algorithm for Testing Convexity of Histogram

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Abstract

This paper is concerned with the convexity of histogram. First the definition of convexity and the necessary and sufficient criterion of histogram convexity are presented. Then it is proved that this criterion is generally global i.e. the whole histogram must be tested altogether. Finally are treated algorithms based on this criterion.

Key words: Convex histogram, construction of convex histogram interpolant.

1991 Mathematics Subject Classification: 41A15, 65D05

1 Introduction

In various applications it is often necessary to construct a smooth function that interpolates prescribed data and preserves some shape properties of them. In the last years many papers were devoted to such problems. The majority of them treats the problems of positive (see e.g. [7], [10], [14]), monotone (see e.g. [1], [2], [3], [4], [5], [6], [20]) or convex (see e.g. [3], [4], [8], [11], [13], [17]) spline interpolation of prescribed function values. Only few papers (see e.g. [9], [12], [15], [16], [18]) were devoted to problems of shape preserving interpolation of histogram. They suggest that problems of positive or monotone histopolation are only a little more complicated than equivalent problems of function values interpolation and in case of polynomial splines they can be transformed to problems of monotone or convex function values interpolation by splines with order increased by one (see e.g. [9], [12], [19]). But the convex histopolation seems to be more difficult. The first question which arrises is when we can say that histogram is convex. And the solūtion of such problem is the subject of this paper.

The natural approach to that problem is based on existence of convex function interpolating given histogram. But this definition is too general and some more simple criterion is needed. In [15], [16] the so called *histogram in convex position* is defined as histogram which can be interpolated by convex linear spline (the mesh is same as for histogram). But we can see (Fig. 1) that histogram $G = \{1.25, 0.75, 0.25, 0.375, 3, 6\}$ on the mesh $\{0, 1, 2, 3, 4, 5, 6\}$ is not convex according to this criterion although there exist convex linear splines on refined mesh $\{0, 1, 2, 3, 3.5, 4, 5, 6\}$ which interpolate G (see Fig. 2).

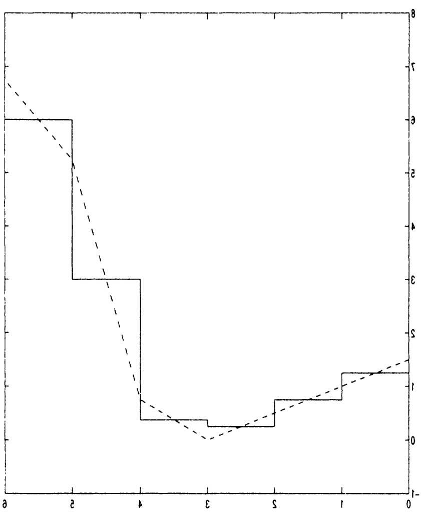

Figure 1: The nonconvexity of linear interpolatory spline on original mesh

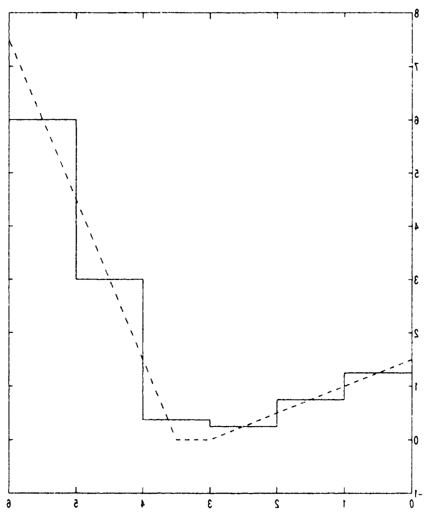

Figure 2: The convexity of linear interpolatory spline on refined mesh

It suggests that better criterion of histogram convexity is existence of convex linear spline on refined mesh with one added knot to any interval of original mesh. In section 2 is proved that this criterion is equivalent to the definition which uses continuous function. Another question which arises is if previous criterion is local or global. It is known that criterion of function values convexity is local (all the second differences of date must be non-negative). But on the contrary the criterion of histogram convexity cannot be decomposed in such way and is global which is proved in section 2. In section 3 some properties of convex linear splines interpolating histogram on refined mesh are proved. There are also shown the algorithms for testing histogram convexity and finding interpolatory convex linear spline on refined mesh based on criterion from section 2.

2 Convexity of histogram

Let us have given histogram $G = \{g_i\}_{i=0}^{n}$ on the mesh

$$(\Delta x): \quad x_0 < x_1 < \ldots < x_n < x_{n+1}, \quad \text{with } h_i = x_{i+1} - x_i,\ i = 0(1)n.$$

Let us denote $(\Delta^\alpha x) = \{x_i\}_{i=0}^{n+1} \cup \{x_i + \alpha_i h_i\}_{i=0}^{n}$ with $0 < \alpha_i < 1$ and let $S_{11}(\Delta^\alpha x)$ be space of linear splines on the refined mesh $(\Delta^\alpha x)$.

Definition 2.1 We say that histogram G is convex if there exists convex continuous function f interpolating histogram G (i.e. $\int_{x_i}^{x_{i+1}} f(x)dx = h_i g_i$ for $i = 0(1)n$) on mesh (Δx).

Theorem 2.2 (Necessary and sufficient criterion of convexity) *Histogram G is convex if and only if there exist set of numbers $\{\alpha_i\}_{i=0}^{n}$ and corresponding function $p(x) \in S_{11}(\Delta^\alpha x)$ which interpolates histogram G.*

Proof 1. Let us have convex histogram $G = \{g_i\}_{i=0}^{n}$ on the mesh (Δx) and let f be convex continuous function which interpolates histogram G. Let us denote

$$f_i = f(x_i), \quad d_i^- = \lim_{x \to x_i -} f'(x), \quad d_i^+ = \lim_{x \to x_i +} f'(x),$$
$$\beta_i = (f_i + h_i d_{i+1}^- - f_{i+1})/(h_i(d_{i+1}^- - d_i^+))$$

Then we can define

$$p_i(x) = f_i + (x - x_i)(f_{i+1} - f_i)/h_i$$
$$p_i^{\beta_i}(x) = \begin{cases} f_i + (x - x_i)d_i^+ & \text{if } x \le x_i + h_i\beta_i \\ f_{i+1} - (x_{i+1} - x)d_{i+1}^- & \text{others} \end{cases}$$
$$p_i^{\alpha_i}(x) = \begin{cases} f_i + (x - x_i)d_i^+ & \text{if } x \le x_i + h_i\alpha_i \\ f_{i+1} - (x_{i+1} - x)(f_{i+1} - f_i - \alpha_i h_i d_i^+)/((1 - \alpha_i)h_i) & \text{others} \end{cases}$$

with $\alpha_i \in [0, \beta_i]$ (see Fig 3).

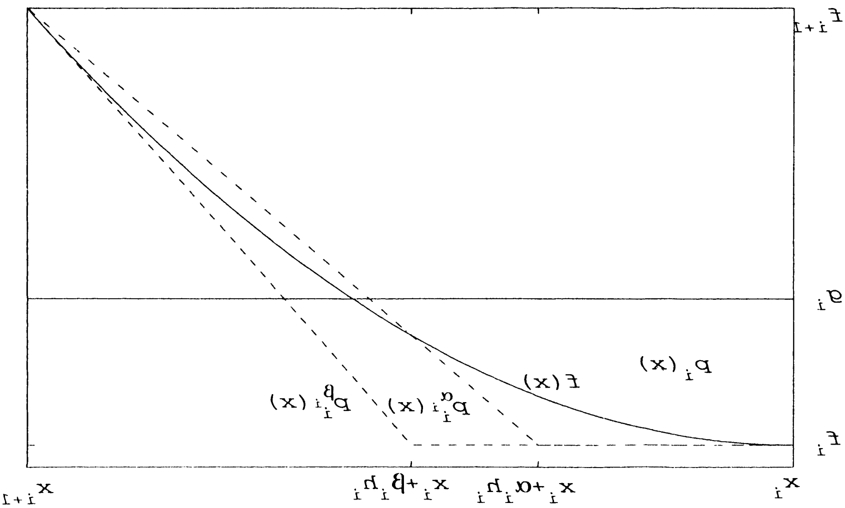

Figure 3

The function $p_i^{\alpha_i}(x)$ has the following properties:

1. $p_i^{\beta_i}(x) \leq p_i^{\alpha_i}(x) \leq p_i(x)$ for all $x \in [x_i, x_{i+1}]$ and all $\alpha_i \in [0, \beta_i]$
2. $p_i^{\alpha_i}$ is continuous with respect to parameter α_i for all $x \in [x_i, x_{i+1}]$
3. if $\alpha_i = 0$ then $p_i^{\alpha_i}(x) = p_i(x)$ for all $x \in [x_i, x_{i+1}]$
4. if $\alpha_i = \beta_i$ then $p_i^{\alpha_i}(x) = p_i^{\beta_i}(x)$ for all $x \in [x_i, x_{i+1}]$

From convexity of function $f(x)$ we obtain that $p_i^{\beta_i}(x) \leq f(x) \leq p_i(x)$ for all $x \in [x_i, x_{i+1}]$. It implies that $\int_{x_i}^{x_{i+1}} p_i^{\beta_i}(x)dx \leq h_i g_i \leq \int_{x_i}^{x_{i+1}} p_i(x)dx$. This result and properties of $p_i^{\alpha_i}$ imply that there exists $\alpha_i \in [0, \beta_i]$ such that $p_i^{\alpha_i}$ interpolate mean value g_i and function values f_i and f_{i+1}.

Then we can define linear spline $p(x) = p_i^{\alpha_i}(x)$ if $x \in [x_i, x_{i+1}]$. This spline is convex function because $p_i^{\alpha_i}$ is convex on $[x_i, x_{i+1}]$ for all $i = 0(1)n$ and

$$\lim_{x \to x_{i+1}^-} p'(x) \leq d_{i+1}^- \leq d_{i+1}^+ = \lim_{x \to x_{i+1}^+} p'(x).$$

2. Convex $p(x) \in S_{11}(\Delta^\alpha x)$ which interpolate G is continues too. □

Theorem 2.3 *There exists no local criterion of histogram convexity i.e. there does not exist $m \in N$ such that for all $n > m$ the convexity of histograms $G_j = \{g_i\}_{i=j}^{m+j}$, $j = 0(1)n - m$ implies the convexity of $G = \{g_i\}_{i=0}^n$.*

Proof Let us denote $\lfloor z \rfloor$ the nearest integer less than or equal $z \in R$. Then for all $k \in N$, $k \geq 3$, $n = 2k$ and for the mesh (Δx) with $x_i = i$ for $i = 0(1)n$ we can construct histogram $G = \{g_i\}_{i=0}^n$ such that $g_0 = 10$, $g_1 = 6$, $g_2 = 2$, $g_3 = 1$, $g_i = g_{i-1} + 2^{\lfloor (i-4)/2 \rfloor} + 2^{\lfloor (i-3)/2 \rfloor}$ for $i = 4(1)n - 2$, $g_{n-1} = g_{n-2} + 2^{k-2} + 1$, $g_n = g_{n-1} + 2^{k-2} + 1$. Histograms $G_A = \{g_i\}_{i=0}^{n-2}$ and $G_B = \{g_i\}_{i=n-2}^n$ have unique convex interpolants in $S_{11}(\Delta^\alpha x)$ given by following sets of breakpoints

$$B_A = \{(0, 12), (3, 0)\} \cup \{(3 + 2i, \sum_{j=1}^{i} 2^{j+1})\}_{i=1}^{k-2},$$

$$B_B = \{(n - 2, \sum_{j=1}^{k-2} 2^{j+1} - 2^{k-2} - 0.5), (n + 1, \sum_{j=1}^{k-2} 2^{j+1} + 2^{k-1} + 2.5)\}.$$

The uniqueness of these interpolants and their different function values on interval $[x_{n-2}, x_{n-1}]$ imply the nonexistence of convex interpolant of histogram G. But for histograms $G_1 = \{g_i\}_{i=0}^{n-1}$ and $G_2 = \{g_i\}_{i=1}^n$ there exist convex interpolatory linear splines given for example by following sets of breakpoints

$$B_1 = B_A \cup \{(n, \sum_{j=1}^{k-2} 2^{j+1} + 2^{k-2} + 2)\},$$

$$B_2 = \{(1, 8.5), (2, 3.5), (3, 0.5)\} \cup \{(3 + i, \sum_{j=1}^{i} 2^{\lfloor (j+1)/2 \rfloor} + (-1)^i/2\}_{i=1}^{n-6} \cup B_B.$$

□

Remark The previous proof (with $k = 4$) is illustrated on following figures.

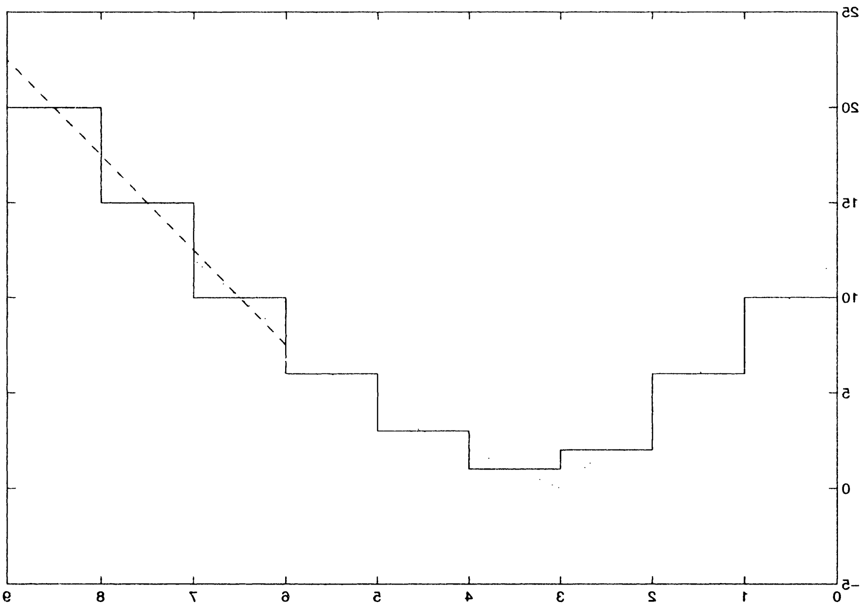

Figure 4: The nonconvexity of G: interpolant of G_A is dotted, interpolant of G_B is dashed

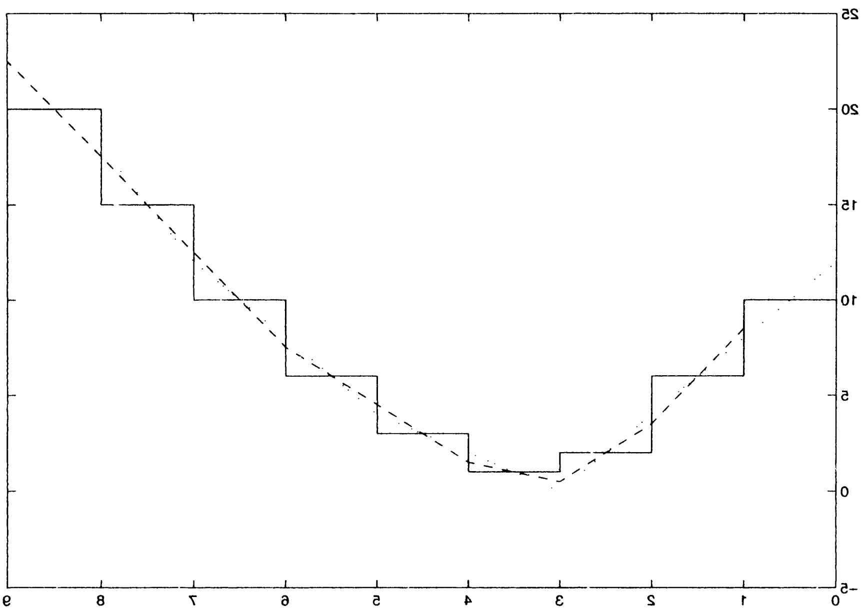

Figure 5: The convexity of G_1 and G_2: interpolant of G_1 is dotted, interpolant of G_2 is dashed

3 Algorithm of convexity testing

In this section we will show the relations between the function values and the first one-sided derivatives in original knots which are consequences of necessary and sufficient criterion of histogram convexity given in the theorem 2.2. Using these relations we will foremost show the algorithm for convexity testing and than the algorithm for finding convex linear spline on refined mesh which interpolate the convex histogram.

Let us have given histogram $G = \{g_i\}_{i=0}^{n}$ on the mesh (Δx) and let $p(x) \in S_{11}(\Delta^\alpha x)$ interpolate histogram G. Let us denote

$$s_i = p(x_i) \tag{1}$$

$$m_i^+ = p'(x_i + 0) \equiv \lim_{x \to x_i^+} p'(x) \tag{2}$$

$$m_i^- = p'(x_i - 0) \equiv \lim_{x \to x_i^-} p'(x) \tag{3}$$

$$f_m^i(s_i, s_{i+1}, m_i^+) = \frac{s_{i+1}^2 - 2(h_i m_i^+ + s_i)s_{i+1} + 2g_i h_i m_i^+ + s_i^2}{h_i(2g_i - 2s_i - h_i m_i^+)} \tag{4}$$

$$F_m^i(s_i, s_{i+1}, m_{i+1}^-) = \frac{s_i^2 - 2s_i(s_{i+1} - h_i m_{i+1}^-) + s_{i+1}^2 - 2h_i g_i m_{i+1}^-}{h_i(2s_{i+1} - 2g_i - h_i m_{i+1}^-)} \tag{5}$$

3.1 Some properties of interpolatory linear spline on the refined mesh

The relations between parameters s_i, m_i^+, g_i, s_{i+1}, m_{i+1}^- will be specified in the following lemmas and consequences.

Lemma 3.1 *Let us have given s_i, s_{i+1}, m_i^+ such that $s_i + s_{i+1} \neq 2g_i$ and $m_i^+ \neq 2(g_i - s_i)/h_i$ for $i \in \{0, 1, \ldots, n\}$. Then*

$$\alpha_i = (s_{i+1} + s_i - 2g_i)/(s_{i+1} - s_i - h_i m_i^+) \tag{6}$$

$$m_{i+1}^- = f_m^i(s_i, s_{i+1}, m_i^+) \tag{7}$$

Proof From formula $m_{i+1}^- = (s_{i+1} - s_i - \alpha_i h_i m_i^+)/(h_i(1 - \alpha_i))$ and from interpolatory condition

$$\int_0^{\alpha_i h_i} (s_i + m_i^+ x)dx + \int_0^{(1-\alpha_i)h_i} (s_i + m_i^+ \alpha_i h_i + m_{i+1}^- x)dx = h_i g_i$$

we obtain (6) and using it in formula for derivative m_{i+1}^- we obtain (7). □

Lemma 3.2 *Let us have given s_i, s_{i+1}, m_{i+1}^- such that $s_i + s_{i+1} \neq 2g_i$ and $m_i^+ \neq 2(g_i - s_i)/h_i$ for $i \in \{0, 1, \ldots, n\}$. Then*

$$\alpha_i = (2s_{i+1} - 2g_i - h_i m_{i+1}^-)/(s_{i+1} - s_i - h_i m_{i+1}^-) \tag{8}$$

$$m_i^+ = F_m^i(s_i, s_{i+1}, m_{i+1}^-) \tag{9}$$

Proof Can be followed similar way as in proof of lemma 3.1 from interpolatory condition and from $m_i^+ = (s_{i+1} - s_i - (1-\alpha_i)h_i m_{i+1}^-)/(h_i\alpha_i)$. □

Lemma 3.3 *The necessary conditions of convexity of $p(x)$ on $[x_i, x_{i+1}]$ are*

1. *for unknown parameters s_{i+1}, m_{i+1}^-:*

$$s_{i+1} > 2g_i - s_i \quad and \quad m_{i+1}^- > (2s_{i+1} - 2g_i)/h_i \tag{10}$$

or

$$s_{i+1} = 2g_i - s_i \quad and \quad m_{i+1}^- = (2s_{i+1} - 2g_i)/h_i \tag{11}$$

2. *for unknown parameters s_i, m_i^+:*

$$s_i > 2g_i - s_{i+1} \quad and \quad m_i^+ < (2g_i - 2s_i)/h_i \tag{12}$$

or

$$s_i = 2g_i - s_{i+1} \quad and \quad m_i^+ = (2g_i - 2s_i)/h_i \tag{13}$$

Proof
a) If $p(x)$ is line on $[x_i, x_{i+1}]$ then from interpolatory condition $(s_{i+1}+s_i)/2 = g_i$ and from formulas for derivatives $m_i^+ = m_{i+1}^- = (s_{i+1}-s_i)/h_i$ are obtained (11) and (13).

b) From convexity condition $m_i^+ < m_{i+1}^- = (s_{i+1}-s_i-\alpha_i h_i m_i^+)/(h_i(1-\alpha_i))$ and from condition $0 < \alpha_i < 1$ we obtain (10). Similarly from convexity condition $(s_{i+1}-s_i-(1-\alpha_i)h_i m_{i+1}^-)/(h_i\alpha_i) = m_i^+ < m_{i+1}^-$ and from condition $0 < \alpha_i < 1$ we obtain (12). □

Let us denote as Df_m^i the following domain of parameters:

$$Df_m^i = \{(s_i, , s_{i+1}, m_i^+); s_{i+1} > 2g_i - s_i, \quad m_i^+ < 2(g_i - s_i)/h_i\} \in R^3 \tag{14}$$

Lemma 3.4 *The functions f_m^i have following properties:*

1. f_m^i *are continuous on* Df_m^i (15)
2. f_m^i *are increasing with respect to all parameters on* Df_m^i (16)
3. $f_m^i(s_i, s_{i+1}, m_i^+) \geq 2(s_{i+1} - g_i)/h_i$ (17)
4. $\lim\limits_{m_i^+ \to (2(g_i-s_i)/h_i)^-} f_m^i(s_i, s_{i+1}, m_i^+) = \infty$ (18)
5. $\lim\limits_{m_i^+ \to -\infty} f_m^i(s_i, s_{i+1}, m_i^+) = 2(s_{i+1} - g_i)/h_i$ (19)
6. $\lim\limits_{s_i \to (2g_i - s_{i+1})^+} f_m^i(s_i, s_{i+1}, m_i^+) = f_m^i(2g_i - s_{i+1}, s_{i+1}, m_i^+)$

$$= 2(s_{i+1} - g_i)/h_i \tag{20}$$

Proof 1. Function f_m^i is not continuous only when $m_i^+ = 2(g_i - s_i)/h_i$ but such points are not from Df_m^i.
2. Using inequalities $s_{i+1} > 2g_i - s_i$, $m_i^+ < 2(g_i - s_i)/h_i$ and their consequence $s_{i+1} - s_i - h_i m_i^+ > 0$ we obtain

$$\frac{\partial f_m^i}{\partial s_i} = \frac{2(s_{i+1} + s_i - 2g_i)(s_{i+1} - s_i - h_i m_i^+)}{h_i(2g_i - 2s_i - h_i m_i^+)^2} > 0 \text{ on } Df_m^i$$

$$\frac{\partial f_m^i}{\partial s_{i+1}} = \frac{2(s_{i+1} - s_i - h_i m_i^+)}{h_i(2g_i - 2s_i - h_i m_i^+)} > 0 \text{ on } Df_m^i$$

$$\frac{\partial f_m^i}{\partial m_i^+} = \frac{s_{i+1} + s_i - 2g_i}{(2g_i - 2s_i - h_i m_i^+)^2} > 0 \text{ on } Df_m^i$$

3. $$f_m^i(s_i, s_{i+1}, m_i^+) - 2(s_{i+1} - g_i)/h_i = \frac{(s_{i+1} + s_i - 2g_i)^2}{h_i(2g_i - 2s_i - h_i m_i^+)} > 0 \text{ on } Df_m^i$$

Remaining statements can be proved simply by substitutions. □

Consequence 3.5

1. *Let s_i, $m_i^d < 2(g_i - s_i)/h_i$ be given. Then for any $s_{i+1} > 2g_i - s_i$ and any $m_{i+1}^- \geq f_m^i(s_i, s_{i+1}, m_i^d)$ there exists one $m_i^+ \in [m_i^d, 2(g_i - s_i)/h_i)$ such that $p(x) \in S_{11}(\Delta^\alpha x)$ defined by s_i, s_{i+1}, m_i^+, m_{i+1}^- on $[x_i, x_{i+1}]$ is convex here and interpolates mean value g_i.*

2. *Let s_i, $m_i^d = 2(g_i - s_i)/h_i$ be given. Then $p(x) \in S_{11}(\Delta^\alpha x)$ is convex on $[x_i, x_{i+1}]$ and interpolates mean value g_i here only if $s_{i+1} = 2g_i - s_i$, $m_i^+ = m_{i+1}^- = m_i^d$.*

Consequence 3.6

1. *Let s_i and $m_i^d < 2(g_i - s_i)/h_i$ be given. Then for any $s_{i+1} \geq 2g_i - s_i$, $m_{i+1}^+ \geq f_m^i(s_i, s_{i+1}, m_i^d)$, $m_{i+1}^+ \leq 2(g_{i+1} - s_{i+1})/h_{i+1}$ there exists $m_i^+ \in [m_i^d, 2(g_i - s_i)/h_i]$ and convex $p(x) \in S_{11}(\Delta^\alpha x)$ interpolating mean values g_i, g_{i+1} such that $p(x_i) = s_i$, $p(x_i + 0) = m_i^+$, $p(x_{i+0}) = s_{i+1}$, $p(x_{i+1} + 0) = m_{i+1}^+$.*

2. *Let s_i and $m_i^d = 2(g_i - s_i)/h_i$ be given. Then for $s_{i+1} = 2g_i - s_i$, $m_i^+ = m_i^d$ and any $m_{i+1}^+ \geq m_i^d$, $m_{i+1}^+ \leq 2(g_{i+1} - s_{i+1})/h_{i+1}$ there exists convex $p(x) \in S_{11}(\Delta^\alpha x)$ interpolating mean values g_i, g_{i+1} such that $p(x_i) = s_i$, $p(x_i + 0) = m_i^+$, $p(x_{i+0}) = s_{i+1}$, $p(x_{i+1} + 0) = m_{i+1}^+$.*

3.2 Algorithm for testing histogram convexity

We will use the algorithm based on criterion from theorem 2.2 and on similar idea as so called staircase algorithm (see [3], [13] and [17]). Let us have given histogram $G = \{g_i\}_{i=0}^n$ on the mesh (Δx) and let us denote $G_j = \{g_i\}_{i=0}^j$ for $j = 0(1)n$.

Algorithm 3.7 Let the sets $W_j \subset R^2$, $j = 0(1)n+1$ be constructed according to following rules:

1.

$$W_0 = \left\{ \begin{array}{l} (s_0, m_0^+) : \exists \text{ convex } p_0(x) \in S_{11}(\Delta^\alpha x) \text{ interpolating } G_0 \\ \quad \text{such that } p_0(x_0) = s_0,\ p_0'(x_0+0) = m_0^+ \end{array} \right\}$$

2. for $j = 1(1)n$:

$$W_j = \left\{ \begin{array}{l} (s_j, m_j^+) : \exists \text{ convex } p_j(x) \in S_{11}(\Delta^\alpha x) \text{ interpolating } G_j \\ \quad \text{and } \exists (s_i, m_i^+) \in W_i \text{ for } i = 0(1)j-1 \text{ such that} \\ \quad p_i(x_i) = s_i,\ p_i'(x_i+0) = m_i^+ \text{ for } i = 0(1)j \end{array} \right\}$$

3.

$$W_{n+1} = \left\{ \begin{array}{l} (s_{n+1}, m_{n+1}^-) : \exists \text{ convex } p_{n+1}(x) \in S_{11}(\Delta^\alpha x) \\ \quad \text{interpolating } G \text{ and } \exists (s_i, m_i^+) \in W_i \text{ for } i = 0(1)n \\ \quad \text{such that } p_i(x_i) = s_i,\ p_i'(x_i+0) = m_i^+ \text{ and} \\ \quad p_{n+1}(x_{n+1}) = s_{n+1},\ p_{n+1}'(x_{n+1}-0) = m_{n+1}^- \end{array} \right\}$$

If all $W_j \neq \emptyset$, $j = 0(1)n+1$ then there exists convex $p(x) \in S_{11}(\Delta^\alpha x)$ interpolating histogram G.

Theorem 3.8 *The sets W_j from algorithm 3.7 can be rewritten as:*

1.

$$W_0 = \{(s_0, m_0^+) :\ m_0^+ \leq 2(g_0 - s_0)/h_0\} \tag{21}$$

2. for $j = 1(1)n$:

$$W_j = \left\{ \begin{array}{l} (s_j, m_j^+) : \exists (s_{j-1}, m_{j-1}^+) \in W_{j-1} \textit{ such that} \\ \quad s_j \geq 2g_{j-1} - s_{j-1},\ m_j^+ \geq f_m^{j-1}(s_{j-1}, s_j, m_{j-1}^+), \\ \quad m_j^+ \leq 2(g_j - s_j)/h_j \end{array} \right\} \tag{22}$$

3.

$$W_{n+1} = \left\{ \begin{array}{l} (s_{n+1}, m_{n+1}^-) : \exists (s_n, m_n^+) \in W_n \textit{ such that} \\ \quad ((s_{n+1} > 2g_n - s_n) \wedge (m_{n+1}^- \geq f_m^n(s_n, s_{n+1}, m_n^+)) \\ \quad \vee ((s_{n+1} = 2g_n - s_n) \wedge (m_{n+1}^- = 2(g_n - s_n)/h_n)) \end{array} \right\} \tag{23}$$

Proof The statement (21) is consequence of necessary conditions in lemma 3.3. The statement (22) (or (23)) is implied by consequence 3.6 (or consequence 3.5). □

The more precise description of W_i will be given in the following consequences. This description depends on properties of previous set W_{i-1}. First the description of W_1 will be given.

Consequence 3.9

$$W_1 = \{(s_1, m_1^+) : \ m_1^+ \geq 2(s_1 - g_0)/h_0, \ m_1^+ \leq 2(g_1 - s_1)/h_1\} \tag{24}$$

Proof In (22) we have

$$W_1 = \left\{ \begin{array}{l} (s_1, m_1^+) : \exists (s_0, m_0^+) \in W_0 \text{ such that } s_1 \geq 2g_0 - s_0, \\ \qquad\qquad m_1^+ \geq f_m^0(s_0, s_1, m_0^+), \ m_1^+ \leq 2(g_1 - s_1)/h_1 \end{array} \right\}$$

From (21) is obtained that $s \in R$ and $m \in (-\infty, 2(g_0 - s_0)/h_0]$. If $s_0 \to \infty$ then using $s_1 \geq 2g_0 - s_0$ we obtain $s_1 > \infty$. If $m_0 \to -\infty$ then using $m_1^+ \geq f_m^0(s_0, s_1, m_0^+)$ and (19) we obtain $m_1 \geq 2(s_1 - g_0)/h_0$. □

Now for $i \leq n$ and W_{i-1} such that $W_{i-1} \neq \emptyset$ and $\operatorname{int} W_{i-1} = \emptyset$ we obtain following two consequences.

Consequence 3.10 *Let us have given $i \leq n$, s_{i-1}^d, $m_{i-1}^d \leq 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ and the set*

$$W_{i-1} = \{(s_{i-1}, m_{i-1}^+) : \ s_{i-1} = s_{i-1}^d, \ m_{i-1}^+ \in [m_{i-1}^d, 2(g_{i-1} - s_{i-1}^d)/h_{i-1}]\}$$

Then the following implications hold:

1. *If $2(g_i - (2g_{i-1} - s_{i-1}^d))/h_i < 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ then $W_i = \emptyset$.*
2. *If $2(g_i - (2g_{i-1} - s_{i-1}^d))/h_i = 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ then*

$$W_i = \{(2g_{i-1} - s_{i-1}^d, 2(g_{i-1} - s_{i-1}^d)/h_{i-1})\} \tag{25}$$

3. *If $2(g_i - (2g_{i-1} - s_{i-1}^d))/h_i > 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ and $m_{i-1}^d = 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ then*

$$W_i = \left\{ \begin{array}{l} (s_i, m_i^+) : \ s_i = 2g_{i-1} - s_{i-1}^d, \ m_i^+ \geq m_{i-1}^d, \\ \quad m_i^+ \leq 2(g_i - s_i^d)/h_i)) \end{array} \right\} \tag{26}$$

4. *If $2(g_i - (2g_{i-1} - s_{i-1}^d))/h_i > 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ and $m_{i-1}^d < 2(g_{i-1} - s_{i-1}^d)/h_{i-1}$ then*

$$W_i = \left\{ \begin{array}{l} (s_i, m_i^+) : \ s_i \geq 2g_{i-1} - s_{i-1}^d, \ m_i^+ \leq 2(g_i - s_i)/h_i, \\ \quad m_i^+ \geq f_m^{i-1}(s_{i-1}^d, s_i, m_{i-1}^d) \end{array} \right\} \tag{27}$$

Proof The lemma 3.4 implies that

$$\begin{aligned} m_{i-1}^{min} &:= \min\{m_{i-1}^+ : \ \exists s_{i-1} \text{ such that } (s_{i-1}, m_{i-1}^+) \in W_{i-1}\} \\ &= 2(g_{i-1} - s_{i-1}^d)/h_{i-1}, \\ s_i^{min} &:= \min\{s_i : \ \exists m_i^+ \text{ such that } (s_i, m_i^+) \in W_i\} = 2g_{i-1} - s_{i-1}^d, \\ m_i^{max} &:= \min\{m_i^+ : \ \exists s_i \text{ such that } (s_i, m_i^+) \in W_i\} = 2(g_i - s_i^{min})/h_i \\ &= 2(g_i - (2g_{i-1} - s_{i-1}^d))/h_i. \end{aligned}$$

If $m_{i-1}^{min} > m_i^{max}$ then there can not exist convex interpolant and $W_i = \emptyset$.

If $m_{i-1}^{min} = m_i^{max}$ then the only convex interpolant on interval $[x_{i-1}, x_{i+1}]$ is line and this implies (25).

If $m_{i-1}^{min} < m_i^{max}$ then (26) and (27) are obtained from consequence 3.6. □

Consequence 3.11 *Let us have given s_n^d, $m_n^d \le 2(g_n - s_n^n)/h_n$ and the set*

$$W_n = \{(s_n, m_n^+ : \; s_n = s_n^d, \; m_n^+ \in [m_n^d, 2(g_n - s_n^d)/h_n]\}$$

Then the following implications hold:

1. *If $m_n^d = 2(g_n - s_n^d)/h_n$ then*

$$W_{n+1} = \{(2g_n - s_n^d, 2(g_n - s_n^d)/h_n\} \tag{28}$$

2. *If $m_n^d < 2(g_n - s_n^d)/h_n$ then*

$$W_{n+1} = \left\{ \begin{array}{l} (s_{n+1}, m_{n+1}^-) : \; ((s_{n+1} = 2g_n - s_n^d) \\ \quad \wedge (m_{n+1}^- = 2(g_n - s_n^n)/h_n)) \vee ((s_{n+1} > 2g_n - s_d^d) \wedge \\ \quad (m_{n+1}^- \ge f_m^n(s_n^d, s_{n+1}, m_n^d)) \end{array} \right\} \tag{29}$$

Proof Can be followed similar way as in proof of consequence 3.10, using consequence 3.5 instead of consequence 3.6. □

Now for $i \le n$ and W_{i-1} such that $\operatorname{int} W_{i-1} \ne \emptyset$ we obtain following two consequences.

Consequence 3.12 *Let us have given $i \le n$, $-\infty \le s_{i-1}^d < \infty$, the increasing continuous function $r_{i-1}(s_{i-1})$ such that $r_{i-1}(s_{i-1}^d) < 2(g_{i-1} - s_{i-1}^d)/h_i$, $m_{i-1}^d = r_{i-1}(s_{i-1}^d)$ and the set*

$$W_{i-1} = \left\{ \begin{array}{l} (s_{i-1}, m_{i-1}^+) : \; s_{i-1} \ge s_{i-1}^d, \; m_{i-1}^+ \le 2(g_{i-1} - s_{i-1})/h_{i-1}, \\ \quad m_{i-1}^+ \ge r_{i-1}(s_{i-1}) \end{array} \right\}$$

Let be s_{i-1}^m such that $r_{i-1}(s_{i-1}^m) = 2(g_{i-1} - s_{i-1}^m)/h_{i-1}$. Then the following implications hold:

1. *If $2(g_i - (2g_{i-1} - s_{i-1}^m))/h_i < 2(g_{i-1} - s_{i-1}^m)/h_{i-1}$ then $W_i = \emptyset$.*
2. *If $2(g_i - (2g_{i-1} - s_{i-1}^m))/h_i = 2(g_{i-1} - s_{i-1}^m)/h_{i-1}$ then*

$$W_i = \{(2g_{i-1} - s_{i-1}^m, 2(g_{i-1} - s_{i-1}^m)/h_{i-1})\} \tag{30}$$

3. *If $2(g_i - (2g_{i-1} - s_{i-1}^m))/h_i > 2(g_{i-1} - s_{i-1}^m)/h_{i-1}$ then*

$$W_i = \left\{ \begin{array}{l} (s_i, m_i^+) : \; s_i > 2g_{i-1} - s_{i-1}^m, \; m_i^+ \le 2(g_i - s_i)/h_i, \\ \quad \text{if } s_i \le 2g_{i-1} - s_{i-1}^d \text{ then } m_i^+ \ge 2(s_i - g_{i-1})/h_{i-1} \\ \qquad\qquad\qquad\qquad \text{else } m_i^+ \ge f_m^{i-1}(s_{i-1}^d, s_i, m_{i-1}^d) \end{array} \right\} \tag{31}$$

Proof The lemma 3.4 implies that

$$\begin{aligned} m_{i-1}^{min} &:= \min\{m_{i-1}^+ : \exists s_{i-1} \text{ such that } (s_{i-1}, m_{i-1}^+) \in W_{i-1}\} \\ &= 2(g_{i-1} - s_{i-1}^m)/h_{i-1}, \\ s_i^{min} &:= \min\{s_i : \exists m_i^+ \text{ such that } (s_i, m_i^+) \in W_i\} = 2g_{i-1} - s_{i-1}^m, \\ m_i^{max} &:= \min\{m_i^+ : \exists s_i \text{ such that } (s_i, m_i^+) \in W_i\} = 2(g_i - s_i^{min})/h_i \\ &= 2(g_i - (2g_{i-1} - s_{i-1}^m))/h_i. \end{aligned}$$

If $m_{i-1}^{min} > m_i^{max}$ then there can not exist convex interpolant and $W_i = \emptyset$.

If $m_{i-1}^{min} = m_i^{max}$ then the only convex interpolant on interval $[x_{i-1}, x_{i+1}]$ is line and this implies (30).

If $m_{i-1}^{min} < m_i^{max}$ then we obtain (31) from consequence 3.6 and from properties of f_m^{i-1} given in lemma 3.4. If $s_i > 2g_{i-1} - s_{i-1}^d$ then $s_i > 2g_{i-1} - s_{i-1}^m$ and for all $(s_{i-1}, m_{i-1}^-) \in W_{i-1}$ we can compute $f_m^{i-1}(s_{i-1}, s_i, m_{i-1}^+)$. From (16) we obtain that $f_m^{i-1}(s_{i-1}, s_i, m_{i-1}^+) \geq f_m^{i-1}(s_{i-1}^d, s_i, m_{i-1}^d)$ for all $(s_{i-1}, m_{i-1}^-) \in W_{i-1}$. If $2g_{i-1} - s_{i-1}^m \leq s_i \leq 2g_{i-1} - s_{i-1}^d$ than we can compute $f_m^{i-1}(s_{i-1}, s_i, m_{i-1}^+)$ for all $(s_{i-1}, m_{i-1}^-) \in W_{i-1}$ such that $s_{i-1} \geq 2g_{i-1} - s_i$. From (20) it is obtained that $m_i^+ \geq 2(s_i - g_{i-1})/h_{i-1}$. □

Consequence 3.13 *Let us have given* $-\infty \leq s_n^d < \infty$, *the increasing continuous function* $r_n(s_n)$ *such that* $r_n(s_n^d) < 2(g_n - s_n^d)/h_n$, $m_n^d = r_n(s_n^d)$ *and the set*

$$W_n = \left\{ \begin{array}{l} (s_n, m_n^+) : \ s_n \geq s_n^d, \ m_n^+ \leq 2(g_n - s_n)/h_n, \\ \quad m_n^+ \geq r_n(s_n) \end{array} \right\}$$

Let be s_n^m *such that* $r_n(s_n^m) = 2(g_n - s_n^m)/h_n$. *Then*

$$\begin{aligned} W_{n+1} &= \{2g_n - s_n^m, 2(g_n - s_n)/h_n\} \\ &\cup \left\{ \begin{array}{l} (s_{n+1}, m_{n+1}^-) : \ s_{n+1} > 2g_n - s_n^m, \\ \quad \textit{if } s_{n+1} \leq 2g_n - s_n^d \ \textit{ then } \ m_{n+1}^+ \geq 2(s_{n+1} - g_n)/h_n \\ \qquad\qquad\qquad\qquad\quad \textit{else } \ m_{n+1}^+ \geq f_m^n(s_n^d, s_{n+1}, m_n^d) \end{array} \right\} \end{aligned} \tag{32}$$

Proof Can be followed the similar way as in proof of consequence 3.12, using consequence 3.5 instead of consequence 3.6. □

Remark The functions $r_i(s_i)$ for $i = 1(1)n$ in previous consequences can be given by one of the following rules:

1. $r_i(s_i) = 2(s_i - g_{i-1})/h_{i-1}$
2. $r_i(s_i) = f_m^{i-1}(s_{i-1}^d, s_i, m_{i-1}^d)$
3. $r_i(s_i) = \left\{ \begin{array}{ll} f_m^{i-1}(s_{i-1}^d, s_i, m_{i-1}^d) & \text{if } s_i > 2g_{i-1} - s_{i-1}^d \\ 2(s_i - g_{i-1})/h_{i-1} & \text{others} \end{array} \right.$

where s_{i-1}^d and m_{i-1}^d are given by set W_{i-1}

3.3 Algorithm for finding convex $p(x) \in S_{11}(\Delta^\alpha x)$ interpolating G

Let us have given sets $W_i \neq \emptyset$ for $i = 0(1)n+1$ from algorithm 3.7 and functions F^i_m from (5) for $i = 0(1)n$. Let us denote

$$s^d_i = \min\{s_i : \exists m^+_i \text{ such that } (s_i, m^+_i) \in W_i\} \text{ for } i = 0(1)n+1 \quad (33)$$

$$m^d_i = \min\{m^+_i : (s^d_i, m^+_i) \in W_i\} \text{ for } i = 0(1)n \quad (34)$$

$$m^d_{n+1} = \min\{m^-_{n+1} : (s^d_{n+1}, m^-_{n+1}) \in W_{n+1}\} \quad (35)$$

Algorithm 3.14

1. *Choose some* $(s_{n+1}, m^-_{n+1}) \in W_{n+1}$

2. *for* $j = n(-1)1$ *do:*
   choose some $(s_j, m^+_j) \in W_j \cap F^j_m(s_j, s_{j+1}, m^-_{j+1})$
   if $s_j = s^d_j$ *then put* $m^-_j = m^d_j$ *else choose* m^-_j *such that* $(s_j, m^-_j) \in W_j$

3. *Choose some* $(s_0, m^+_0) \in W_0 \cap F^0_m(s_0, s_1, m^-_1)$

Remark The choices in algorithm can be done for example in the following way:

1. The choice of s_{n+1} and m^-_{n+1}: $s_{n+1} = s^d_{n+1}$, $m_{n+1} = 2g_n - s_{n+1}$

2. The choice of s_j and m^+_j for $j = 0(1)n$:
   if $2g_j - s_{j+1} \le s^d_j$ then $s_j = s^d_j$ else $s_j = 2g_j - s_{j+1}$
   $m^+_j = F^j_m(s_j, s_{j+1}, m^-_{j+1})$

3. The choice of m^-_j for $j = 1(1)n$: $m^-_j = m^+_j$

4 Numerical example

Example 4.1 The histogram G was obtained as mean values of function $\sin(x)$ on mesh $(\Delta x) = \{\pi + i\pi/10\}_{i=0}^{12}$. The algorithm find that histograms is convex on interval $[\pi, 2\pi + \pi/10]$ in which the interval of convexity of function $\sin(x)$ is contained (see Fig 6).

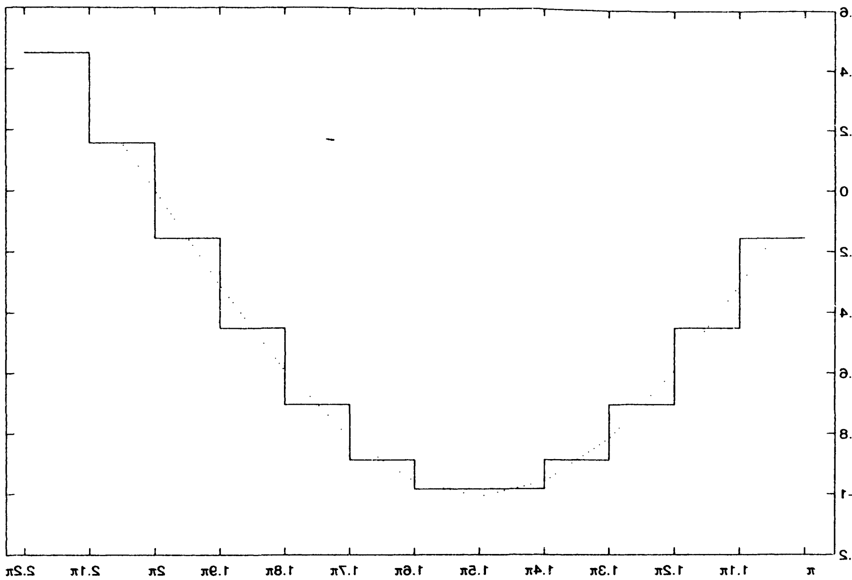

Figure 6

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