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Projections of Tensor Spaces *

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Abstract

The problem of projections of tensor spaces is studied in this paper. It is shown that projections of tensor spaces of symmetric tensors are either the identity mapping or its coefficients are equal zero or its coefficients are linear combination of the Kronecker's tensors.

Key words: Absolute invariant tensor, equivariant mapping, projection.

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1 Introduction

Let $T_k^r \mathbb{R}^n$ be the vector space of tensors of type (r,k) on $\mathbb{R}^n$. This space has a canonical basis. Denote $\zeta_{q_1 q_2 \cdots q_k}^{m_1 m_2 \cdots m_r}$ components of any tensor $\zeta \in T_k^r \mathbb{R}^n$ with respect to this basis.

Let L_n^1 be the first differential group and let ν: $L_n^1 \times T_k^r \mathbb{R}^n \to T_k^r \mathbb{R}^n$ be the action of L_n^1 on $T_k^r \mathbb{R}^n$. The mapping f : $T_k^r \mathbb{R}^n \to T_k^r \mathbb{R}^n$ is called L_n^1*-equivariant* if the next equation is true:

$$f \circ \nu = \nu \circ f.$$

Every equivariant mapping Φ : $T_k^r \mathbb{R}^n \to T_k^r \mathbb{R}^n$ is expressed by the equation

$$\zeta_{q_1 q_2 \cdots q_k}^{m_1 m_2 \cdots m_r} = \Phi_{q_1 q_2 \cdots q_k s_1 s_2 \cdots s_r}^{m_1 m_2 \cdots m_r p_1 p_2 \cdots p_k} \cdot \zeta_{p_1 p_2 \cdots p_k}^{s_1 s_2 \cdots s_r}$$

where $\Phi_{q_1 q_2 \cdots q_k s_1 s_2 \cdots s_r}^{m_1 m_2 \cdots m_r p_1 p_2 \cdots p_k}$ are components of the absolute invariant tensor.

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The absolute invariant tensor we define by the next way. Let $GL(\mathbb{R}^n)$ is the general linear group. For any element $A \in GL(\mathbb{R}^n)$ we define an action $\mu : GL(\mathbb{R}^n) \times T_k^r \mathbb{R}^n \to T_k^r \mathbb{R}^n$ of $GL(\mathbb{R}^n)$ on $T_k^r \mathbb{R}^n$ by the formula

$$\mu(A, \zeta) = A \cdot \zeta.$$

Tensor $\zeta \in T_k^r \mathbb{R}^n$ is called an *absolute invariant* if for each element $A \in GL(\mathbb{R}^n)$, $A \cdot \zeta = \zeta$. Components of absolute invariant tensor do not depend on the basis. It is easy to check if $r \neq k$ then the tensor $\zeta \in T_k^r \mathbb{R}^n$ is the absolute invariant if and only if $\zeta = 0$ [2].

We will define L_n^1-equivariant projection of tensor spaces. The mapping $\pi : T_k^r \mathbb{R}^n \to T_k^r \mathbb{R}^n$ is said to be an *L_n^1-equivariant projection* if it is L_n^1-equivariant and the equation $\pi \circ \pi = \pi$ is true.

Now we can describe projections of the tensor space $\mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*}$. Let $\mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*}$ denote the subspace of $T_k^1 \mathbb{R}^n$ of tensors symmetric in subscripts.

For these projections the following theorem holds. This theorem is mentioned in [2].

Theorem 1 *Each L_n^1-equivariant projection $\pi : \mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*} \to \mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*}$ is expressed by the equation*

$$\bar{\xi}^m_{q_1 q_2 \cdots q_k} = t^{m p_1 p_2 \cdots p_k}_{q_1 q_2 \cdots q_k s} \cdot \xi^s_{p_1 p_2 \cdots p_k}$$

where

$$k! \cdot t^{m p_1 p_2 \cdots p_k}_{q_1 q_2 \cdots q_k s} = A \sum_\sigma \delta^m_s \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + B \Big(\sum_\sigma \delta^{p_1}_s \delta^m_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_k}_{q_{\sigma(k)}}$$

$$+ \sum_\sigma \delta^{p_2}_s \delta^{p_1}_{q_{\sigma(1)}} \delta^m_{q_{\sigma(2)}} \delta^{p_3}_{q_{\sigma(3)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots + \sum_\sigma \delta^{p_k}_s \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-1}}_{q_{\sigma(k-1)}} \delta^m_{q_{\sigma(k)}} \Big),$$

σ denote permutations of the set $\{1, 2, \ldots, k\}$ and the coefficients $A, B \in \mathbb{R}$ obey one of the following four possibilities:

a) $A = 0,\ B = 0$;
b) $A = 0,\ B = \frac{1}{n+k-1}$;
c) $A = 1,\ B = 0$;
d) $A = 1,\ B = \frac{-1}{n+k-1}$.

The next part of this paper deals with the generalization of this theorem.

2 Projections of the space $S^r \mathbb{R}^n \otimes S^k \mathbb{R}^{n*}$

Let $\zeta^{m_1 m_2 \cdots m_k}_{q_1 q_2 \cdots q_k}$ be a tensor of type (r, k). Then we will denote $\zeta^{\cdots j \cdots}_{\cdots j \cdots}$ contraction of this tensor. If we will contract this tensor twice we will this double contraction denote $\zeta^{\cdots j_1 \cdots j_2 \cdots}_{\cdots j_1 \cdots j_2 \cdots}$, etc.

Let $S^r \mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*}$ denote the subspace of $T_k^r \mathbb{R}^n$ of tensors symmetric in superscripts and subscripts.

The following theorem is the generalization of the theorem 1.

Theorem 2 *Each L_n^1-equivariant projection $\pi : S^r\mathbb{R}^n \bigotimes S^k\mathbb{R}^{n*} \to S^r\mathbb{R}^n \bigotimes S^k\mathbb{R}^{n*}$, $r \leq k$, is expressed by the equation*

$$\bar{\xi}^{m_1 m_2 \cdots m_r}_{q_1 q_2 \cdots q_k} = t^{m_1 m_2 \cdots m_r p_1 p_2 \cdots p_k}_{q_1 q_2 \cdots q_k s_1 s_2 \cdots s_r} \cdot \xi^{s_1 s_2 \cdots s_r}_{p_1 p_2 \cdots p_k} \tag{1}$$

where

$$r!k! \cdot t^{m_1 m_2 \cdots m_r p_1 p_2 \cdots p_k}_{q_1 q_2 \cdots q_k s_1 s_2 \cdots s_r} = \tag{2}$$

$$\begin{aligned}
&= A_0 \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} \\
&+ A_1 \Big(\sum_\rho \sum_\sigma \delta^{p_1}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{m_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{p_2}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{m_1}_{q_{\sigma(2)}} \delta^{p_3}_{q_{\sigma(3)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{p_k}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-1}}_{q_{\sigma(k-1)}} \delta^{m_1}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{p_1}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{m_2}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{p_2}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{m_2}_{q_{\sigma(2)}} \delta^{p_3}_{q_{\sigma(3)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{p_k}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-1}}_{q_{\sigma(k-1)}} \delta^{m_2}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_{r-1}}_{s_{\rho(r-1)}} \delta^{p_1}_{s_{\rho(r)}} \delta^{m_r}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_{r-1}}_{s_{\rho(r-1)}} \delta^{p_2}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{m_r}_{q_{\sigma(2)}} \delta^{p_3}_{q_{\sigma(3)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \delta^{m_2}_{s_{\rho(2)}} \cdots \delta^{m_{r-1}}_{s_{\rho(r-1)}} \delta^{p_k}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-1}}_{q_{\sigma(k-1)}} \delta^{m_r}_{q_{\sigma(k)}} \Big) \\
&+ A_2 \Big(\sum_\rho \sum_\sigma \delta^{p_1}_{s_{\rho(1)}} \delta^{p_2}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{m_1}_{q_{\sigma(1)}} \delta^{m_2}_{q_{\sigma(2)}} \delta^{p_3}_{q_{\sigma(3)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{p_1}_{s_{\rho(1)}} \delta^{p_3}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{m_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \delta^{m_2}_{q_{\sigma(3)}} \delta^{p_4}_{q_{\sigma(4)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{p_1}_{s_{\rho(1)}} \delta^{p_k}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{m_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-1}}_{q_{\sigma(k-1)}} \delta^{m_2}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{p_{k-1}}_{s_{\rho(1)}} \delta^{p_k}_{s_{\rho(2)}} \delta^{m_3}_{s_{\rho(3)}} \cdots \delta^{m_r}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \cdots \delta^{p_{k-2}}_{q_{\sigma(k-2)}} \delta^{m_1}_{q_{\sigma(k-1)}} \delta^{m_2}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \cdots \delta^{m_{r-2}}_{s_{\rho(r-2)}} \delta^{p_1}_{s_{\rho(r-1)}} \delta^{p_2}_{s_{\rho(r)}} \delta^{m_{r-1}}_{q_{\sigma(1)}} \delta^{m_r}_{q_{\sigma(2)}} \delta^{p_3}_{q_{\sigma(3)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \cdots \delta^{m_{r-2}}_{s_{\rho(r-2)}} \delta^{p_1}_{s_{\rho(r-1)}} \delta^{p_3}_{s_{\rho(r)}} \delta^{m_{r-1}}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \delta^{m_r}_{q_{\sigma(3)}} \delta^{p_4}_{q_{\sigma(4)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \cdots \delta^{m_{r-2}}_{s_{\rho(r-2)}} \delta^{p_1}_{s_{\rho(r-1)}} \delta^{p_k}_{s_{\rho(r)}} \delta^{m_{r-1}}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-1}}_{q_{\sigma(k-1)}} \delta^{m_r}_{q_{\sigma(k)}} \\
&+ \sum_\rho \sum_\sigma \delta^{m_1}_{s_{\rho(1)}} \cdots \delta^{m_{r-2}}_{s_{\rho(r-2)}} \delta^{p_{k-1}}_{s_{\rho(r-1)}} \cdots \delta^{p_k}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \cdots \delta^{p_{k-2}}_{q_{\sigma(k-2)}} \delta^{m_{r-1}}_{q_{\sigma(k-1)}} \delta^{m_r}_{q_{\sigma(k)}} \Big) + \cdots \\
&+ A_r \Big(\sum_\rho \sum_\sigma \delta^{p_1}_{s_{\rho(1)}} \delta^{p_2}_{s_{\rho(2)}} \cdots \delta^{p_r}_{s_{\rho(r)}} \delta^{m_1}_{q_{\sigma(1)}} \delta^{m_2}_{q_{\sigma(2)}} \cdots \delta^{m_r}_{q_{\sigma(r)}} \delta^{p_{r+1}}_{q_{\sigma(r+1)}} \cdots \delta^{p_k}_{q_{\sigma(k)}} + \cdots \\
&+ \sum_\rho \sum_\sigma \delta^{p_{k-r+1}}_{s_{\rho(1)}} \delta^{p_{k-r+2}}_{s_{\rho(2)}} \cdots \delta^{p_k}_{s_{\rho(r)}} \delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \cdots \delta^{p_{k-r}}_{q_{\sigma(k-r)}} \delta^{m_1}_{q_{\sigma(k-r+1)}} \cdots \delta^{m_r}_{q_{\sigma(k)}} \Big),
\end{aligned}$$

σ and ρ are permutation of the sets $\{1, 2, \ldots k\}$ and $\{1, 2, \ldots r\}$ respectively.

The coefficients A_0, A_h; $(h = 1, 2, \dots, r)$ are obey one of the following possibilities:

$$1)\ A_0 = A_1 = \cdots = A_r = 0;$$

$$2)\ A_0 = A_1 = \cdots = A_{i-1} = 0, \quad (i = 1, 2, \dots, r),$$

$$A_h = \frac{(-1)^{h-i} i(i+1)(i+2)\cdots(h-1)\cdot h!}{(h-i)!(x-i-1)(x-i-2)\cdots(x-i-h)}, \quad (h = i, i+1, \dots, r);$$

or

$$3)\ A_0 = 1,\ A_1 = \cdots = A_r = 0;$$

$$4)\ A_0 = 1,\ A_1 = \cdots = A_{i-1} = 0; \quad (i = 2, 3, \dots, r),$$

$$A_h = \frac{(-1)^{h-i+1} i(i+1)(i+2)\cdots(h-1)\cdot h!}{(h-i)!(x-i-1)(x-i-2)\cdots(x-i-h)}, \quad (h = i, i+1, \dots, r),$$

where $x = n + k + r$.

Proof Let $\pi : S^r \mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*} \to S^r \mathbb{R}^n \bigotimes S^k \mathbb{R}^{n*}$ be an L^1_n-equivariant projection expressed by the equation (1). Tensor t has expression (2). From the equality $\pi \circ \pi = \pi$ we obtain following equation

$$t^{m_1 m_2 \cdots m_r p_1 p_2 \cdots p_k}_{q_1 q_2 \cdots q_k s_1 s_2 \cdots s_r} \cdot t^{s_1 s_2 \cdots s_r i_1 i_2 \cdots i_k}_{p_1 p_2 \cdots p_k j_1 j_2 \cdots j_r} = t^{m_1 m_2 \cdots m_r i_1 i_2 \cdots i_k}_{q_1 q_2 \cdots q_k j_1 j_2 \cdots j_r} \tag{3}$$

Now we substitute t in (3) from (2) and we obtain difficulty equation. After some computation and when we compare left-hand side and right-hand side we get following system of equations. For h even we have

$$A_0^2 = A_0; \tag{4}$$

$$\begin{aligned}
&2A_0 A_h + \tfrac{1}{h!}(x-h-1)(x-h-2)\cdots(x-2h)A_h^2 \\
&+ \sum_{i=[\frac{h}{2}]+1}^{h-1} \frac{h^2(h-1)^2\cdots(i+1)^2}{(2i-h)![(h-i)!]^2}(x-h-1)(x-h-2)\cdots(x-2i)A_i^2 \\
&+ 2\sum_{i=1}^{h-1} \frac{h(h-1)\cdots(h-i+1)}{(i!)^2}(x-h-1)(x-h-2)\cdots(x-h-i)A_i A_h \\
&+ 2\sum_{*} \frac{h^2(h-1)^2\cdots(j+1)^2 j(j-1)\cdots(h-i+1)}{i!(h-j)!(i+j-h)!}(x-h-1)(x-h-2)\cdots(x-i-j)A_i A_j \\
&+ \frac{h^2(h-1)^2\cdots(\frac{h}{2}+1)^2}{(\frac{h}{2}!)^2} A_{\frac{h}{2}}^2 + 2\sum_{i=[\frac{h}{2}]+1}^{h-1} \frac{h^2(h-1)^2\cdots(i+1)^2}{[(h-i)!]^2} A_{h-i} A_i = A_h
\end{aligned}$$

$$* = \begin{cases} j = 2, \dots, h-1 \\ i = 1, \dots, h-2, \quad i < j,\ i+j > h, \end{cases} \qquad x = n + k + r,\ h = 1, 2, \dots, r.$$

For h odd we have

$$A_0^2 = A_0; \tag{5}$$

$$2A_0A_h + \tfrac{1}{h!}(x-h-1)(x-h-2)\cdots(x-2h)A_h^2$$
$$+\sum_{i=[\frac{h}{2}]+1}^{h-1} \tfrac{h^2(h-1)^2\cdots(i+1)^2}{(2i-h)![(h-i)!]^2}(x-h-1)(x-h-2)\cdots(x-2i)A_i^2$$
$$+2\sum_{i=1}^{h-1} \tfrac{h(h-1)\cdots(h-i+1)}{(i!)^2}(x-h-1)(x-h-2)\cdots(x-h-i)A_iA_h$$
$$+2\sum_{*} \tfrac{h^2(h-1)^2\cdots(j+1)^2j(j-1)\cdots(h-i+1)}{i!(h-j)!(i+j-h)!}(x-h-1)(x-h-2)\cdots(x-i-j)A_iA_j$$
$$+2\sum_{i=[\frac{h}{2}]+1}^{h-1} \tfrac{h^2(h-1)^2\cdots(i+1)^2}{[(h-i)!]^2}A_{h-i}A_i = A_h$$

$$* = \begin{cases} j = 2,\cdots,h-1 \\ i = 1,\cdots,h-2, \quad i<j,\ i+j>h, \end{cases} \qquad x = n+k+r,\ h = 1,2,\ldots,r.$$

And to prove the converse we must check that each of the mappings expressed by the equation (2) where coefficients $A_0, A_1, \ldots A_r$ are defined by (4) or (5), is an L_n^1-equivariant projection.

Equations (4) and (5) have a trivial solution and coefficients of the identity mapping are also the solution of these equations. These two solutions give mappings which are obviously L_n^1-equivariant projections. Now from the first equation we get A_0 is 0 or 1. For these two coefficients we get mappings which sums will be the identity mapping. Therefore we now compute only mapping for which $A_0 = 0$.

We take the mapping which is given by the equation

$$r!k!\cdot\bar{\xi}^{m_1m_2\cdots m_r}_{q_1q_2\cdots q_k} = \sum_h\Big[A_h\cdot\xi^{s_1\cdots s_r}_{p_1\cdots p_k} \tag{6}$$

$$\cdot\Big(\sum_\rho\sum_\sigma \delta^{p_1}_{s_{\rho(1)}}\cdots\delta^{p_h}_{s_{\rho(h)}}\delta^{m_{h+1}}_{s_{\rho(h+1)}}\cdots\delta^{m_r}_{s_{\rho(r)}}\delta^{m_1}_{q_{\sigma(1)}}\cdots\delta^{m_h}_{q_{\sigma(h)}}\delta^{p_{h+1}}_{q_{\sigma(h+1)}}\cdots\delta^{p_k}_{q_{\sigma(k)}}+\cdots$$
$$+\sum_\rho\sum_\sigma \delta^{m_1}_{s_{\rho(1)}}\cdots\delta^{m_{r-h}}_{s_{\rho(h)}}\delta^{p_1}_{s_{\rho(h+1)}}\cdots\delta^{p_h}_{s_{\rho(r)}}\delta^{m_{r-h+1}}_{q_{\sigma(1)}}\cdots\delta^{m_r}_{q_{\sigma(h)}}\delta^{p_{h+1}}_{q_{\sigma(h+1)}}\cdots\delta^{p_k}_{q_{\sigma(k)}}+\cdots$$
$$+\sum_\rho\sum_\sigma \delta^{p_{k-h+1}}_{s_{\rho(1)}}\cdots\delta^{p_k}_{s_{\rho(h)}}\delta^{m_{h+1}}_{s_{\rho(h+1)}}\cdots\delta^{m_r}_{s_{\rho(r)}}\delta^{p_1}_{q_{\sigma(1)}}\cdots\delta^{p_{k-h}}_{q_{\sigma(h)}}\delta^{m_1}_{q_{\sigma(h+1)}}\cdots\delta^{m_h}_{q_{\sigma(k)}}+\cdots$$
$$+\sum_\rho\sum_\sigma \delta^{m_1}_{s_{\rho(1)}}\cdots\delta^{m_{r-h}}_{s_{\rho(h)}}\delta^{p_{k-h+1}}_{s_{\rho(h+1)}}\cdots\delta^{p_k}_{s_{\rho(r)}}\delta^{p_1}_{q_{\sigma(1)}}\cdots\delta^{p_{k-h}}_{q_{\sigma(h)}}\delta^{m_{r-h+1}}_{q_{\sigma(h+1)}}\cdots\delta^{m_r}_{q_{\sigma(k)}}\Big)\Big]$$
$$=\sum_h A_h\Big(\sum_\rho\sum_\sigma \delta^{m_1}_{q_{\sigma(1)}}\cdots\delta^{m_h}_{q_{\sigma(r)}}\cdot\xi^{m_1\cdots j_1\cdots j_h\cdots m_r}_{q_{\sigma_1}\cdots j_1\cdots j_h\cdots q_{\sigma_k}}+\cdots$$
$$+\sum_\rho\sum_\sigma \delta^{m_1}_{q_{\sigma(k-r+1)}}\cdots\delta^{m_h}_{q_{\sigma(k)}}\cdot\xi^{m_1\cdots j_1\cdots j_h\cdots m_r}_{q_{\sigma_1}\cdots j_1\cdots j_h\cdots q_{\sigma_k}}+\cdots$$
$$+\sum_\rho\sum_\sigma \delta^{m_{r-h+1}}_{q_{\sigma(1)}}\cdots\delta^{m_r}_{q_{\sigma(r)}}\cdot\xi^{m_1\cdots j_1\cdots j_h\cdots m_r}_{q_{\sigma_1}\cdots j_1\cdots j_h\cdots q_{\sigma_k}}+\cdots$$
$$+\sum_\rho\sum_\sigma \delta^{m_{r-h+1}}_{q_{\sigma(k-r+1)}}\cdots\delta^{m_r}_{q_{\sigma(k)}}\cdot\xi^{m_1\cdots j_1\cdots j_h\cdots m_r}_{q_{\sigma_1}\cdots j_1\cdots j_h\cdots q_{\sigma_k}}\Big)$$

where $h = i, i+1, \ldots, r$; $i = 1, \ldots, h$. To prove that this mapping is a projection we must check if

$$\bar{\bar{\xi}}^{m_1m_2\cdots m_r}_{q_1q_2\cdots q_k} = \bar{\xi}^{m_1m_2\cdots m_r}_{q_1q_2\cdots q_k}.$$

Contracting equation (6) we obtain

$$\bar{\xi}^{m_1 m_2 \cdots m_{r-i} j_1 j_2 \cdots j_i}_{q_1 q_2 \cdots q_{k-i} j_1 j_2 \cdots j_i} = \xi^{m_1 m_2 \cdots m_{r-i} j_1 j_2 \cdots j_i}_{q_1 q_2 \cdots q_{k-i} j_1 j_2 \cdots j_i}$$

$$+ \Big(A_i \frac{(x-i-2)(x-i-3)\cdots(x-2i)}{(i-1)!} + A_{i+1} \frac{(x-i-2)(x-i-3)\cdots(x-2i-1)}{(i+1)!} \Big) \xi^{m_1 \cdots m_{r-i-1} j_1 \cdots j_{i+1}}_{q_1 \cdots q_{k-i-1} j_1 \cdots j_{i+1}}$$

$$+ \Big(A_i \frac{i(i-1)(x-i-3)(x-i-4)\cdots(x-2i)}{2!j!} + A_{i+1} \frac{i(x-i-3)(x-i-4)\cdots(x-2i-1)}{(i+1)!}$$

$$+ A_{i+2} \frac{(x-i-3)(x-i-4)\cdots(x-2i-2)}{(i+2)!} \cdot \xi^{m_1 \cdots m_{r-i-1} j_1 \cdots j_{i+2}}_{q_1 \cdots q_{k-i-1} j_1 \cdots j_{i+2}} + \cdots$$

$$+ \Big(A_i \frac{1}{i!} + A_{i+1} \frac{i(x-2i-1)}{(i+1)!} + A_{i+2} \frac{i(i-1)(x-2i-1)(x-2i-2)}{2!(i+2)!} + \cdots$$

$$+ A_{2i-2} \frac{i(i-1)(x-2i-1)(x-2i-2)\cdots(x-3i-2)}{2!(2i-2)!} + A_{2i-1} \frac{i(x-2i-1)(x-2i-2)\cdots(x-3i+1)}{(2i-1)!}$$

$$+ A_{2i} \frac{(x-2i-1)(x-2i-2)\cdots(x-3i)}{(2i)!} \Big) \cdot \xi^{m_1 \cdots m_{r-i-1} j_1 \cdots j_{2i}}_{q_1 \cdots q_{k-i-1} j_1 \cdots j_{2i}}$$

$$+ \Big(A_{i+1} \frac{1}{(i+1)!} + A_{i+2} \frac{(i+1)(x-2i-2)}{(i+2)!} + A_{i+3} \frac{(i+1)i(x-2i-2)(x-2i-3)}{2!(i+3)!} + \cdots$$

$$+ A_{2i-1} \frac{(i+1)i(x-2i-2)(x-2i-3)\cdots(x-3i-1)}{2!(2i-1)!} + A_{2i} \frac{(i+1)(x-2i-2)(x-2i-3)\cdots(x-3i)}{(2i)!}$$

$$+ A_{2i+1} \frac{(x-2i-2)(x-2i-3)\cdots(x-3i-1)}{(2i+1)!} \Big) \cdot \xi^{m_1 \cdots m_{r-i-1} j_1 \cdots j_{2i+1}}_{q_1 \cdots q_{k-i-1} j_1 \cdots j_{2i+1}} + \cdots$$

$$+ \Big(A_{h-i} \frac{1}{(h-i)!} + A_{h-i+1} \frac{(h-i)(x-h-1)}{(h-i+1)!}$$

$$+ A_{h-i+2} \frac{(h-i)(h-i-1)(x-h-1)(x-h-2)}{2!(h-i+2)!} + \cdots$$

$$+ A_{h-2} \frac{(h-i)(h-i-1)(x-h-1)(x-h-2)\cdots(x-h-i+2)}{2!(h-2)!}$$

$$+ A_{h-1} \frac{(h-i)(x-h-1)(x-h-2)\cdots(x-h-i+1)}{(h-1)!}$$

$$+ A_h \frac{(x-h-1)(x-h-2)\cdots(x-h-i)}{h!} \Big) \cdot \xi^{m_1 \cdots m_{r-i-1} j_1 \cdots j_h}_{q_1 \cdots q_{k-i-1} j_1 \cdots j_h}$$

where $i = 1, \ldots, h$.

If π is the projection then $\pi \circ \pi = \pi$, i.e.

$$\bar{\bar{\xi}}^{m_1 m_2 \cdots m_{r-i} j_1 j_2 \cdots j_i}_{q_1 q_2 \cdots q_{k-i} j_1 j_2 \cdots j_i} = \bar{\xi}^{m_1 m_2 \cdots m_{r-i} j_1 j_2 \cdots j_i}_{q_1 q_2 \cdots q_{k-i} j_1 j_2 \cdots j_i} \tag{7}$$

We can see that the equation (7) is true if next equations hold.

$$A_i \frac{(x-i-2)(x-i-3)\cdots(x-2i)}{(i-1)!} + A_{i+1} \frac{(x-i-2)(x-i-3)\cdots(x-2i-1)}{(i+1)!} = 0,$$

$$A_i \frac{i(i-1)(x-i-3)(x-i-4)\cdots(x-2i)}{2!j!} + A_{i+1} \frac{i(x-i-3)(x-i-4)\cdots(x-2i-1)}{(i+1)!}$$

$$+ A_{i+2} \frac{(x-i-3)(x-i-4)\cdots(x-2i-2)}{(i+2)!} = 0,$$

....

$$A_i \frac{1}{i!} + A_{i+1} \frac{i(x-2i-1)}{(i+1)!} + A_{i+2} \frac{i(i-1)(x-2i-1)(x-2i-2)}{2!(i+2)!} + \cdots$$

$$+ A_{2i-2} \frac{i(i-1)(x-2i-1)(x-2i-2)\cdots(x-3i-2)}{2!(2i-2)!}$$

$$+ A_{2i-1} \frac{i(x-2i-1)(x-2i-2)\cdots(x-3i+1)}{(2i-1)!}$$

$$+ A_{2i} \frac{(x-2i-1)(x-2i-2)\cdots(x-3i)}{(2i)!} = 0,$$

$$A_{i+1}\tfrac{1}{(i+1)!}+A_{i+2}\tfrac{(i+1)(x-2i-2)}{(i+2)!}+A_{i+3}\tfrac{(i+1)i(x-2i-2)(x-2i-3)}{2!(i+3)!}+\cdots$$
$$+A_{2i-1}\tfrac{(i+1)i(x-2i-2)(x-2i-3)\cdots(x-3i-1)}{2!(2i-1)!}+A_{2i}\tfrac{(i+1)(x-2i-2)(x-2i-3)\cdots(x-3i)}{(2i)!}$$
$$+A_{2i+1}\tfrac{(x-2i-2)(x-2i-3)\cdots(x-3i-1)}{(2i+1)!}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{h-i}\tfrac{1}{(h-i)!}+A_{h-i+1}\tfrac{(h-i)(x-h-1)}{(h-i+1)!}+A_{h-i+2}\tfrac{(h-i)(h-i-1)(x-h-1)(x-h-2)}{2!(h-i+2)!}+\cdots$$
$$+A_{h-2}\tfrac{(h-i)(h-i-1)(x-h-1)(x-h-2)\cdots(x-h-i+2)}{2!(h-2)!}$$
$$+A_{h-1}\tfrac{(h-i)(x-h-1)(x-h-2)\cdots(x-h-i+1)}{(h-1)!}$$
$$+A_{h}\tfrac{(x-h-1)(x-h-2)\cdots(x-h-i)}{h!}=0.$$

For $A_0=A_1=\cdots=A_{i-1}=0$ we can compute A_i. From equations (4) we get one nonzero solution

$$A_i=\frac{i!}{(x-i-1)(x-i-2)\cdots(x-2i)}\,. \tag{8}$$

When we know A_i we compute A_h for $h=i+1,i+2,\dots,r$:

$$A_h=\frac{(-1)^{h-i}i(i+1)(i+2)\cdots(h-1)\cdot h!}{(h-i)!(x-i-1)(x-i-2)\cdots(x-i-h)}. \tag{9}$$

Solutions (8) and (9) we can write in the next form

$$A_h=\frac{(-1)^{h-i}i(i+1)(i+2)\cdots(h-1)\cdot h!}{(h-i)!\,(x-i-1)(x-i-2)\cdots(x-i-h)},\quad (h=i,i+1,\dots,r).$$

We obtain solutions which are also solutions of the system of equations (4) and the proof is complete.

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