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One Multivariate Linear Model with Nuisance Parameters *

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Abstract

The characterization for a class of functions of useful parameters which are estimable under the model with nuisance parameters and under the model, where the nuisance parameters are neglected and estimators of which have the same variance in both mentioned models is given in the paper.

Key words: Multivariate linear model, nuisance parameters, BLUE.

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1 Introduction

Let R^n denote the space of all n-dimensional real vectors, let $\boldsymbol{u}_p$ and $\boldsymbol{A}_{m,n}$ denote a real column p-dimensional vector and a real $m \times n$ matrix, respectively. The symbols $\boldsymbol{A}'$, $\boldsymbol{A}^{(j)}$, $\mathscr{R}(\boldsymbol{A})$, $\mathscr{N}(\boldsymbol{A})$, $r(\boldsymbol{A})$, $Tr(\boldsymbol{A})$ will denote transpose, j-th column, range, null space, rank and trace of the matrix $\boldsymbol{A}$, respectively. Further $\text{vec}(\boldsymbol{A})$ will denote the column vector $((\boldsymbol{A}^{(1)})', \ldots, (\boldsymbol{A}^{(n)})')'$ created by the columns of the matrix $\boldsymbol{A}$. The symbol $\boldsymbol{A} \otimes \boldsymbol{B}$ will denote the Kronecker (tensor) product of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$. $\boldsymbol{A}^-$ will denote an arbitrary generalized inverse of $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$), $\boldsymbol{A}^+$ will denote the Moore–Penrose generalized inverse of $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$, $(\boldsymbol{A}\boldsymbol{A}^+)' = \boldsymbol{A}\boldsymbol{A}^+$,

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$(A^+A)' = A^+A)$. Moreover P_A and Q_A will stand for the ortogonal projector onto $\mathscr{R}(A)$ and $\mathscr{R}^{\perp}(A) = \mathscr{N}(A')$, respectively. $A^{\perp}$ will stand for any matrix such that $\mathscr{R}(A^{\perp}) = \mathscr{R}^{\perp}(A)$. The symbol I denotes the identity matrix.

If $\mathscr{R}(A) \subset \mathscr{R}(S)$, S p.s.d., then the symbol $P_A^{S^-}$ denotes the projector projecting vectors in $\mathscr{R}(S)$ onto $\mathscr{R}(A)$ along $\mathscr{R}(SA^{\perp})$. A general representation of all such projectors $P_A^{S^-}$ is given by $A(A'S^-A)^-A'S^- + F(I - SS^-)$, where F is arbitrary, see [4], (2.14). $Q_A^{S^-} = I - P_A^{S^-}$.

Let

$$Y_{n,m} = X_{n,k}B_{k,l}Z_{l,m} + \varepsilon_{n,m} \tag{1}$$

be a multivariate linear model under consideration.

Here Y is an observation matrix, X, Z, are known nonzero matrices, ε is a random matrix and B is a matrix of unknown parameters

$$B = (B_1, B_2),$$

where B_1 is a $k \times r$ matrix of useful parameters which (or their functions) have to be estimated from the observation matrix Y and B_2 is a $k \times s$ matrix of nuisance parameters. Thus we consider the model

$$Y = X(B_1, B_2)\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \varepsilon. \tag{2}$$

Similarly as in [3] (where the linear model $Y = XBZ + SGT + \varepsilon$ under the assumption $\mathrm{var}(\mathrm{vec}(Y)) = I \otimes \Sigma$ was considered), the purpose of this paper is to characterize the class of all linear functions of the useful parameters $\mathrm{vec}(B_1)$ which are unbiasedly estimable under the model with nuisance parameters and under the model, where the nuisance parameters are neglected and estimators of which have the same variance in both models mentioned.

A parametric function $p'\,\mathrm{vec}(B_1)$ is said to be unbiasedly estimable under the model (2) if there exists an estimator $L'\,\mathrm{vec}(Y)$, $L \in R^{mn}$, such that $E[L'\,\mathrm{vec}(Y)] = p'\,\mathrm{vec}(B_1)$, $\forall\,\mathrm{vec}(B_1), \forall\,\mathrm{vec}(B_2)$.

2 Auxiliary statements

Lemma 1 *The model (2) can be equivalently written in the form*

$$\mathrm{vec}(Y) = [Z_1' \otimes X, Z_2' \otimes X]\begin{pmatrix} \mathrm{vec}(B_1) \\ \mathrm{vec}(B_2) \end{pmatrix} + \mathrm{vec}(\varepsilon). \tag{3}$$

where a $r \times m$ matrix Z_1 and a $s \times m$ matrix Z_2 are known nonzero matrices.

Proof is obvious by virtue of the following statement

$$\mathrm{vec}(ABC) = (C' \otimes A)\,\mathrm{vec}(B), \tag{4}$$

valid for all matrices of corresponding types. □

Suppose that the observation vector $\mathrm{vec}(\boldsymbol{Y})$ has the mean value

$$E(\mathrm{vec}(\boldsymbol{Y})) = [\boldsymbol{Z}_1' \otimes \boldsymbol{X}, \boldsymbol{Z}_2' \otimes \boldsymbol{X}] \begin{pmatrix} \mathrm{vec}(\boldsymbol{B}_1) \\ \mathrm{vec}(\boldsymbol{B}_2) \end{pmatrix},$$

and the covariance matrix

$$\mathrm{var}[\mathrm{vec}(\boldsymbol{Y})] = \Sigma_{m,m} \otimes \boldsymbol{I}_{n,n},$$

where Σ (the covarince matrix of any row of the matrix $\boldsymbol{Y}$) is obviously at least p.s.d.

In this paper we consider the linear model (see [4])

$$\mathscr{M}_a(\Sigma \otimes \boldsymbol{I}) = \left[\mathrm{vec}(\boldsymbol{Y}), (\boldsymbol{Z}_1' \otimes \boldsymbol{X}, \boldsymbol{Z}_2' \otimes \boldsymbol{X}) \begin{pmatrix} \mathrm{vec}(\boldsymbol{B}_1) \\ \mathrm{vec}(\boldsymbol{B}_2) \end{pmatrix}, \Sigma \otimes \boldsymbol{I}\right],$$

with nuisance parameters and the linear model

$$\mathscr{M}(\Sigma \otimes \boldsymbol{I}) = \left[\mathrm{vec}(\boldsymbol{Y}), (\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \mathrm{vec}(\boldsymbol{B}_1), \Sigma \otimes \boldsymbol{I}\right],$$

where nuisance parameters are neglected.

Assume Σ be such that

$$\mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}, \boldsymbol{Z}_2' \otimes \boldsymbol{X}) \subset \mathscr{R}(\Sigma \otimes \boldsymbol{I}). \tag{5}$$

This condition is warranted by

$$\mathscr{R}(\boldsymbol{Z}_1') \subset \mathscr{R}(\Sigma) \quad \& \quad \mathscr{R}(\boldsymbol{Z}_2') \subset \mathscr{R}(\Sigma), \tag{6}$$

that will be supposed throughout. Under the assumption (5)

$$\mathrm{vec}(\boldsymbol{Y}) \in \mathscr{R}(\Sigma \otimes \boldsymbol{I}) \ (a.s.).$$

Notation 1 Let, according to [4], $\mathscr{E}_a$ and $\mathscr{E}$ denote the sets of all linear functions of $\mathrm{vec}(\boldsymbol{B}_1)$ which are unbiasedly estimable under the model $\mathscr{M}_a(\Sigma \otimes \boldsymbol{I})$ and $\mathscr{M}(\Sigma \otimes \boldsymbol{I})$, respectively. The index a will indicate, that the estimator is considered in the complete model, i.e. in the model with nuisance parameters.

Obviously

$$\mathscr{E} = \{\boldsymbol{p}' \mathrm{vec}(\boldsymbol{B}_1) : \boldsymbol{p} \in \mathscr{R}(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\}. \tag{7}$$

Remark 1 The considered function $\boldsymbol{p}' \mathrm{vec}(\boldsymbol{B}_1)$ can be expressed in another form: $\boldsymbol{p} = (\boldsymbol{p}_1', \dots, \boldsymbol{p}_r')'$, where $\boldsymbol{p}_j$ are k-dimensional vectors, $j = 1, \dots, r$. Let $\boldsymbol{P}' = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_r)$. Using the fact that

$$(\mathrm{vec}(\boldsymbol{A}'))' \mathrm{vec}(\boldsymbol{B}) = Tr(\boldsymbol{AB}),$$

we have

$$\boldsymbol{p}' \mathrm{vec}(\boldsymbol{B}_1) = (\mathrm{vec}(\boldsymbol{P}'))' \mathrm{vec}(\boldsymbol{B}_1) = Tr(\boldsymbol{P}\boldsymbol{B}_1).$$

Remark 2 In view of the relation (4)

$$\boldsymbol{p} \in \mathscr{R}(\boldsymbol{Z}_1 \otimes \boldsymbol{X}') \iff \exists A_{m,n},\ \boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\operatorname{vec}(\boldsymbol{A}') \Leftrightarrow \exists \boldsymbol{A}_{m,n},\ \boldsymbol{P} = \boldsymbol{Z}_1\boldsymbol{A}\boldsymbol{X}.$$

Let us consider the class $\mathscr{E}_a$.

$$\begin{aligned}\mathscr{E}_a = \{&\boldsymbol{p}'\operatorname{vec}(\boldsymbol{B}_1) : \boldsymbol{p} \in R^{kr},\ \exists \boldsymbol{L} \in R^{nm},\ \forall \operatorname{vec}(\boldsymbol{B}_1) \in R^{kr},\\ &\forall \operatorname{vec}(\boldsymbol{B}_2) \in R^{ks},\ E\left[\boldsymbol{L}'\operatorname{vec}(\boldsymbol{Y})\right] = \boldsymbol{p}'\operatorname{vec}(\boldsymbol{B}_1)\}.\end{aligned}$$

The equality

$$E[\boldsymbol{L}'\operatorname{vec}(\boldsymbol{Y})] = \boldsymbol{L}'(\boldsymbol{Z}_1' \otimes \boldsymbol{X})\operatorname{vec}(\boldsymbol{B}_1) + \boldsymbol{L}'(\boldsymbol{Z}_2' \otimes \boldsymbol{X})\operatorname{vec}(\boldsymbol{B}_2) = \boldsymbol{p}'\operatorname{vec}(\boldsymbol{B}_1),$$

$$\forall \operatorname{vec}(\boldsymbol{B}_1), \forall \operatorname{vec}(\boldsymbol{B}_2),$$

is fulfiled if and only if

$$\boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{L} \quad \& \quad (\boldsymbol{Z}_2 \otimes \boldsymbol{X}')\boldsymbol{L} = o,$$

which is equivalent to

$$\boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{Q}_{Z_2' \otimes X}\boldsymbol{u}, \quad \boldsymbol{u} \in R^{mn}.$$

Thus

Lemma 2

$$\begin{aligned}\mathscr{E}_a &= \{\boldsymbol{p}'\operatorname{vec}(\boldsymbol{B}_1) : \boldsymbol{p} \in \mathscr{R}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{Q}_{Z_2' \otimes X}]\}\\ &= \{\boldsymbol{p}'\operatorname{vec}(\boldsymbol{B}_1) : \boldsymbol{p} \in \mathscr{R}(\boldsymbol{Z}_1\boldsymbol{Q}_{Z_2'} \otimes \boldsymbol{X}')\}.\end{aligned} \tag{8}$$

Remark 3 Using the matrix $\boldsymbol{P}$ from Remark 1 we get

$$\boldsymbol{p} \in \mathscr{R}(\boldsymbol{Z}_1\boldsymbol{Q}_{Z_2'} \otimes \boldsymbol{X}') \iff \exists \boldsymbol{A}_{m,n} \text{ such that } \boldsymbol{P} = \boldsymbol{Z}_1\boldsymbol{Q}_{Z_2'}\boldsymbol{A}\boldsymbol{X}.$$

Comparing (7) and (8) it is obvious that

$$\mathscr{E}_a \subset \mathscr{E}.$$

Moreover,

Lemma 3

$$\mathscr{E}_a = \mathscr{E} \iff \mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{Z}_2' \otimes \boldsymbol{X}) = \{o\} \iff \mathscr{R}(\boldsymbol{Z}_1') \cap \mathscr{R}(\boldsymbol{Z}_2') = \{o\}.$$

Proof Under the condition $\mathscr{E}_a \subset \mathscr{E}$

$$\begin{aligned}\mathscr{E}_a = \mathscr{E} &\iff 0 = r(\boldsymbol{Z}_1 \otimes \boldsymbol{X}') - r[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{Q}_{Z_2' \otimes X}]\\ &= \dim\left[\mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \cap \mathscr{R}^{\perp}(\boldsymbol{Q}_{Z_2' \otimes X})\right] = \dim\left[\mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{Z}_2' \otimes \boldsymbol{X})\right],\end{aligned}$$

using the fact that (see [4], (2.4))

$$r(\boldsymbol{A}) - r(\boldsymbol{AB}) = \dim[\mathscr{R}(\boldsymbol{A}') \cap \mathscr{R}^{\perp}(\boldsymbol{B})]. \tag{9}$$

$$\begin{aligned} \mathscr{E}_a = \mathscr{E} &\iff 0 = r(\boldsymbol{Z}_1 \otimes \boldsymbol{X}') - r(\boldsymbol{Z}_1 \boldsymbol{Q}_{Z_2'} \otimes \boldsymbol{X}') \\ &\iff 0 = r(\boldsymbol{Z}_1) - r(\boldsymbol{Z}_1 \boldsymbol{Q}_{Z_2'}) = \\ \dim\left[\mathscr{R}(\boldsymbol{Z}_1') \cap \mathscr{R}^{\perp}(\boldsymbol{Q}_{Z_2'})\right] &= \dim\left[\mathscr{R}(\boldsymbol{Z}_1') \cap \mathscr{R}(\boldsymbol{Z}_2')\right], \end{aligned}$$

where $r(\boldsymbol{A} \otimes \boldsymbol{B}) = r(\boldsymbol{A})r(\boldsymbol{B})$ was taken into account. □

We assume throughout that

$$\mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \not\subset \mathscr{R}(\boldsymbol{Z}_2' \otimes \boldsymbol{X}).$$

If $\mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \subset \mathscr{R}(\boldsymbol{Z}_2' \otimes \boldsymbol{X})$, then $\mathscr{R}(\boldsymbol{Z}_1 \boldsymbol{Q}_{Z_2'} \otimes \boldsymbol{X}') = \{o\}$.

Notation 2 Denote $\widehat{\operatorname{vec}(\boldsymbol{B}_1)}$ and $\widehat{\operatorname{vec}(\boldsymbol{B}_1)}_a$ an $(\Sigma^- \otimes \boldsymbol{I})$-LS estimator of the parameter $\operatorname{vec}(\boldsymbol{B}_1)$ computed under the linear model $\mathscr{M}(\Sigma \otimes \boldsymbol{I})$ and $\mathscr{M}_a(\Sigma \otimes \boldsymbol{I})$, respectively (see [1], p. 161).

According to the assumption (6) $\boldsymbol{p}' \widehat{\operatorname{vec}(\boldsymbol{B}_1)}$ and $\boldsymbol{p}' \widehat{\operatorname{vec}(\boldsymbol{B}_1)}_a$ is the BLUE of the function $\boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}$ and $\boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}_a$, respectively (see [1], Theorem 5.3.2., p. 162).

Lemma 4

$$\begin{aligned} \boldsymbol{p}' \widehat{\operatorname{vec}(\boldsymbol{B}_1)} &= \boldsymbol{p}' \left[(\boldsymbol{Z}_1 \Sigma^- \boldsymbol{Z}_1')^- \boldsymbol{Z}_1 \Sigma^- \otimes (\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{X}'\right] \operatorname{vec}(\boldsymbol{Y}), \\ & \quad \textit{if } \boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}, \end{aligned} \tag{10}$$

$$\begin{aligned} \boldsymbol{p}' \widehat{\operatorname{vec}(\boldsymbol{B}_1)}_a &= \boldsymbol{p}' \left\{[\boldsymbol{Z}_1 \Sigma^- \boldsymbol{Q}_{Z_2'}^{\Sigma^-} \boldsymbol{Z}_1']^- \boldsymbol{Z}_1 \Sigma^- \boldsymbol{Q}_{Z_2'}^{\Sigma^-} \otimes (\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{X}'\right\} \operatorname{vec}(\boldsymbol{Y}), \\ & \quad \textit{if } \boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}_a, \end{aligned} \tag{11}$$

$$\begin{aligned} \operatorname{var}[\boldsymbol{p}' \widehat{\operatorname{vec}(\boldsymbol{B}_1)}] &= \boldsymbol{p}' \left[(\boldsymbol{Z}_1 \Sigma^- \boldsymbol{Z}_1')^- \otimes (\boldsymbol{X}'\boldsymbol{X})^-\right] \boldsymbol{p}, \\ & \quad \textit{if } \boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}, \end{aligned} \tag{12}$$

$$\begin{aligned} \operatorname{var}[\boldsymbol{p}' \widehat{\operatorname{vec}(\boldsymbol{B}_1)}_a] &= \boldsymbol{p}' \left[(\boldsymbol{Z}_1 \Sigma^- \boldsymbol{Q}_{Z_2'}^{\Sigma^-} \boldsymbol{Z}_1')^- \otimes (\boldsymbol{X}'\boldsymbol{X})^-\right] \boldsymbol{p}, \\ & \quad \textit{if } \boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}_a. \end{aligned} \tag{13}$$

These expressions are invariant to the choice of g-inverse matrices.

Proof Under $\mathscr{M}_a$ we have

$$\begin{pmatrix} \widehat{\operatorname{vec}(\boldsymbol{B}_1)}_a \\ \widehat{\operatorname{vec}(\boldsymbol{B}_2)}_a \end{pmatrix} =$$

$$= \left[(Z_1' \otimes X, Z_2' \otimes X)'(\Sigma \otimes I)^-(Z_1' \otimes X, Z_2' \otimes X)\right]^- \begin{pmatrix} Z_1 \otimes X' \\ Z_2 \otimes X' \end{pmatrix}$$

$$\times (\Sigma \otimes I)^- \operatorname{vec}(Y)$$

$$= \begin{bmatrix} Z_1\Sigma^- Z_1' \otimes X'X, & Z_1\Sigma^- Z_2' \otimes X'X \\ Z_2\Sigma^- Z_1' \otimes X'X, & Z_2\Sigma^- Z_2' \otimes X'X \end{bmatrix}^- \begin{pmatrix} Z_1\Sigma^- \otimes X' \\ Z_2\Sigma^- \otimes X' \end{pmatrix} \operatorname{vec}(Y). \quad (14)$$

The estimate obtained by a substitution of this expression into unbiasedly estimable function is given uniquely.

Using the following Rohde's formula for generalized inverse of partitioned p.s.d. matrix (see [2], Lemma 13, p. 68)

$$\begin{pmatrix} A, & B \\ B', & C \end{pmatrix}^- =$$

$$= \begin{pmatrix} A^- + A^-B(C - B'A^-B)^-B'A^-, & -A^-B(C - B'A^-B)^- \\ -(C - B'A^-B)^-B'A^-, & (C - B'A^-B)^- \end{pmatrix}$$

$$= \begin{pmatrix} (A - BC^-B')^-, & -(A - BC^-B')^-BC^- \\ -C^-B'(A - BC^-B')^-, & C^- + C^-B'(A - BC^-B')^-BC^- \end{pmatrix},$$

we get the first row A_{11}, A_{12} of the g-inverse matrix in (14):

$$\begin{aligned} A_{11} &= \Big[(Z_1\Sigma^- Z_1' \otimes X'X) \\ &\quad - (Z_1\Sigma^- Z_2' \otimes X'X)(Z_2\Sigma^- Z_2' \otimes X'X)^-(Z_2\Sigma^- Z_1' \otimes X'X)\Big]^- \\ &= \Big[(Z_1\Sigma^-(I - P_{Z_2'}^{\Sigma^-})Z_1' \otimes X'X)\Big]^- = \Big[Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1' \otimes X'X\Big]^-. \end{aligned}$$

In view of (6) we can choose Σ^- p.d., so we have $P_{Z_2'}^{\Sigma^-} = Z_2'(Z_2\Sigma^- Z_2')^- Z_2\Sigma^-$.

$$\begin{aligned} A_{12} &= -\Big[(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- \otimes (X'X)^-\Big] \\ &\quad \times (Z_1\Sigma^- Z_2' \otimes X'X)\left[(Z_2\Sigma^- Z_2')^- \otimes (X'X)^-\right] \\ &= -\Big[(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- Z_1\Sigma^- Z_2'(Z_2\Sigma^- Z_2')^- \otimes (X'X)^- X'X(X'X)^-\Big]. \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{vec}(\widehat{B}_1)_a &= \Big\{\Big[(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- \otimes (X'X)^-\Big](Z_1\Sigma^- \otimes X') \\ &\quad - \Big[(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- Z_1\Sigma^- Z_2'(Z_2\Sigma^- Z_2')^- \otimes (X'X)^- X'X(X'X)^-\Big] \\ &\quad \times (Z_2\Sigma^- \otimes X')\Big\} \operatorname{vec}(Y) \\ &= \Big[(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} \otimes (X'X)^- X'\Big] \operatorname{vec}(Y). \end{aligned}$$

We have proved (11).

$$\mathrm{var}[p'\widehat{\mathrm{vec}(B_1)_a}] = p'\Big\{[(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} \otimes (X'X)^- X'][\Sigma \otimes I]$$

$$\times [(Q_{Z_2'}^{\Sigma^-})'\Sigma^- Z_1'(Z_1(Q_{Z_2'}^{\Sigma^-})'\Sigma^- Z_1')^- \otimes X(X'X)^-]\Big\}p$$

$$= p'\Big\{[Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1']^-[Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1'][Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1']^-$$

$$\otimes (X'X)^- X'X(X'X)^-\Big\}p = p'\{(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- \otimes (X'X)^-\}p,$$

where the assertion

$$u \in \mathscr{R}(B) \subset \mathscr{R}(A) \;\Rightarrow\; u'A^- AA^- u = u'A^- u,$$

was utilized. The assumption $p \in \mathscr{R}(Z_1 Q_{Z_2'} \otimes X') \subset \mathscr{R}(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1' \otimes X'X)$ is satisfied. The invariance of (10)–(13) to the choice of g-inverse matrices can be proved if we take the following assertion (see [2], Lemma 8, p. 65)

$$AB^- C \text{ is invariant to the choice of the } g\text{-inverse } B^-$$

$$\Longleftrightarrow \mathscr{R}(A') \subset \mathscr{R}(B') \quad \& \quad \mathscr{R}(C) \subset \mathscr{R}(B), \tag{15}$$

into account.

Remark 4 (10)–(13) are equivalent to

$$Tr(\widehat{PB_1}) = Tr\left[P(X'X)^- X'Y\Sigma^- Z_1'(Z_1\Sigma^- Z_1')^-\right], \text{ if } Tr(PB_1) \in \mathscr{E},$$

$$Tr(\widehat{PB_1})_a = Tr\left[P(X'X)^- X'YQ_{Z_2'}^{\Sigma^-}\Sigma^- Z_1'(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^-\right] \text{ if } Tr(PB_1) \in \mathscr{E}_a,$$

$$\mathrm{var}[\widehat{Tr(PB_1)}] = Tr[P(X'X)^- P'(Z_1\Sigma^- Z_1')^-], \text{ if } Tr(PB_1) \in \mathscr{E}.$$

$$\mathrm{var}[\widehat{Tr(PB_1)_a}] = Tr[P(X'X)^- P'(Z_1\Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^-], \text{ if } Tr(PB_1) \in \mathscr{E}_a.$$

3 Efficiently estimable functions

Let, according to [4], $\mathscr{E}_0(\Sigma \otimes I)$ denote the subset of $\mathscr{E}_a$ consisting of all those functions of $p'\,\mathrm{vec}(B_1)$ for which the BLUE under model $\mathscr{M}_a(\Sigma \otimes I)$ posseses the same variance as the BLUE under model $\mathscr{M}(\Sigma \otimes I)$, i.e.

$$\mathscr{E}_0(\Sigma \otimes I) = \{p'\,\mathrm{vec}(B_1) \in \mathscr{E}_a : \mathrm{var}[p'\widehat{\mathrm{vec}(B_1)}] = \mathrm{var}[p'\widehat{\mathrm{vec}(B_1)_a}]\}.$$

Theorem 1 *If $p'\,\mathrm{vec}(B_1) \in \mathscr{E}_a$, i.e. if there exists a matrix U_0 such that $P = Z_1 Q_{Z_2'} U_0 X$, then*

$$p'\,\mathrm{vec}(B_1) \in \mathscr{E}_0(\Sigma \otimes I) \iff X'U_0'Q_{Z_2'}P_{Z_1'}^{\Sigma^-} Z_2' = O.$$

Proof Let $p' \operatorname{vec}(B_1) \in \mathscr{E}_a$, by (8) it is equivalent to $p = (Z_1 Q_{Z_2'} \otimes X')u_0$ for some vector $u_0 \in R^{mn}$. Under this condition the equality of variances

$$\operatorname{var}[p' \widehat{\operatorname{vec}(B_1)}] = \operatorname{var}[p' \widehat{\operatorname{vec}(B_1)_a}], \tag{16}$$

stands for

$$u_0'(Z_1 Q_{Z_2'} \otimes X')' \left\{ \left[(Z_1 \Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- - (Z_1 \Sigma^- Z_1')^- \right] \otimes (X'X)^- \right\} \\ \times (Z_1 Q_{Z_2'} \otimes X')u_0 = 0. \tag{17}$$

Let us denote

$$W = (Z_1 \Sigma^- Q_{Z_2'}^{\Sigma^-} Z_1')^- - (Z_1 \Sigma^- Z_1')^-.$$

Using the implication (see Rohde's formula)

$$\begin{pmatrix} A, & B \\ B', & C \end{pmatrix} \text{ p.s.d.} \quad \Longrightarrow \quad (A - BC^- B')^- = A^- + A^- B(C - B'A^- B)^- B'A^-,$$

to the matrix

$$\begin{pmatrix} Z_1 \Sigma^- Z_1', & Z_1 \Sigma^- Z_2' \\ Z_2 \Sigma^- Z_1', & Z_2 \Sigma^- Z_2' \end{pmatrix},$$

we obtain

$$W = (Z_1 \Sigma^- Z_1')^- Z_1 \Sigma^- Z_2' [Z_2 \Sigma^- Q_{Z_1'}^{\Sigma^-} Z_2']^- Z_2 \Sigma^- Z_1' (Z_1 \Sigma^- Z_1')^-.$$

In view of (6), (15) and [2], Lemma 16, p. 69

$$Z_2 \Sigma^- Q_{Z_1'}^{\Sigma^-} Z_2' = Z_2[\Sigma^+ - \Sigma^+ Z_1'(Z_1 \Sigma^+ Z_1')^- Z_1 \Sigma^+] Z_2' = Z_2 (Q_{Z_1'} \Sigma Q_{Z_1'})^+ Z_2',$$

i.e. this matrix is p.s.d.

One of the choices of the matrices

$$V^- = (Z_2 \Sigma^- Q_{Z_1'}^{\Sigma^-} Z_2')^- \qquad \text{and} \qquad U^- = (X'X)^-$$

can be p.d. (i.e. regular) matrices. Thus $V^- = JJ'$, $U^- = KK'$, where J, K are regular. Therefore (17) is valid if and only if

$$[Z_2 (P_{Z_1'}^{\Sigma^-})' Q_{Z_2'} \otimes X'] u_0 = o. \tag{18}$$

Let $u_0 = \operatorname{vec}(U_0')$. Using matrix P (cf. Remark 1) and (4) we see that (18) is equivalent to

$$X' U_0' Q_{Z_2'} P_{Z_1'}^{\Sigma^-} Z_2' = O. \qquad \square$$

Theorem 2 *The class* $\mathscr{E}_0(\Sigma \otimes I)$ *is given by*

$$\mathscr{E}_0(\Sigma \otimes I) = \left\{ p' \operatorname{vec}(B_1) : p \in \mathscr{R}[(Z_1 \Sigma^- Z_1' \otimes X'X) Q_{Z_1 \Sigma^- Z_2' \otimes X'X}] \right\}$$

$$= \left\{ Tr(PB_1) : P = Z_1 \Sigma^- Z_1' Q_{Z_1 \Sigma^- Z_2'} V X'X, \ \textit{for arbitrary matrix } V \right\}.$$

Proof The class $\mathscr{E}_0(\Sigma \otimes \boldsymbol{I})$ includes functions $\boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) \in \mathscr{E}_a$ (i.e. functions, where $\boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{Q}_{Z_2' \otimes X}\boldsymbol{u}$, $\boldsymbol{u} \in R^{mn}$), satisfying (16).

By (17) and by the proof of the Theorem 1 the equality (16) holds for such functions if and only if

$$\boldsymbol{u}'\boldsymbol{Q}_{Z_2'\otimes X}(\boldsymbol{Z}_1' \otimes \boldsymbol{X})\{[(\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_2'\boldsymbol{J}\boldsymbol{J}'\boldsymbol{Z}_2\Sigma^-\boldsymbol{Z}_1'(\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_1')^-] \otimes \boldsymbol{K}\boldsymbol{K}'\}$$
$$\times (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{Q}_{Z_2'\otimes X}\boldsymbol{u} = 0, \tag{19}$$

where $\boldsymbol{K}$, $\boldsymbol{J}$, are regular. It is equivalent to

$$\boldsymbol{u}'\boldsymbol{Q}_{Z_2'\otimes X}[\boldsymbol{Z}_1'(\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_2' \otimes \boldsymbol{X}] = 0.$$

By using $\boldsymbol{X} = \boldsymbol{P}_X\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{X}$ we see that (19) holds if and only if

$$\boldsymbol{Q}_{Z_2'\otimes X}\boldsymbol{u} \perp \mathscr{R}\Big\{(\boldsymbol{Z}_1' \otimes \boldsymbol{X})[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')(\Sigma^- \otimes \boldsymbol{I})(\boldsymbol{Z}_1' \otimes \boldsymbol{X})]^-$$
$$\times (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')(\Sigma^- \otimes \boldsymbol{I})(\boldsymbol{Z}_2' \otimes \boldsymbol{X})\Big\}$$
$$= \mathscr{R}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \cap [\mathscr{R}^\perp((\Sigma^- \otimes \boldsymbol{I})(\boldsymbol{Z}_1' \otimes \boldsymbol{X})) + \mathscr{R}(\boldsymbol{Z}_2' \otimes \boldsymbol{X})].$$

The last equality follows by [4], Lemma 2.1. and (2.15).

Thus

$$\boldsymbol{Q}_{Z_2'\otimes X}\boldsymbol{u} \in \mathscr{R}(\boldsymbol{Q}_{Z_1'\otimes X}) + [\mathscr{R}(\Sigma^-\boldsymbol{Z}_1' \otimes \boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{Q}_{Z_2'\otimes X})].$$

It implies that

$$\boldsymbol{Q}_{Z_2'\otimes X}\boldsymbol{u} = \boldsymbol{Q}_{Z_1'\otimes X}\boldsymbol{a} + \boldsymbol{Q}_{Z_2'\otimes X}\boldsymbol{b} = \boldsymbol{Q}_{Z_1'\otimes X}\boldsymbol{a} + (\Sigma^-\boldsymbol{Z}_1' \otimes \boldsymbol{X})\boldsymbol{c}.$$

Since

$$(\boldsymbol{Z}_2 \otimes \boldsymbol{X}')(\Sigma^-\boldsymbol{Z}_1' \otimes \boldsymbol{X})\boldsymbol{c} = 0,$$

we have $\boldsymbol{c} \in \mathscr{R}(\boldsymbol{Q}_{Z_1\Sigma^-Z_2'\otimes X'X})$, and so

$$\boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{Q}_{Z_2'\otimes X}\boldsymbol{u} = (\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X})\boldsymbol{Q}_{Z_1\Sigma^-Z_2'\otimes X'X}\boldsymbol{v}, \quad \boldsymbol{v} \in R^{ks},$$

i.e.

$$\mathscr{E}_0(\Sigma \otimes \boldsymbol{I}) = \{\boldsymbol{p}' \operatorname{vec}(\boldsymbol{B}_1) : \boldsymbol{p} \in \mathscr{R}[(\boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X})\boldsymbol{Q}_{Z_1\Sigma^-Z_2'\otimes X'X}]\}. \tag{20}$$

By the matrix $\boldsymbol{P}$ (cf. Remark 1)

$$\mathscr{E}_0(\Sigma \otimes \boldsymbol{I}) =$$
$$= \{Tr(\boldsymbol{P}\boldsymbol{B}_1) : \boldsymbol{P} = \boldsymbol{Z}_1\Sigma^-\boldsymbol{Z}_1'\boldsymbol{Q}_{Z_1\Sigma^-Z_2'}\boldsymbol{V}\boldsymbol{X}'\boldsymbol{X}, \text{ for arbitrary matrix } \boldsymbol{V}\}. \quad \square$$

Theorem 3 $\dim \mathscr{E}_0(\Sigma \otimes \boldsymbol{I}) = r(\boldsymbol{X})[r(\boldsymbol{Z}_1) - r(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_2')]$.

Proof In view of (9) and (20)

$$\dim \mathscr{E}_0(\Sigma \otimes \boldsymbol{I}) = r\left[(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X})\boldsymbol{Q}_{Z_1\Sigma^- Z_2' \otimes X'X}\right]$$

$$= r(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X}) - \dim\left[\mathscr{R}(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X}) \cap \mathscr{R}^{\perp}(\boldsymbol{Q}_{Z_1\Sigma^- Z_2' \otimes X'X})\right]$$

$$= r(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X}) - \dim\left[\mathscr{R}(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_1' \otimes \boldsymbol{X}'\boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_2' \otimes \boldsymbol{X}'\boldsymbol{X})\right].$$

The assumption (6) yields that $\mathscr{R}(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_1') = \mathscr{R}(\boldsymbol{Z}_1)$. Thus

$$\dim \mathscr{E}_0(\Sigma \otimes \boldsymbol{I}) =$$

$$= r(\boldsymbol{Z}_1)r(\boldsymbol{X}') - \dim\left[\mathscr{R}(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_2' \otimes \boldsymbol{X}'\boldsymbol{X})\right] = r(\boldsymbol{Z}_1)r(\boldsymbol{X}') - r(\boldsymbol{Z}_1\Sigma^- \boldsymbol{Z}_2')r(\boldsymbol{X}').$$

The assertion

$$\mathscr{R}(\boldsymbol{A}) \subset \mathscr{R}(\boldsymbol{B}) \ \& \ \mathscr{R}(\boldsymbol{C}) \subset \mathscr{R}(\boldsymbol{D}) \Rightarrow \mathscr{R}(\boldsymbol{A} \otimes \boldsymbol{C}) \subset \mathscr{R}(\boldsymbol{B} \otimes \boldsymbol{D}),$$

was utilized. □

Remark 5 In this paper we have used the same procedures as in [3]. Model (1) is the special case of the linear model considered there. It is impossible to rewrite directly the results given in Lemma 4, Theorem 1 and Theorem 2 because different var(vec(Y)) were assumed. In comparison with [3] new result about the dimension of the class $\mathscr{E}_0(\Sigma \otimes \boldsymbol{I})$ was proved (see Theorem 3).

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