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On the Relationship between the Initial and the Multipoint Boundary Value Problems for n-th Order Linear Differential Equations of Neutral Type

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Abstract

In this paper we give some relationship between the initial problem and the Haščák's boundary value problems for linear differential equation of neutral type.

Key words: Linear differential equations with delay, initial value problem for differential equations with delay, multipoint boundary value problem for linear differential equation with delay.

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1 Introduction

Boundary value problems (BVP-s) for ordinary differential equations have been studied in many papers under various types of boundary conditions. However, the corresponding theory for differential equations with delays has not yet been built up. The main problem is the formulation of BVP for these equations. Some very interesting formulations of BVP appear in the last time (see [2]–[5], [9]–[12]). The purpose of this paper is to give some relationship between the initial value problem and the Haščák's BVP-s for linear differential equation of neutral type ([3], [4]).

Consider the following n-th order differential equation with delay of neutral type

$$x^{(n)}(t) + a(t)x^{(n)}(t - \triangle_0(t)) + \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}(t)x^{(n-i)}(t - \triangle_{ij}(t)) = 0, \ n \geq 1 \quad (1)$$

with continuous coefficients $a(t)$, $b_{ij}(t)$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ and delays $\triangle_0(t) > 0$, $\triangle_{ij}(t) \geq 0$ on interval $\langle a, b\rangle$.

Moreover let

$$|a(t)| \leq A < 1, \qquad |b_{ij}(t)| \leq B_{ij} \quad (2)$$

for all t in a compact interval $I = \langle a, b\rangle$ $i = 1, \ldots, n$; $j = 1, \ldots, m$.

Define the function χ by

$$\chi(h) := \sum_{i=1}^{n} \frac{\sum_{i=1}^{m} B_{ij}}{i[\frac{i-1}{2}]![\frac{i}{2}]!} h^i. \quad (3)$$

The *initial value problem* (IVP) for (1) is defined as follows: Let $t_0 \in \langle a, b)$ and let a continuous initial vector function $\Phi(t) = (\phi_0(t), \phi_1(t), \ldots, \phi_n(t))$ be given on the initial set

$$E_{t_0} := \bigcup_{i=1}^{n}\bigcup_{j=1}^{m} E_{t_0}^{ij} \cup E_{t_0}^{0},$$

where

$$E_{t_0}^{ij} := \{t - \triangle_{ij}(t) : \ t - \triangle_{ij}(t) < t_0, \ t \in \langle t_0, b)\} \cup \{t_0\},$$

$i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ and

$$E_{t_0}^{0} := \{t - \triangle_0(t) : \ t - \triangle_0(t) \leq t_0, \ t \in \langle t_0, b)\} \cup \{t_0\}.$$

We have to find the solution $x(t)$ of the equation (1) defined on interval $\langle t_0, b)$ which satisfies the conditions

$$\begin{gathered} x^{(k)}(t_0) = \phi_k(t_0), \quad k = 0, 1, \ldots, n-1, \\ x^{(k)}(t - \triangle_{ij}(t)) = \phi_k(t - \triangle_{ij}(t)) \quad \text{if } t - \triangle_{ij}(t) < t_0, \\ i = 1, \ldots, n; \quad j = 1, \ldots, m; \quad k = 0, 1, \ldots, n-1, \end{gathered} \quad (4)$$

$$\begin{gathered} x^{(n)}(t_0) = \phi_n(t_0), \\ x^{(n)}(t - \triangle_0(t)) = \phi_n(t - \triangle_0(t)) \quad \text{if } t - \triangle_0(t) < t_0. \end{gathered} \quad (5)$$

Paper [5] presents following theorem:

Theorem 1 *Let the coefficients $a(t)$, $b_{ij}(t)$, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$ and the delays $\triangle_0(t) > 0$, $\triangle_{ij}(t) \geq 0$ of (1) be continuous on $\langle t_0, b)$ and let the initial vector function $\Phi(t)$ be continuous and bounded on E_{t_0}. Then the initial value problem (1), (4), (5) has exactly one solution on the interval $\langle t_0, b)$.*

We shall now introduce some definitions and notations which will be needed on the sequel:

A vector function Φ is called admissible if it is continuous and bounded on its domain of definition.

Let $t_0 \in \langle a, b)$, let an admissible vector function

$$\Phi(t) = (\phi_0(t), \phi_1(t), \ldots, \phi_n(t))$$

defined on E_{t_0} be given and let $r_0 \in \{0, 1 \ldots, n\}$. Then

$$H(t_0, \Phi, r_0) := \{(\phi_0(t), \ldots, \phi_{r_0-1}(t), \phi_{r_0}(t) + c_0, \ldots, \phi_n(t) + c_{n-r_0}) : \\ c_i \in \mathbf{R}, i = 0, 1, \ldots, n - r_0\}.$$

Let $x(t)$ be a solution of (1). Then we shall write $x \in H(t_0, \Phi, r_0)$ iff there are constants $\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{n-r_0} \in \mathbf{R}$ such that $x(t)$ is determined by initial vector function $(\phi_0(t), \ldots, \phi_{r_0-1}(t), \phi_{r_0}(t) + \bar{c}_0, \ldots, \phi_n(t) + \bar{c}_{n-r_0})$.

Following BVP for (1) is formulated in papers [5], [6]:

Let $J \subset \langle a, b)$ be interval, let

$$\tau_0, \tau_1, \ldots, \tau_p \in J, \quad \tau_0 < \tau_1 \leq \ldots \leq \tau_p \quad (p \leq n),$$

$$r_0 \in \{0, 1, \ldots, n\}, \; r_1, \ldots, r_p \in N, \quad r_0 + r_1 + \cdots + r_p = n + 1$$

and let

$$\beta_0^{(1)}, \ldots, \beta_0^{(r_0)}, \ldots, \beta_1^{(1)}, \ldots, \beta_1^{(r_1)}, \ldots, \beta_p^{(1)}, \ldots, \beta_p^{(r_p)} \in \mathbf{R},$$

$\Phi(t) = (\phi_0(t), \phi_1(t), \ldots, \phi_n(t))$ be an admissible function defined on E_{t_0} such that

$$\phi_{i-1}(\tau_0) = \beta_0^{(i)}, \quad i = 1, \ldots, r_0.$$

The problem is to find a solution of (1), which satisfies the conditions

$$x^{(l_k-1)}(\tau_k) = \beta_k^{(l_k)}, \quad l_k = 1, \ldots, r_k; \quad k = 1, \ldots, p; \quad x \in H(\tau_0, \Phi, r_0). \qquad (6)$$

Definition 1 (Haščák [5]) Equation (1) is strictly disconjugate on the interval J, iff each its nontrivial solution which is determined by initial point $\tau_0 \in J$ and constant initial vector function has at most n zero point (including multiplicity).

Following theorems are concerning with BVP for (1). Theorem 2 is some reformulation of Theorem 7 of [5] and Theorem 3 is a consequence of Theorem 3 of [6].

Theorem 2 *Equation (1) is strictly disconjugate on the interval J iff each BVP for (1) has exactly one solution.*

Theorem 3 *Let function $a(t)$, $b_{ij}(t)$ satisfies assumption (2) and*

$$\chi(b - a) < 1 - A. \qquad (7)$$

Then the differential equation (1) is strictly disconjugate on $\langle a, b)$.

The purpose of this note is to show a relation between the initial value problem (1), (4), (5) and BVP (1), (6).

Now we shall introduce some notation which will be needed in the sequel.

Let a function $f(t)$ in the interval $\langle a, b\rangle$ be given. Consider the points

$$a < \tau_0 < \tau_1 < \ldots < \tau_n < b.$$

Denote

$$\beta_i = f(\tau_i), \quad i = 0, 1, \ldots, n.$$

By difference quotient of the n-th order we shall understand (see [1] p. 17)

$$D^{n+1}(\tau_0, \ldots, \tau_n; \beta_0, \ldots, \beta_n) = [\tau_0, \ldots, \tau_n] = \sum_{i=0}^{n} \beta_i \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{1}{(\tau_i - \tau_j)}.$$

If the function f has continuous derivatives to the n-th order (including the n-th order) in $\langle a, b\rangle$, then there are numbers ξ_k, $k = 0, \ldots, n$, such that

$$\tau_0 < \xi_k < \tau_{k+1}$$

and

$$D^{k+1}(\tau_0, \ldots, \tau_k; \beta_0, \ldots, \beta_k) = \frac{f^{(k)}(\xi_k)}{k!}, \quad k = 0, 1, \ldots, n \tag{8}$$

holds.

Theorem 4 *Let*

$$\chi(b - a) < 1 - A, \tag{9}$$

$t_0 \in \langle a, b)$ and let admissible function $\Phi(t)$ defined on E_{t_0} be given.

Let the boundary conditions

$$\tau_{v0}, \tau_{v1}, \ldots, \tau_{vn}; \quad \beta_{v0}, \beta_{v1}, \ldots, \beta_{vn}, \quad v = 1, 2, \ldots \tag{$\bar{\tau}_v, \bar{\beta}_v$}$$

be such that

$$\tau_{v0} < \tau_{v1} < \ldots < \tau_{vn}, \quad v = 1, 2, \ldots,$$
$$\lim_{v \to \infty} \tau_{vi} = t_0, \quad i = 1, \ldots, n \tag{10}$$

and

$$\lim_{v \to \infty} D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk}) = \frac{\phi_k(t_0)}{k!}, \quad k = 0, 1, \ldots, n. \tag{11}$$

Then the sequence $\varphi(t; \bar{\tau}_v, \bar{\beta}_v)$, $v = 1, 2, \ldots$ of solutions of the boundary value problem (1), (6) and the sequence $\varphi^{(k)}(t; \bar{\tau}_v, \bar{\beta}_v)$, $k = 1, \ldots, n$; $v = 1, 2, \ldots$ of their derivatives converge uniformly to the solution $\varphi(t; t_0, \Phi)$ of the initial value problem (1), (4), (5) resp. to its derivatives $\varphi^{(k)}(t; t_0, \Phi)$, $k = 1, \ldots, n$ on $\langle t_0, b)$ as $v \to \infty$.

Proof Without loss of generality, we shall assume, that

$$0 < \tau_{vi} - t_0 < h < \min\{b - t_0, 1, \tfrac{1-A}{nmL}\}, \quad i = 0, 1, \ldots, n; \; v = 1, 2, \ldots,$$

where $L = \max\{A, B_{11}, \ldots, B_{nm}\}$ [see (10)].

From (11) we conclude that $D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk})$, $k = 0, 1, \ldots, n$; $v = 1, 2, \ldots$ are bounded and there is a positive number M such that

$$|n! D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk})| \leq M, \quad k = 0, \ldots, n; \quad v = 1, 2, \ldots \tag{12}$$

By (8) there are numbers

$$\xi_{vk}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk}) \in (\tau_{v0}, \tau_{vn}), \quad v = 1, 2, \ldots, \tag{13}$$

such that

$$D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk}) = \frac{\varphi^{(k)}(\xi_{vk}; \bar{\tau}_v, \bar{\beta}_v)}{k!}, \tag{14}$$

$k = 0, 1, \ldots, n; \quad v = 1, 2, \ldots$

By (14) we have

$$|\varphi^{(k)}(t_0; \bar{\tau}_v, \bar{\beta}_v) - \phi_k(t_0)| \leq |\varphi^{(k)}(t_0; \bar{\tau}_v, \bar{\beta}_v) - \varphi^{(k)}(\xi_{vk}; \bar{\tau}_v, \bar{\beta}_v)|$$

$$+ |k! D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk}) - \phi_k(t_0)|, \quad k = 0, \ldots, n; \quad v = 1, 2, \ldots$$

from where by Mean Value Theorem we have

$$|\varphi^{(k)}(t_0; \bar{\tau}_v, \bar{\beta}_v) - \phi_k(t_0)| \leq (\tau_{vn} - t_0) \max_{t \in \langle t_0, t_0 + h \rangle} |\varphi^{(k+1)}(t; \bar{\tau}_v, \bar{\beta}_v)|$$

$$+ |k! D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk}) - \phi_k(t_0)|, \tag{15}$$

$k = 0, \ldots, n-1; \; v = 1, 2, \ldots$

Further, by Theorems 1–3 for each $(\bar{\tau}_v, \bar{\beta}_v)$, $v = 1, 2, \ldots$ there is unique function $\psi_v \in H(t_0, \Phi, r_0)$, $\psi_v = (\psi_{v0}, \ldots, \psi_{vn})$ sucn that $\varphi(t; t_0, \psi_v) = \varphi(t; \bar{\tau}_v, \bar{\beta}_v)$, $t \in \langle t_0, b \rangle$, $v = 1, 2, \ldots$ i.e. there are constants c_{vk}, $k = 0, 1, \ldots, n$; $v = 1, 2, \ldots$ such that

$$\psi_{vk}(t) = \phi_k(t) + c_{vk}, \quad t \in E_{t_0}, \quad k = 0, 1, \ldots, n; \quad v = 1, 2, \ldots \tag{16}$$

Thus the equality

$$\psi_{vk}(t_0) = \phi_k(t_0) + c_{vk}, \quad k = 0, 1, \ldots, n; \quad v = 1, 2, \ldots \tag{17}$$

holds. By (15), (16), (17) we get

$$|\psi_{vk}(t) - \phi_k(t)| = |\psi_{vk}(t_0) - \phi_k(t_0)| \leq (\tau_{vn} - t_0) \max_{t \in \langle t_0, t_0 + h \rangle} |\varphi^{(k+1)}(t; \bar{\tau}_v, \bar{\beta}_v)|$$

$$+ |k! D^{k+1}(\tau_{v0}, \ldots, \tau_{vk}; \beta_{v0}, \ldots, \beta_{vk}) - \phi_k(t_0)|, \quad t \in E_{t_0}, \tag{18}$$

$k = 0, \ldots, n-1; \; v = 1, 2, \ldots$

To show that $\psi_{vk}(t)$, $k=0,1,\dots,n-1$; $v=1,2,\dots$ uniformly converge to $\phi_k(t)$ on E_{t_0} as $v\to\infty$ it suffices to show that there is a real constants C which is not dependent on $\bar{\tau}_v,\bar{\beta}_v$ such that

$$p_i(\bar{\tau}_v,\bar{\beta}_v):=\max_{t\in\langle t_0,t_0+h\rangle}|\varphi^{(i)}(t;\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(i)}(t;t_0,\Phi)|\le C, \tag{19}$$

$i=0,1,\dots,n$; $v=1,2,\dots$ By (14) we have

$$\left|\varphi^{(k)}(t;\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(k)}(t;t_0,\Phi)\right|\le$$

$$\le\left|[\varphi^{(k)}(t;\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(k)}(t;t_0,\Phi)]-[\varphi^{(k)}(\xi_{vk};\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(k)}(\xi_{vk};t_0,\Phi)]\right|$$
$$+\left|k!D^{k+1}(\tau_{v0},\dots,\tau_{vk};\beta_{v0},\dots,\beta_{vk})\right|+\left|\varphi^{(k)}(\xi_{vk};t_0,\Phi)\right|, \tag{20}$$

$k=0,1,\dots,n-1$. Further by (12) there are constants M, M_k such that

$$|k!D^{k+1}(\tau_{v0},\dots,\tau_{vk};\beta_{v0},\dots,\beta_{vk})|\le M,$$
$$|\varphi^{(k)}(t;t_0,\Phi)|\le M_k,\quad t\in\langle t_0,t_0+h\rangle.$$

Thus by (19), (20) we get

$$p_k(\bar{\tau}_v,\bar{\beta}_v)\le K+hp_{k+1}(\bar{\tau}_v,\bar{\beta}_v),\quad k=0,1,\dots,n-1 \tag{21}$$

where $K=M+\max\{M_0,M_1,\dots,M_{n-1}\}$. From (21) we get

$$p_k(\bar{\tau}_v,\bar{\beta}_v)\le K(1+h+\dots+h^{n-k-1})+h^{n-k}p_n(\bar{\tau}_v,\bar{\beta}_v)\le nK+hp_n(\bar{\tau}_v,\bar{\beta}_v) \tag{22}$$

and

$$\sum_{k=0}^{n-1}p_k(\bar{\tau}_v,\bar{\beta}_v)\le Kn^2+nhp_n(\bar{\tau}_v,\bar{\beta}_v). \tag{23}$$

On the other hand $\varphi(t;\bar{\tau}_v,\bar{\beta}_v)$, $\varphi(t;t_0,\Phi)$ are solutions of (1). By (1) we have

$$[\varphi^{(n)}(t;\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(n)}(t;t_0,\Phi)]+a(t)[\varphi^{(n)}(t-\Delta_0(t);\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(n)}(t-\Delta_0(t);t_0,\Phi)]$$
$$=-\sum_{i=1}^{n}\sum_{j=1}^{m}b_{ij}(t)[\varphi^{(n-i)}(t-\Delta_{ij}(t);\bar{\tau}_v,\bar{\beta}_v)-\varphi^{(n-i)}(t-\Delta_{ij}(t);t_0,\Phi)].$$

By (11), (19) we have

$$(1-A)p_n(\bar{\tau}_v,\bar{\beta}_v)\le mL\sum_{k=0}^{n-1}p_k(\bar{\tau}_v,\bar{\beta}_v),\quad v=1,2,\dots \tag{24}$$

Thus by (23) we get

$$(nh)^{-1}\sum_{k=0}^{n-1}p_k(\bar{\tau}_v,\bar{\beta}_v)-\frac{Kn}{h}\le(1-A)^{-1}mL\sum_{k=0}^{n-1}p_k(\bar{\tau}_v,\bar{\beta}_v)$$

and

$$\sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \bar{\beta}_v) \leq C = \frac{n^2 K(1-A)}{1-A-nmLh}.$$

From this and (24) we conclude that

$$p_k(\bar{\tau}_v, \bar{\beta}_v) \leq C, \quad k = 0, 1, \ldots, n-1; \quad v = 1, 2, \ldots,$$

$$p_n(\bar{\tau}_v, \bar{\beta}_v) \leq CmL(1-A)^{-1}.$$

By Theorems 1–3 for each $(\bar{\tau}_{v_l}, \bar{\beta}_{v_l})$, $l = 1, 2, \ldots$ there is unique solution of BVP $\varphi(t; \bar{\tau}_{v_l}, \bar{\beta}_{v_l})$ and solution of IVP

$$\varphi(t; t_0, \psi_{v_l}) = \varphi(t; t_0, \phi_0(t) + c_{l0}, \ldots, \phi_{n-1}(t) + c_{l,n-1}, \phi_n(t) + c_{ln})$$

such that $\varphi(t; \bar{\tau}_{v_l}, \bar{\beta}_{v_l}) = \varphi(t; t_0, \phi_0(t) + c_{l0}, \ldots, \phi_{n-1}(t) + c_{l,n-1}, \phi_n(t) + c_{ln})$, $t \in \langle t_0, t_0 + h\rangle$. We proved that

$$\lim_{l \to \infty} \varphi^{(k)}(t_0; \bar{\tau}_{v_l}, \bar{\beta}_{v_l}) = \varphi^{(k)}(t_0; t_0, \phi_0(t_0), \ldots, \phi_{n-1}(t_0), \phi_n(t_0) + c_n),$$

$k = 1, \ldots, n-1$. Further for $(\bar{\tau}_{v_l}, \bar{\beta}_{v_l})$, $l = 1, 2, \ldots$; $\varphi^{(n)}(t; \bar{\tau}_{v_l}, \bar{\beta}_{v_l})$, $l = 1, 2, \ldots$ are equicontinuous and uniformly bounded on $\langle t_0, t_0 + h\rangle$. From Arzel–Ascoli Theorem functions $\varphi^{(n)}(t; \bar{\tau}_{v_l}, \bar{\beta}_{v_l})$, $l = 1, 2, \ldots$ uniformly converge on $\langle t_0, t_0 + h\rangle$. By (11), (14) we have

$$\phi_n(t_0) = n! \lim_{l \to \infty} D^{n+1}(\tau_{v_l 0}, \ldots, \tau_{v_l n}; \beta_{v_l 0}, \ldots, \beta_{v_l n}) = \lim_{l \to \infty} \varphi^{(n)}(\xi_{v_l}; \bar{\tau}_{v_l}, \bar{\beta}_{v_l})$$

$$= \varphi^{(n)}(t_0; t_0, \phi_0(t_0), \ldots, \phi_{n-1}(t_0), \phi_n(t_0) + c_n) = \phi_n(t_0) + c_n.$$

From this we have $c_n = 0$. Thus $\psi_{vk}(t)$ uniformly converge to $\phi_k(t)$ on E_{t_0} for $k = 0, 1, \ldots, n$ as $v \to \infty$. From this fact by theorem on continuous dependents of solutions on initial conditions we have that $\varphi^{(k)}(t; \bar{\tau}_v, \bar{\beta}_v)$, $k = 0, 1, \ldots, n$ uniformly converge to $\varphi^{(k)}(t; t_0, \Phi)$ on $\langle t_0, b\rangle$.

The proof of theorem is complete. □

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