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Disjoint Unions of Incidence Structures and Complete Lattices *

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Abstract

Conceptual lattices of disjoint unions of incidence structures are studied in this paper. It is showed that these lattices are disjoint unions of their complete sublattices. An isomorphism between the ordered set of all disjoint unions of an incidence structure and the ordered set of all equivalence classes of disjoint unions of corresponding conceptual lattice is described.

Key words: Incidence structures, conceptual lattices, disjoint union of complete lattices.

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Definition 1 Let G and M be sets and $I \subseteq G \times M$. Then the triple (G, M, I) is called an *incidence structure.* If $A \subseteq G$, $B \subseteq M$ are non-empty sets, then we denote

$$A^{\uparrow} = \{m \in M \mid gIm \ \ \forall g \in A\}, \quad B^{\downarrow} = \{g \in G \mid gIm \ \ \forall m \in B\}.$$

For the empty set we put $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$. And moreover, we denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$, $g^{\uparrow} := \{g\}^{\uparrow}$, $m^{\downarrow} := \{m\}^{\downarrow}$ for $A \subseteq G$, $B \subseteq M$ and $g \in G$, $m \in M$.

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Theorem 1 *Let $\mathcal{J} = (G, M, I)$ be an incidence structure. If we put*

$$\mathcal{G}(\mathcal{J}) = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\},$$

then $\mathcal{L}(\mathcal{J}) = (\mathcal{G}(\mathcal{J}), \subseteq) = (\mathcal{G}(\mathcal{J}), \wedge, \vee)$ is a complete lattice in which

$$\bigwedge_{i\in Q} A_i = \bigcap_{i\in Q} A_i \quad and \quad \bigvee_{i\in Q} A_i = \Big(\bigcap_{i\in Q} A_i^{\uparrow}\Big)^{\downarrow}$$

for $A_i \in \mathcal{G}(\mathcal{J})$, $i \in Q$. (See [1, 3].)

Remark 1 If we denote $\mathcal{M}(\mathcal{J}) = \{B \subseteq M \mid B = B^{\downarrow\uparrow}\}$, then ${}^1\mathcal{L}(\mathcal{J}) = (\mathcal{M}(\mathcal{J}), \subseteq)$ is a complete lattice too. The lattices $\mathcal{L}(\mathcal{J})$ and ${}^1\mathcal{L}(\mathcal{J})$ are dually isomorphic. The lattices $\mathcal{L}(\mathcal{J})$ and ${}^1\mathcal{L}(\mathcal{J})$ are called *conceptual lattices* of the incidence structure $\mathcal{J}$.

Definition 2 An incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is a *substructure* of $\mathcal{J} = (G, M, I)$ if $G_1 \subseteq G, M_1 \subseteq M$ and $I_1 = I \cap (G_1 \times M_1)$.

In what follows we suppose that the sets G and M are not empty and $g^{\uparrow} \neq \emptyset$, $m^{\downarrow} \neq \emptyset$ for all $g \in G, m \in M$.

Definition 3 Let $\mathcal{J} = (G, M, I)$ be an incidence structure. The sequence $(g_0, m_0, g_1, m_1, \ldots, g_{r-1}, m_{r-1}, g_r)$ where $g_i \in G$ for $i \in \{0, \ldots, r\}$, $m_j \in M$ for $j \in \{0, \ldots, r-1\}$ and $g_j I m_j, g_{j+1} I m_j$ for all $j \in \{0, \ldots, r-1\}$, is called a *join* of elements g_0, g_r. If a join of elements $g_0, g_r \in G$ exists, then these elements are *joinable.*

In a similar way we can define a join of elements of M.

We say that an incidence structure $\mathcal{J}$ is *irreducible* if every two elements of G are joinable.

If we define a relation $\sim$ on G by setting $g \sim h \Leftrightarrow g, h$ are joinable, then $\sim$ is equivalence relation and it determines a decomposition $\widetilde{G}$ of G. In a similar way we define an equivalence relation $\sim$ on M assigning a decomposition $\widetilde{M}$.

Definition 4 An incidence structure $\mathcal{J} = (G, M, I)$ is called a *disjoint union* of its substructures $\mathcal{J}_t = (G_t, M_t, I_t), t \in T$ if $\overline{G} = \{G_t \mid t \in T\}, \overline{M} = \{M_t \mid t \in T\}$ and $\overline{I} = \{I_t \mid t \in T\}$ are decompositions of G, M, I. It will be denoted by $\mathcal{J} = \dot{\bigcup}_{t\in T} \mathcal{J}_t$.

Lemma 1 *Let $\mathcal{J} = \dot{\bigcup}_{t\in T} \mathcal{J}_t$ and $|T| > 1$. We will write the operators $\uparrow$ and $\downarrow$ on the right-hand side of the set symbol in $\mathcal{J}$ and on the left-hand side in substructures $\mathcal{J}_t$. Then the following statements are valid:*

1. *If $A \subseteq G$ and $A \not\subseteq G_t$ for any $t \in T$, then $A^{\uparrow} = \emptyset$ and $A^{\uparrow\downarrow} = G$.*
2. *Let $A \subseteq G_t$ for certain $t \in T$.*
   (a) *If $A \neq \emptyset$, then $A^{\uparrow} = {}^{\uparrow}A$. Moreover, $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ if and only if $A^{\uparrow} \neq \emptyset$.*
   (b) *If $A = \emptyset$, then $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ if and only if ${}^{\downarrow}M_t = \emptyset$.*

Analogous statements are valid for $B \subseteq M$.

Theorem 2 *Let $\mathcal{J} = (G, M, I)$ be an incidence structure, $\overline{G} = \{G_t \mid t \in T\}$ a decomposition of G, $\{M_t \mid t \in T\}$ a system of subsets of M and $\mathcal{J}_t = (G_t, M_t, I_t)$, $t \in T$, substructures of $\mathcal{J}$. The following conditions are equivalent:*

(i) $\mathcal{J}$ is a disjoint union of the substructures $\mathcal{J}_t$.

(ii) The decomposition $\widetilde{G}$ is a refinement of $\overline{G}$ ($\widetilde{G} \leq \overline{G}$) and $M_t = \bigcup_{g \in G_t} g^{\uparrow}$ for all $t \in T$.

Proof $(i) \Rightarrow (ii)$ By Definition 4, $\overline{M} = \{M_t \mid t \in T\}$ and $\overline{I} = \{I_t \mid t \in T\}$ are decompositions of sets M and I. Let $m \in M_t$ for $t \in T$. Since $m^{\downarrow} \neq \emptyset$, there exists $g \in G$ such that gIm. Hence gI_tm and $g \in G_t$ which yields $m \in \bigcup_{g \in G_t} g^{\uparrow}$. Conversely, if $m \in \bigcup_{g \in G_t} g^{\uparrow}$, then gI_tm for certain $g \in G_t$ and $m \in M_t$. Thus $M_t = \bigcup_{g \in G_t} g^{\uparrow}$.

Let $G' \in \widetilde{G}$ and $g \in G'$. Since $\overline{G}$ is a decomposition of G, there exists precisely one $t \in T$ such that $g \in G_t$. Consider an arbitrary $h \in G'$. Then g and h are joinable elements and there exists a sequence $(g, m_0, g_1, \ldots, g_{r-1}, m_{r-1}, h)$ with the properties from Definition 3. Thus gIm_0 and because of $g \in G_t$ we get $m_0 \in M_t$. Similarly, from g_1Im_0 and $m_0 \in M_t$ we obtain $g_1 \in G_t$ and so on. At the end from hIm_{r-1} and $m_{r-1} \in M_t$ we get $h \in G_t$. Hence $G' \subseteq G_t$ and $\widetilde{G} \leq \overline{G}$.

$(ii) \Rightarrow (i)$ a) $\overline{M} = \{M_t \mid t \in T\}$ is a decomposition of M: Consider $m \in M$. There exists $g \in G$ such that gIm, $g \in G_t$ for precisely one $t \in T$ and $m \in g^{\uparrow}$. Since $M_t = \bigcup_{g \in G_t} g^{\uparrow}$, we get $m \in M_t$ and $M = \bigcup_{t \in T} M_t$. Let us suppose that $m \in M_{t_1} \cap M_{t_2}$ for $t_1, t_2 \in T$. Then $m \in g_1^{\uparrow}, m \in g_2^{\uparrow}$ for certain $g_1 \in G_{t_1}, g_2 \in G_{t_2}$ and the elements g_1, g_2 are joinable. From $\widetilde{G} \leq \overline{G}$ we get $G_{t_1} = G_{t_2}$ and the definition of the sets M_{t_1}, M_{t_2} implies $M_{t_1} = M_{t_2}$.

b) $\overline{I} = \{I_t \mid t \in T\}$ is a decomposition of I: Consider $(g, m) \in I$. Then $g \in G_t$ for certain $t \in T$ and $m \in g^{\uparrow}$. Hence $m \in M_t$. We obtain $(g, m) \in I_t$. If $(g, m) \in I_{t_1} \cap I_{t_2}$, then $g \in G_{t_1}, g \in G_{t_2}$. Thus $t_1 = t_2$ and $I_{t_1} = I_{t_2}$. □

Let X be a non-empty set and $R(X)$ the set of all decompositions of X. We can define an ordering on $R(X)$ by setting $\overline{A} \leq \overline{B}$ iff $\overline{A}$ is a refinement of $\overline{B}$ for $\overline{A}, \overline{B} \in R(X)$. Then $L(X) = (R(X), \leq)$ is a (complete) partition lattice with the greatest element $\overline{A} = \{X\}$ and the least element $\overline{\overline{A}} = \{\{x\} \mid x \in X\}$. (See [2] (IV,4).)

Let us denote the set of all disjoint unions of an incidence structure $\mathcal{J} = (G, M, I)$ by $DS(\mathcal{J})$. We can define an ordering on $DS(\mathcal{J})$ by the following formula:

$$\dot{\bigcup_{t \in T}}{}^1 \mathcal{J}_t \leq \dot{\bigcup_{k \in K}}{}^2 \mathcal{J}_k \quad \text{iff} \quad \overline{G} \leq \overline{\overline{G}}$$

where $\overline{G}, \overline{\overline{G}}$ are decompositions of G belonging to $\dot{\bigcup}_{t \in T}{}^1 \mathcal{J}_t, \dot{\bigcup}_{k \in K}{}^2 \mathcal{J}_k$.

From Theorem 2 we obtain:

Theorem 3 *The ordered set $(DS(\mathcal{J}), \leq)$ is isomorphic to the upper cone of the element $\widetilde{G} \in R(G)$ in the lattice $L(G)$.*

Remark 2 $(DS(\mathcal{J}), \leq)$ is a complete lattice. The greatest element is a trivial disjoint union $\dot{\bigcup}_{t\in T}\mathcal{J}_t$ for $|T| = 1$, the least element is a disjoint union generated by the decomposition $\widetilde{G}$ of G.

$(DS(\mathcal{J}), \leq)$ is a one-element lattice if and only if $\mathcal{J}$ is irreducible and it is isomorphic to $L(G)$ if and only if $\mathcal{J}$ is completely irreducible, i.e. $|g^{\uparrow}| = 1$ for all $g \in G$.

Let us consider a complete lattice $\mathcal{L} = (L, \wedge, \vee) = (L, \leq)$ with the least element 0 and the greatest element 1. We can assign an incidence structure $\mathcal{J}(\mathcal{L}) = (L, L, I)$ to $\mathcal{L}$ where $xIy \Leftrightarrow x \leq y$. Denote by $\mathcal{L}(\mathcal{J}(\mathcal{L})) = (\mathcal{G}(\mathcal{J}(\mathcal{L})), \subseteq)$ its conceptual lattice. Then $A \in \mathcal{G}(\mathcal{J}(\mathcal{L}))$ iff $A = D(x)$ where $D(x)$ is the lower cone of certain element $x \in L$.

Lemma 2 *The mapping $x \mapsto D(x)$, $x \in L$, is an isomorphism of complete lattices $\mathcal{J}$ and $\mathcal{L}(\mathcal{J}(\mathcal{L}))$.*

Lemma 3 *Let $\mathcal{L} = (L, \leq)$ be a complete lattice and consider the incidence structure $\mathcal{J}(\mathcal{L})$. Let us put $L' = L - \{0,1\}$ and consider a substructure $\mathcal{J}(\mathcal{L}') = (L', L', \leq')$ of $\mathcal{J}(\mathcal{L})$. The conceptual lattices $\mathcal{L}(\mathcal{J}(\mathcal{L}))$ and $\mathcal{L}(\mathcal{J}(\mathcal{L}'))$ are isomorphic if and only if the ordered set $(L', \leq')$ does not have the greatest element.*

Remark 3 If $(L', \leq')$ does not have the greatest element, then the mapping assigning $D(x) \mapsto D(x) - \{0\}$ for $x \neq 1$ and $D(1) = L \mapsto L'$ is an isomorphism of the lattices $\mathcal{L}(\mathcal{J}(\mathcal{L}))$ and $\mathcal{L}(\mathcal{J}(\mathcal{L}'))$, and also $x \mapsto D(x) - \{0\}$ for $x \neq 1$, $1 \mapsto L'$ is an isomorphism of the lattices $\mathcal{L}$, $\mathcal{L}(\mathcal{J}(\mathcal{L}'))$.

Definition 5 Let $\mathcal{L} = (L, \leq)$ be a complete lattice (0 is the least and 1 is the greatest element in it) and $\mathcal{L}_t = (L_t, \leq_t)$, $t \in T$, be complete sublattices of $\mathcal{L}$. We put $L' = L - \{0,1\}$, $L'_t = L_t - \{0,1\}$, $\leq' = \leq \cap (L' \times L')$ and $\leq'_t = \leq \cap (L'_t \times L'_t)$ for $t \in T$. The complete lattice $\mathcal{L}$ is called a *disjoint union* of the complete sublattices $\mathcal{L}_t$ if $\overline{L'} = \{L'_t \mid t \in T\}$ is a decomposition of L' and $\overline{\leq'} = \{\leq'_t \mid t \in T\}$ is a decomposition of $\leq'$. We will write $\mathcal{L} = \dot{\bigcup}_{t\in T}\mathcal{L}_t$.

Remark 4 If $\mathcal{L} = \dot{\bigcup}_{t\in T}\mathcal{L}_t$, $|T| > 1$, then the ordered set $(L', \leq')$ does not have the greatest element.

Remark 5 Let $\mathcal{L} = \dot{\bigcup}_{t\in T}\mathcal{L}_t$. Then $(L_t \cup \{0\}, \leq'')$, $(L_t \cup \{1\}, \leq'')$ and $(L_t \cup \{0,1\}, \leq'')$, $t \in T$, are also complete sublattices of $\mathcal{L}$ where $\leq''$ denotes the restriction of the relation $\leq$ to the corresponding sets.

Definition 6 Disjoint unions $\dot{\bigcup}_{t\in T}{}^1\mathcal{L}_t$ and $\dot{\bigcup}_{k\in K}{}^2\mathcal{L}_k$ of a complete lattice $\mathcal{L}$ where ${}^1\mathcal{L}_t = ({}^1L_t, \leq^1_t)$ for all $t \in T$ and ${}^2\mathcal{L}_k = ({}^2L_k, \leq^2_k)$ for all $k \in K$ are *equivalent* if $\{{}^1L'_t \mid t \in T\} = \{{}^2L'_k \mid k \in K\}$.

Theorem 4 *Let an incidence structure $\mathcal{J} = (G, M, I)$ be a disjoint union of its substructures $\mathcal{J}_t = (G_t, M_t, I_t)$ for $t \in T$ and $|T| > 1$. Let $\mathcal{L}(\mathcal{J}) = (\mathcal{G}(\mathcal{J}), \subseteq)$ and $\mathcal{L}(\mathcal{J}_t) = (\mathcal{G}(\mathcal{J}_t), \subseteq)$, $t \in T$, be conceptual lattices of $\mathcal{J}$ and $\mathcal{J}_t$. Moreover, let 0, 1, and $0_t, 1_t$ be their least and greatest elements, respectively.*

- *If $G_t^{\uparrow} \neq \emptyset$, then $\mathcal{L}(\mathcal{J}_t)$ is a complete sublattice in $\mathcal{L}(\mathcal{J})$.*
- *If $G_t^{\uparrow} = \emptyset$, then $\mathcal{L}^*(\mathcal{J}_t) = ((\mathcal{G}(\mathcal{J}_t) - \{1_t\}) \cup \{1\}, \subseteq) = (\mathcal{G}^*(\mathcal{J}_t), \subseteq)$ is a complete sublattice in $\mathcal{L}(\mathcal{J})$.*

  Moreover $\mathcal{L}(\mathcal{J}) = \dot{\bigcup}_{t \in T} \widehat{\mathcal{L}(\mathcal{J}_t)}$ where $\widehat{\mathcal{L}(\mathcal{J}_t)} = \mathcal{L}(\mathcal{J}_t)$ for $G_t^{\uparrow} \neq \emptyset$ and $\widehat{\mathcal{L}(\mathcal{J}_t)} = \mathcal{L}^*(\mathcal{J}_t)$ for $G_t^{\uparrow} = \emptyset$.

Proof Obviously $1 = G$ and $1_t = G_t$ for all $t \in T$. Because of $|T| > 1$ we get $M^{\downarrow} = \emptyset$, $G^{\uparrow} = \emptyset$ and $\emptyset^{\uparrow\downarrow} = \emptyset$. Thus $\emptyset = 0$. If $t \in T$, then we will write the operators $\uparrow, \downarrow$ in $\mathcal{J}_t$ to the left.

1. Consider $A \in \mathcal{G}(\mathcal{J}_t)$ for $t \in T$. It means that $A = {}^{\downarrow\uparrow}A$. First we suppose that $A \neq G_t$. If $A = \emptyset$, then $A = 0$ and $A \in \mathcal{G}(\mathcal{J})$. Assume $A \neq \emptyset$. According to Lemma 1 we get $A^{\uparrow} = {}^{\uparrow}A$.

If $A^{\uparrow} = \emptyset$, then $A = {}^{\downarrow\uparrow}A = {}^{\downarrow}\emptyset = G_t$. That is a contradiction. If $A^{\uparrow} \neq \emptyset$, then $A = {}^{\downarrow\uparrow}A = A^{\uparrow\downarrow}$ (according to Lemma 1 again) and thus $A \in \mathcal{G}(\mathcal{J})$.

Let $A = G_t$ and $G_t^{\uparrow} \neq \emptyset$. From $G_t \neq \emptyset$ we obtain $G_t = {}^{\downarrow\uparrow}G_t = G_t^{\uparrow\downarrow}$. This implies $1_t = G_t \in \mathcal{G}(\mathcal{J})$ and $\mathcal{G}(\mathcal{J}_t) \subseteq \mathcal{G}(\mathcal{J})$. If $G_t^{\uparrow} = \emptyset$, then $G_t^{\uparrow\downarrow} = G$. Hence $G_t^{\uparrow\downarrow} \neq G_t$ and $1_t \notin \mathcal{G}(\mathcal{J})$. Since $1 \in \mathcal{G}(\mathcal{J})$, we have obtained $\mathcal{G}^*(\mathcal{J}_t) \subseteq \mathcal{G}(\mathcal{J})$.

2. $\mathcal{L}(\mathcal{J}_t)$ and $\mathcal{L}^*(\mathcal{J}_t)$ are sublattices in $\mathcal{L}(\mathcal{J})$:

a) Let $G_t^{\uparrow} \neq \emptyset$. Then $\mathcal{G}(\mathcal{J}_t) \subseteq \mathcal{G}(\mathcal{J})$. Consider subsets $A_i \in \mathcal{G}(\mathcal{J}_t)$, $i \in Q$. Since $\mathcal{L}(\mathcal{J}_t)$ is a conceptual lattice of $\mathcal{J}_t$, we get (by Theorem 1) $\bigcap_{i \in Q} A_i \in \mathcal{G}(\mathcal{J}_t) \subseteq \mathcal{G}(\mathcal{J})$. Hence $\bigwedge_{i \in Q} A_i \in \mathcal{G}(\mathcal{J}_t)$.

Let $G_t^{\uparrow} = \emptyset$. We put ${}^1\mathcal{G}(\mathcal{J}_t) = \mathcal{G}(\mathcal{J}_t) - \{1_t\} \subseteq \mathcal{G}(\mathcal{J}_t)$. Consider $A_i \in \mathcal{G}^*(\mathcal{J}_t)$, $i \in Q$. If there exists $j \in Q$ such that $A_j \in {}^1\mathcal{G}(\mathcal{J}_t)$, then $\bigcap_{i \in Q} A_i \in {}^1\mathcal{G}(\mathcal{J}_t)$ and $\bigwedge_{i \in Q} A_i \in {}^1\mathcal{G}(\mathcal{J}_t) \subseteq \mathcal{G}^*(\mathcal{J}_t)$. If $A_i = G$ for all $i \in Q$, then $\bigwedge_{i \in Q} A_i = G = 1 \in \mathcal{G}^*(\mathcal{J}_t)$.

b) Let $A_i \in \mathcal{G}(\mathcal{J})$, $i \in Q$. Then (by Theorem 1) $\bigvee_{i \in Q} A_i = (\bigcap_{i \in Q} A_i^{\uparrow})^{\downarrow}$. If we put $B = \bigcap_{i \in Q} A_i^{\uparrow}$, then $B \subseteq M$ and $\bigvee_{i \in Q} A_i = B^{\downarrow}$.

First we suppose that $G_t^{\uparrow} \neq \emptyset$. Then $\mathcal{G}(\mathcal{J}_t) \subseteq \mathcal{G}(\mathcal{J})$. Let $A_i \in \mathcal{G}(\mathcal{J}_t)$, $i \in Q$. Because of $A_i \subseteq G_t$ we get $G_t^{\uparrow} \subseteq A_i^{\uparrow}$ for all $i \in Q$ and $G_t^{\uparrow} \subseteq B$. Thus $B \neq \emptyset$. We will assume that there exists $j \in Q$ such that $A_j \neq \emptyset$. By Lemma 1 $A_j^{\uparrow} = {}^{\uparrow}A_j$ and $A_j^{\uparrow} \subseteq M_t$. It means that $B \subseteq M_t$. Since $B \neq \emptyset$, we get ${}^{\downarrow}B = B^{\downarrow}$ and ${}^{\downarrow\uparrow}(B^{\downarrow}) = {}^{\downarrow\uparrow\downarrow}B = {}^{\downarrow}B = B^{\downarrow}$. This implies $B^{\downarrow} = \bigvee_{i \in Q} A_i \in \mathcal{G}(\mathcal{J}_t)$. Let $A_i = \emptyset$ for all $i \in Q$. Then $\emptyset \in \mathcal{G}(\mathcal{J}_t)$ and $A_i^{\uparrow} = M$ for all $i \in Q$. From this $B = M$ and $\bigvee_{i \in Q} A_i = B^{\downarrow} = M^{\downarrow} = \emptyset = 0$. Hence $\bigvee_{i \in Q} A_i \in \mathcal{G}(\mathcal{J}_t)$.

Let us assume that $G_t^{\uparrow} = \emptyset$. In this case $\mathcal{G}^*(\mathcal{J}_t) \subseteq \mathcal{G}(\mathcal{J})$. Consider $A_i \in {}^1\mathcal{G}(\mathcal{J}_t)$. If $B = \emptyset$, then $\bigvee_{i \in Q} A_i = G = 1 \in {}^1\mathcal{G}(\mathcal{J}_t)$. Let $B \neq \emptyset$. Then $B^{\downarrow} \neq G_t$

because from $B^{\downarrow} = G_t$ we get $B \subseteq B^{\downarrow\uparrow} = G_t^{\uparrow} = \emptyset$ which is a contradiction. From $B^{\downarrow} \in \mathcal{G}(\mathcal{J}_t) - \{1_t\}$ we obtain $\bigvee_{i\in Q} A_i \in {}^1\mathcal{G}(\mathcal{J}_t)$.

3. $\mathcal{L}(\mathcal{J})$ is a disjoint union of the complete sublattices $\widehat{\mathcal{L}(\mathcal{J}_t)}$, $t \in T$:

a) Let us put $\widehat{\mathcal{G}(\mathcal{J}_t)} := \mathcal{G}(\mathcal{J}_t)$ if $G_t^{\uparrow} \neq \emptyset$ and $\widehat{\mathcal{G}(\mathcal{J}_t)} := \mathcal{G}^*(\mathcal{J}_t)$ if $G_t^{\uparrow} = \emptyset$. And, moreover, we denote $\mathcal{G}'(\mathcal{J}) = \mathcal{G}(\mathcal{J}) - \{0,1\}$, $\widehat{\mathcal{G}'(\mathcal{J}_t)} = \widehat{\mathcal{G}(\mathcal{J}_t)} - \{0,1\}$. We will prove that $\overline{\mathcal{G}'(\mathcal{J})} = \{\widehat{\mathcal{G}'(\mathcal{J}_t)} \mid t \in T\}$ is a decomposition of $\mathcal{G}'(\mathcal{J})$:

Take $A \in \mathcal{G}'(\mathcal{J})$. Then $A \subseteq G$, $A^{\uparrow\downarrow} = A$, $A \neq G$, $A \neq \emptyset$. Assume that $A^{\uparrow} = \emptyset$. We get $A = A^{\uparrow\downarrow} = G$ and that is a contradiction. Thus $A^{\uparrow} \neq \emptyset$. If A is not contained in any subset G_t for $t \in T$, then $A^{\uparrow\downarrow} = G = A$. This is a contradiction again. Hence $A \subseteq G_t$ for certain $t \in T$. Since $A \neq \emptyset$ we obtain $A = A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ and $A \in \widehat{\mathcal{G}'(\mathcal{J}_t)}$. Suppose that $A \in \widehat{\mathcal{G}'(\mathcal{J}_{t_1})} \cap \widehat{\mathcal{G}'(\mathcal{J}_{t_2})}$. Then $A \subseteq G_{t_1} \cap G_{t_2}$ and because of $A \neq \emptyset$ we obtain $t_1 = t_2$.

b) $\overline{\subseteq'} = \{\subseteq_t' \mid t \in T\}$ is a decomposition of the set $\subseteq'$ (Definition 5): Consider $A, B \in \mathcal{G}'(\mathcal{J})$ such that $A \subseteq B$. This implies $A \subseteq' B$. According to (a) there exists $t \in T$ such that $B \in \widehat{\mathcal{G}'(\mathcal{J}_t)}$. Therefore $B \subseteq G_t$ and $A \subseteq G_t$. Hence $A \in \widehat{\mathcal{G}'(\mathcal{J}_t)}$ and $A \subseteq_t B$. Obviously $\subseteq_{t_1}' \cap \subseteq_{t_2}' = \emptyset$ for distinct $t_1, t_2 \in T$. □

Example 1 Let $\mathcal{J} = (G, M, I)$ be an incidence structure given by its incidence graph (Figure 1). Let us consider substructures $\mathcal{J}_1 = (G_1, M_1, I_1) = (\{a_1, a_2, a_3\}, \{m_1, m_2\}, I_1)$ and $\mathcal{J}_2 = (G_2, M_2, I_2) = (\{a_4, a_5\}, \{m_3, m_4\}, I_2)$ of $\mathcal{J}$. Then $\mathcal{J} = \mathcal{J}_1 \dot\cup \mathcal{J}_2$.

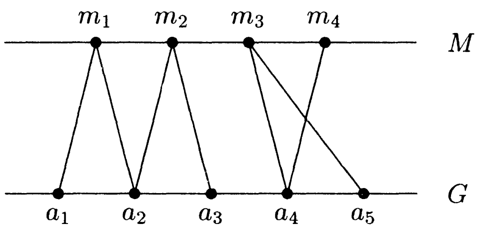


Figure 1

Figure 2 shows conceptual lattices $\mathcal{L}(\mathcal{J})$, $\mathcal{L}(\mathcal{J}_1)$ and $\mathcal{L}(\mathcal{J}_2)$.

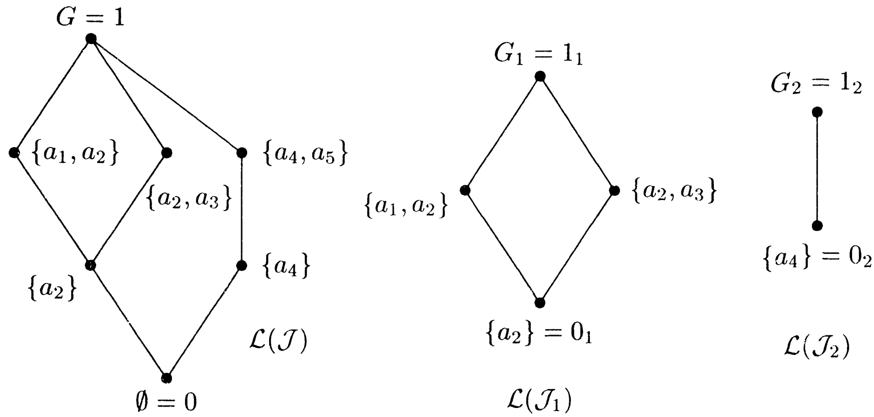


Figure 2

Since $G_2^{\uparrow} = \{m_3\}$, $\mathcal{L}(\mathcal{J}_2)$ is a sublattice of $\mathcal{L}(\mathcal{J})$. Because of $G_1^{\uparrow} = \emptyset$, $\mathcal{L}^*(\mathcal{J}_1) = ((\mathcal{G}(\mathcal{J}_1) - \{1_1\}) \cup \{1\}, \subseteq)$ is a sublattice of $\mathcal{L}(\mathcal{J})$ and $\mathcal{L}(\mathcal{J})$ is a disjoint union of the sublattices $\mathcal{L}^*(\mathcal{J}_1)$, $\mathcal{L}(\mathcal{J}_2)$.

Theorem 5 *Let a complete lattice $\mathcal{L} = (L, \leq)$ be a disjoint union of complete sublattices $\mathcal{L}_t = (L_t, \leq_t)$. We denote $L' = L - \{0, 1\}$, $L'_t = L_t - \{0, 1\}$ for $t \in T$. Then there exists an incidence structure $\mathcal{J}$ and a disjoint union $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$ such that the complete lattices $\mathcal{L}$, $\mathcal{L}(\mathcal{J})$ are isomorphic. Moreover, ordered sets $(L'_t, \leq'_t)$, $(\widehat{\mathcal{G}'(\mathcal{J}_t)}, \subseteq'_t)$ are isomorphic for all $t \in T$.*

Proof Consider incidence structures $\mathcal{J} = \mathcal{J}(\mathcal{L}') = (L', L', \leq')$, $\mathcal{J}_t = \mathcal{J}(\mathcal{L}'_t) = (L'_t, L'_t, \leq'_t)$ for $t \in T$. Then $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$. Let $|T| > 1$. Then, by Remark 4, the set $(L', \leq)$ does not have the greatest element. By Remark 3 the mapping $x \mapsto D(x) - \{0\}$ for $x \neq 1$ and $1 \mapsto L'$ is an isomorphism of the lattices $\mathcal{L}$, $\mathcal{L}(\mathcal{J})$. That induces an isomorphism of the ordered sets $(L'_t, \leq')$ and $(\widehat{\mathcal{G}'(\mathcal{J}_t)}, \subseteq_t)$ for all $t \in T$. The proposition is obvious for $|T| = 1$. $\square$

Let us denote the set of all disjoint unions of a complete lattice $\mathcal{L}$ by $DS(\mathcal{L})$. Consider a relation $\equiv$ on $DS(\mathcal{L})$ assigned by the formula

$$\dot{\bigcup_{t \in T}} {}^1\mathcal{L}_t \equiv \dot{\bigcup_{k \in K}} {}^2\mathcal{L}_k \quad \text{iff} \quad \dot{\bigcup_{t \in T}} {}^1\mathcal{L}_t \ \text{ and } \ \dot{\bigcup_{k \in K}} {}^2\mathcal{L}_k \ \text{ are equivalent.}$$

Then $\equiv$ is equivalence relation and it determines a decomposition $DS(\mathcal{L})/\equiv$ on $DS(\mathcal{L})$.

We define an ordering on the set $DS(\mathcal{L})/\equiv$: Let $\dot{\bigcup}_{t \in T} {}^1\mathcal{L}_t$, $\dot{\bigcup}_{k \in K} {}^2\mathcal{L}_k \in DS(\mathcal{L})$, the corresponding decompositions of L' we denote by $\overline{L'} = \{{}^1L'_t \mid t \in T\}$

and $\overline{\overline{L'}} = \{{}^2L'_k \mid k \in K\}$. For $\overline{\dot{\bigcup}_{t\in T} {}^1\mathcal{L}_t}$ and $\overline{\dot{\bigcup}_{k\in K} {}^2\mathcal{L}_k} \in DS(\mathcal{L})/\equiv$ we define

$$\overline{\dot{\bigcup_{t\in T}} {}^1\mathcal{L}_t} \leq \overline{\dot{\bigcup_{k\in K}} {}^2\mathcal{L}_k} \quad \text{iff} \quad \overline{L'} \quad \text{is a refinement of} \quad \overline{\overline{L'}} \quad (\overline{L'} \leq \overline{\overline{L'}}).$$

Theorem 6 *If $\mathcal{J}$ is a reducible incidence structure, then the ordered sets $DS(\mathcal{J})$ and $DS(\mathcal{L}(\mathcal{J}))/\equiv$ are isomorphic.*

Proof A conceptual lattice $\mathcal{L}(\mathcal{J})$ of a disjoint union $\mathcal{J} = \dot{\bigcup}_{t\in T}\mathcal{J}_t$ is a disjoint union $\dot{\bigcup}_{t\in T}\widehat{\mathcal{L}(\mathcal{J}_t)}$ where $\widehat{\mathcal{L}(\mathcal{J}_t)} = \mathcal{L}(\mathcal{J}_t)$ for $G_t^\uparrow \neq \emptyset$ and $\widehat{\mathcal{L}(\mathcal{J}_t)} = \mathcal{L}^*(\mathcal{J}_t)$ for $G_t^\uparrow = \emptyset$. Let us consider a mapping

$$\varphi : DS(\mathcal{J}) \to DS(\mathcal{L}(\mathcal{J}))/\equiv \, : \dot{\bigcup_{t\in T}} \mathcal{J}_t \mapsto \overline{\dot{\bigcup_{t\in T}} \widehat{\mathcal{L}(\mathcal{J}_t)}}$$

1. The mapping φ is injective: Take $\dot{\bigcup}_{t\in T}{}^1\mathcal{J}_t$, $\dot{\bigcup}_{k\in K}{}^2\mathcal{J}_k \in DS(\mathcal{J})$ and assume that

$$\varphi(\dot{\bigcup_{t\in T}} {}^1\mathcal{J}_t) = \varphi(\dot{\bigcup_{k\in K}} {}^2\mathcal{J}_k) \qquad \text{i.e.} \qquad \dot{\bigcup_{t\in T}} \widehat{\mathcal{L}({}^1\mathcal{J}_t)} \equiv \dot{\bigcup_{k\in K}} \widehat{\mathcal{L}({}^2\mathcal{J}_k)}$$

Hence $\{\widehat{\mathcal{G}'({}^1\mathcal{J}_t)} \mid t \in T\} = \{\widehat{\mathcal{G}'({}^2\mathcal{J}_k)} \mid k \in K\}$ by Definition 6. There exists a bijection $\xi : T \to K$ such that $\mathcal{G}'({}^2\mathcal{J}_{\xi(t)}) = \mathcal{G}'({}^1\mathcal{J}_t)$ for all $t \in T$. Let us put ${}^1\mathcal{J}_t = ({}^1G_t, {}^1M_t, {}^1I_t)$ and ${}^2\mathcal{J}_{\xi(t)} = ({}^2G_{\xi(t)}, {}^2M_{\xi(t)}, {}^2I_{\xi(t)})$.

Take $g \in {}^1G_t$. Then $g^\uparrow \neq \emptyset$ and $g^{\uparrow\downarrow} = {}^{\downarrow\uparrow}g$ by Lemma 1. Obviously $g^{\uparrow\downarrow} \neq \emptyset, G$. Let ${}^1G_t^\uparrow = \emptyset$. If $g^{\uparrow\downarrow} = {}^1G_t$, then $g^\uparrow = {}^1G_t^\uparrow \neq \emptyset$. That is a contradiction. Thus $g^{\uparrow\downarrow} \in \widehat{\mathcal{G}'({}^1\mathcal{J}_t)}$ and $g^{\uparrow\downarrow} \in \mathcal{G}'({}^2\mathcal{J}_{\xi(t)})$. From this $g^{\uparrow\downarrow} \subseteq {}^2G_{\xi(t)}$ and $g \in {}^2G_{\xi(t)}$. Therefore ${}^1G_t \subseteq {}^2G_{\xi(t)}$. In a similar way we show that ${}^2G_{\xi(t)} \subseteq {}^1G_t$ and thus ${}^2G_{\xi(t)} = {}^1G_t$. Obviously ${}^1M_t = {}^2M_{\xi(t)}$ and since $\mathcal{J}_t$, ${}^1\mathcal{J}_t$ are substructures in $\mathcal{J}$, we obtain ${}^1\mathcal{J}_t = {}^2\mathcal{J}_{\xi(t)}$ and $\dot{\bigcup}_{t\in T}{}^1\mathcal{J}_t = \dot{\bigcup}_{t\in T}{}^2\mathcal{J}_{\xi(t)} = \dot{\bigcup}_{k\in K}{}^2\mathcal{J}_k$.

2. φ is surjective: Consider a disjoint union $\dot{\bigcup}_{t\in T}\mathcal{L}_t \in DS(\mathcal{L}(\mathcal{J}))$. We denote $\mathcal{L}(\mathcal{J}) = (\mathcal{G}(\mathcal{J}), \subseteq)$, $\mathcal{G}'(\mathcal{J}) = \mathcal{G}(\mathcal{J}) - \{0, 1\}$, $\mathcal{L}_t = (\mathcal{G}_t, \subseteq)$, $\mathcal{G}'_t = \mathcal{G}_t - \{0, 1\}$. We prove that there exists a disjoint union $\dot{\bigcup}_{t\in T}\mathcal{J}_t \in DS(\mathcal{J})$ such that $\varphi(\dot{\bigcup}_{t\in T}\mathcal{J}_t) = \overline{\dot{\bigcup}_{t\in T}\mathcal{L}_t}$. Let us put $G_t = \bigcup_{A\in\mathcal{G}'_t} A$ for all $t \in T$. We show that $\overline{G} = \{G_t \mid t \in T\}$ is a decomposition of G:

Take $g \in G$. Then $g^{\uparrow\downarrow} \in \mathcal{G}(\mathcal{J})$, $g \in g^{\uparrow\downarrow}$ and $g^{\uparrow\downarrow} \neq \emptyset$. Since $\mathcal{J}$ is reducible, we get $0 = \emptyset$ and $g^{\uparrow\downarrow} \neq 0$. At the same time $G^\uparrow = \emptyset$, thus $g^{\uparrow\downarrow} \neq 1$. We obtain $g^{\uparrow\downarrow} \in \mathcal{G}'(\mathcal{J})$. There exists precisely one $t \in T$ such that $g^{\uparrow\downarrow} \in \mathcal{G}'_t$ because $\{\mathcal{G}'_t \mid t \in T\}$ is a decomposition of $\mathcal{G}'(\mathcal{J})$. This yields $g^{\uparrow\downarrow} \in \mathcal{G}_t$ and $g \in G_t$. If $g \in G_{t_1} \cap G_{t_2}$, then $g^{\uparrow\downarrow} \in \mathcal{G}'_{t_1} \cap \mathcal{G}'_{t_2}$ and because of $g^{\uparrow\downarrow} \neq \emptyset$ we get $t_1 = t_2$.

Let us put $M_t = \bigcup_{g\in G_t} g^\uparrow$ for any $t \in T$. Then $\{M_t \mid t \in T\}$ is a decomposition of M:

Consider $m \in M$. Then $m^{\downarrow} \neq \emptyset, G$ and $m^{\downarrow} \in \mathcal{G}'_t$ for a unique $t \in T$, thus $m^{\downarrow} \subseteq G_t$. We get $(m^{\downarrow})^{\uparrow} = \bigcap_{g \in m^{\downarrow}} g^{\uparrow} \subseteq M_t$, therefore $m \in M_t$.

Let $m \in M_{t_1} \cap M_{t_2}$. Then there exist $g_1 \in G_{t_1}, g_2 \in G_{t_2}$ such that $m \in g_1^{\uparrow} \cap g_2^{\uparrow}$. This yields $g_1^{\uparrow\downarrow}, g_2^{\uparrow\downarrow} \subseteq m^{\downarrow}$ and $g_1, g_2 \in m^{\downarrow}$. However, $m^{\downarrow} \in G_t$ for a unique $t \in T$. Hence $G_{t_1} = G_{t_2} = G_t$ and $t_1 = t_2$.

Consider substructures $\mathcal{J}_t = (G_t, M_t, I_t)$, $t \in T$, in $\mathcal{J}$. The system $\{I_t \mid t \in T\}$ is a decomposition of I:

Take $g\, I\, m$. Then $g \in G_t$ for certain $t \in T$ and $m \in g^{\uparrow}$, thus $m \in M_t$. Since $\mathcal{J}_t$ is a substructure in $\mathcal{J}$, we get $g I_t m$. Obviously $I_{t_1} \cap I_{t_2} = \emptyset$ for $t_1 \neq t_2$ and therefore $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$.

Now we show that $\mathcal{G}'(\mathcal{J}_t) = \mathcal{G}'_t$ for all $t \in T$:

Consider $A \in \mathcal{G}'_t \subseteq \mathcal{G}(\mathcal{J})$. Then $A = A^{\uparrow\downarrow}$, $A \neq \emptyset, G$ and $A \subseteq G_t$. If $A^{\uparrow} = \emptyset$, then $A^{\uparrow\downarrow} = G$. That is a contradiction. Hence $A^{\uparrow} \neq \emptyset$ and $A = A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ by Lemma 1. Thus $A \in \mathcal{G}'(\mathcal{J}_t)$ and $\mathcal{G}'_t \subseteq \mathcal{G}'(\mathcal{J}_t)$.

Conversely, let $A \in \mathcal{G}'(\mathcal{J}_t)$. Then $A = {}^{\downarrow\uparrow}A$. Once again $A^{\uparrow} \neq \emptyset$ and $A = {}^{\downarrow\uparrow}A = A^{\uparrow\downarrow}$. This yields $A \in \mathcal{G}'(\mathcal{J})$ and $A \in \mathcal{G}'_t$ by definition of the set G_t. Hence $\mathcal{G}'(\mathcal{J}_t) \subseteq \mathcal{G}'_t$. We have obtained

$$\dot{\bigcup_{t \in T}} \mathcal{L}_t \equiv \dot{\bigcup_{t \in T}} \mathcal{L}(\mathcal{J}_t) \qquad \text{and} \qquad \varphi(\dot{\bigcup_{t \in T}} \mathcal{J}_t) = \overline{\dot{\bigcup_{t \in T}} \mathcal{L}_t}$$

3. Ordered sets $DS(\mathcal{J})$ and $DS(\mathcal{L}(\mathcal{J}))/\equiv$ are isomorphic: Consider disjoint unions $\dot{\bigcup}_{t \in T} {}^1\mathcal{J}_t, \dot{\bigcup}_{k \in K} {}^2\mathcal{J}_k \in DS(\mathcal{J})$. Let $\overline{G} = \{{}^1G_t \mid t \in T\}$, $\overline{\overline{G}} = \{{}^2G_k \mid k \in K\}$ be corresponding decompositions of G. Assume that $\dot{\bigcup}_{t \in T} {}^1\mathcal{J}_t \leq \dot{\bigcup}_{k \in K} {}^2\mathcal{J}_k$. Then $\overline{G} \leq \overline{\overline{G}}$ and to every $t_0 \in T$ there exists $k_0 \in K$ such that ${}^1G_{t_0} \subseteq {}^2G_{k_0}$.

According to Theorem 4 we consider disjoint unions $\dot{\bigcup}_{t \in T} \mathcal{L}(\widehat{{}^1\mathcal{J}_t})$ and $\dot{\bigcup}_{k \in K} \mathcal{L}(\widehat{{}^2\mathcal{J}_k})$ of $DS(\mathcal{L}(\mathcal{J}))$. Then $\overline{\mathcal{G}'(\mathcal{J})} = \{\mathcal{G}'(\widehat{{}^1\mathcal{J}_t}) \mid t \in T\}$, $\overline{\overline{\mathcal{G}'(\mathcal{J})}} = \{\mathcal{G}'(\widehat{{}^2\mathcal{J}_k}) \mid k \in K\}$ are decompositions of $\mathcal{G}'(\mathcal{J})$.

We will write the operators $\uparrow, \downarrow$ to the right in $\mathcal{J}$, to the left in ${}^1\mathcal{J}_t$. In ${}^2\mathcal{J}_k$ we use symbols $\top, \bot$ instead. Then $\mathcal{G}({}^1\mathcal{J}_t) = \{A \subseteq {}^1G_t \mid A = {}^{\downarrow\uparrow}A\}$, $\mathcal{G}({}^2\mathcal{J}_k) = \{A \subseteq {}^2G_k \mid A = A^{\top\bot}\}$.

Take $t_0 \in T$. By assumption there exists $k_0 \in K$ such that ${}^1G_{t_0} \subseteq {}^2G_{k_0}$. Let $A \in \mathcal{G}'(\widehat{{}^1\mathcal{J}_{t_0}})$. Then $A, A^{\uparrow} \neq \emptyset$: For $A = \emptyset$ we get $A = 0$ (because $\emptyset = 0$). That is a contradiction. We know that ${}^{\uparrow}A = A^{\uparrow}$. Let $A^{\uparrow} = \emptyset$. Then $A = {}^{\downarrow\uparrow}A = {}^{\downarrow}\emptyset = {}^1G_{t_0}$ and $A^{\uparrow} = \emptyset = {}^1G_{t_0}^{\uparrow}$. However, in case of ${}^1G_{t_0}^{\uparrow} = \emptyset$ we get $1_{t_0} = {}^1G_{t_0} \notin \mathcal{G}'(\widehat{{}^1\mathcal{J}_{t_0}})$. That is a contradiction too.

By Lemma 1 we obtain ${}^{\downarrow\uparrow}A = A^{\top\bot}$ and $A \in \mathcal{G}'(\widehat{{}^2\mathcal{J}_{k_0}})$. From this $\mathcal{G}'(\widehat{{}^1\mathcal{J}_{t_0}}) \subseteq \mathcal{G}'(\widehat{{}^2\mathcal{J}_{k_0}})$ follows. Hence $\overline{\mathcal{G}'(\mathcal{J})} \leq \overline{\overline{\mathcal{G}'(\mathcal{J})}}$ and

$$\overline{\dot{\bigcup_{t \in T}} \mathcal{L}(\widehat{{}^1\mathcal{J}_t})} = \varphi(\dot{\bigcup_{t \in T}} {}^1\mathcal{J}_{t_0}) \leq \overline{\dot{\bigcup_{k \in K}} \mathcal{L}(\widehat{{}^2\mathcal{J}_k})} = \varphi(\dot{\bigcup_{k \in k}} {}^2\mathcal{J}_k)$$

Conversely, consider $\overline{\dot{\bigcup}_{t\in T}\mathcal{L}(\widehat{{}^1\mathcal{J}_t})}, \overline{\dot{\bigcup}_{k\in K}\mathcal{L}(\widehat{{}^2\mathcal{J}_k})} \in DS(\mathcal{L}(\mathcal{J}))$. Let $\overline{\mathcal{G}'(\mathcal{J})} = \{\mathcal{G}'(\widehat{{}^1\mathcal{J}_t}) \mid t \in T\}$, $\overline{\overline{\mathcal{G}'(\mathcal{J})}} = \{\mathcal{G}'(\widehat{{}^2\mathcal{J}_k}) \mid k \in K\}$. Assume that

$$\overline{\dot{\bigcup_{t\in T}}\mathcal{L}(\widehat{{}^1\mathcal{J}_t})} \le \overline{\dot{\bigcup_{k\in K}}\mathcal{L}(\widehat{{}^2\mathcal{J}_k})}.$$

Thus $\overline{\mathcal{G}'(\mathcal{J})} \le \overline{\overline{\mathcal{G}'(\mathcal{J})}}$.

Take $\dot{\bigcup}_{t\in T}{}^1\mathcal{J}_t, \dot{\bigcup}_{k\in K}{}^2\mathcal{J}_k \in DS(\mathcal{J})$ where $\overline{G} = \{{}^1G_t \mid t \in T\}$, $\overline{\overline{G}} = \{{}^2G_k \mid k \in K\}$. Let us consider an incidence structure ${}^1\mathcal{J}_{t_0} = ({}^1G_{t_0}, {}^1M_{t_0}, {}^1I_{t_0})$ where

$${}^1G_{t_0} = \bigcup_{A\in\mathcal{G}'(\widehat{{}^1\mathcal{J}_{t_0}})} A \qquad \text{for } t_0 \in T.$$

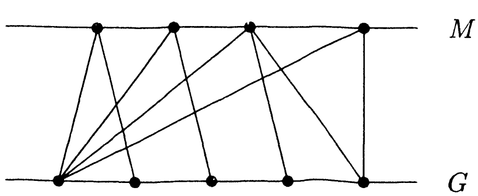
There exists $k_0 \in K$ such that $\mathcal{G}'(\widehat{{}^1\mathcal{J}_{t_0}}) \subseteq \mathcal{G}'(\widehat{{}^2\mathcal{J}_{k_0}})$. We obtain

$${}^2G_{k_0} = \bigcup_{A\in\mathcal{G}'(\widehat{{}^2\mathcal{J}_{k_0}})} A$$

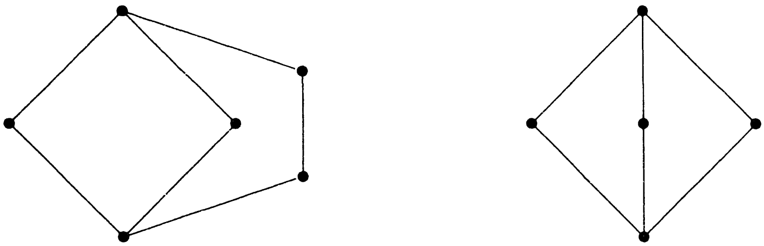
for the incidence structure ${}^2\mathcal{J}_{k_0} = ({}^2G_{k_0}, {}^2M_{k_0}, {}^2I_{k_0})$. Thus ${}^1G_{t_0} \subseteq {}^2G_{k_0}$ and $\overline{G} \le \overline{\overline{G}}$. Hence $\dot{\bigcup}_{t\in T}{}^1\mathcal{J}_t \le \dot{\bigcup}_{k\in K}{}^2\mathcal{J}_k$. □

Remark 6 The ordered set $(DS(\mathcal{L}(\mathcal{J}))/\equiv, \le)$ is a complete lattice with the least element $\varphi(\dot{\bigcup}_{t\in T}\mathcal{J}_t)$ where $\dot{\bigcup}_{t\in T}\mathcal{J}_t$ is a disjoint union of $\mathcal{J}$ with a decomposition $\widetilde{G}$ of G.

Remark 7 Theorem 6 is not valid for irreducible incidence structures: Consider an incidence structure $\mathcal{J}$ given by its incidence graph (Figure 3 (a)). The structure $\mathcal{J}$ is irreducible and thus $DS(\mathcal{J})$ has only one element. See Figure 3 (b) for the conceptual lattice $\mathcal{L}(\mathcal{J})$ and Figure 3 (c) for the ordered set $(DS(\mathcal{L}(\mathcal{J}))/\equiv, \le)$.



(a)



(b) (c)

Figure 3

References

[1] Ganter, B., Wille, R.: *Formale Begriffsanalyse.* Mathematische Grundlagen, Springer-Verlag, 1996.

[2] Grätzer, G.: *General Lattice Theory.* Birkhäuser Verlag, 1998.

[3] Machala, F., Slezák, V.: *Incidence Structures and Closure Spaces.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **36** (1997), 149–156.

[4] Machala, F., Slezák, V.: *Independent Sets in Incidence Structures.* Mathematica Slovaca, (1999) (to appear).

[5] Machala, F., Pomp, M.: *Disjoint and Complete Unions of Incidence Structures.* Mathematica Bohemica, (1996).