Vladimír Slezák

Bases in incidence structures defined on projective spaces

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 37 (1998), No. 1, 113--121

Persistent URL: `http://dml.cz/dmlcz/120378`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **37** (1998) 113–121

Bases in Incidence Structures Defined on Projective Spaces

VLADIMÍR SLEZÁK

Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: slezak@prfnw.upol.cz

(Received February 19, 1998)

Abstract

In this paper an incidence structure on the projective space is defined. The closure spaces induced by that structure are investigated, especially the problems of existence and cardinality of bases in them.

Key words: Closure spaces, incidence structures, projective spaces.

1991 Mathematics Subject Classification: 06B05, 08A35

Definition 1 Let P be a set and $\mathcal{P}$ be a family of its subsets. Then the pair $(P, \mathcal{P})$ is called a *closure space* if $\mathcal{P}$ is closed under intersection and $P \in \mathcal{P}$. The elements of $\mathcal{P}$ are said to be *closed* sets.

If $X \subseteq P$, then the intersection $\langle X \rangle$ of all closed sets containing X is called the *closure* of X.

A closed set A is said to be *generated* by a subset $X \subseteq P$ if $A = \langle X \rangle$.

If $(P, \mathcal{P})$ is a closure space and $X, Y \subseteq P$, then it is obvious that $X \subseteq \langle X \rangle$, $X \subseteq Y \Rightarrow \langle X \rangle \subseteq \langle Y \rangle$, $\langle\langle X \rangle\rangle = \langle X \rangle$.

Definition 2 Let $(P, \mathcal{P})$ be a closure space. A set $X \subseteq P$ is said to be *independent* if $x \notin \langle X - \{x\} \rangle$ for all $x \in X$.

Remark 1 All subsets of any independent set are independent. In what follows, we put $X_a := X - \{a\}$ for $a \in X$.

Definition 3 A set $B \subseteq P$ is called a *basis* of the closure space $(P, \mathcal{P})$ if B is independent and B generates P, i.e. $\langle B \rangle = P$.

Definition 4 If a basis of cardinality κ exists in $(P, \mathcal{P})$ and no other basis has greater cardinality, then we say that $(P, \mathcal{P})$ has *dimension* $\kappa - 1$.

We will write $\dim(P, \mathcal{P}) = \kappa - 1$.

Let G and M be sets and $I \subseteq G \times M$. Then the triple (G, M, I) is called an *incidence structure.* If $A \subseteq G$, $B \subseteq M$ are non-empty sets, then we denote

$$A^{\uparrow} = \{m \in M \mid gIm \quad \forall g \in A\}, \qquad B^{\downarrow} = \{g \in G \mid gIm \quad \forall m \in B\}.$$

For the empty set we put $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$. And moreover, we denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$, $g^{\uparrow} := \{g\}^{\uparrow}$, $m^{\downarrow} := \{m\}^{\downarrow}$ for $A \subseteq G$, $B \subseteq M$ and $g \in G$, $m \in M$.

Let $\mathcal{J} = (G, M, I)$ be an incidence structure. Then it is easy to show (see [1]) that

$$\begin{array}{ll}
A \subseteq C \Rightarrow C^{\uparrow} \subseteq A^{\uparrow} & \text{for } A, C \subseteq G, \\
B \subseteq D \Rightarrow D^{\downarrow} \subseteq B^{\downarrow} & \text{for } B, D \subseteq M, \\
A \subseteq A^{\uparrow\downarrow}, B \subseteq B^{\downarrow\uparrow} & \text{for } A \subseteq G, B \subseteq M, \\
A^{\uparrow\downarrow\uparrow} = A^{\uparrow}, B^{\downarrow\uparrow\downarrow} = B^{\downarrow} & \text{for } A \subseteq G, B \subseteq M, \\
(\bigcup\limits_{i \in L} A_i)^{\uparrow} = \bigcap\limits_{i \in L} A_i{}^{\uparrow} & \text{for } A_i \subseteq G, \\
(\bigcup\limits_{i \in L} B_i)^{\downarrow} = \bigcap\limits_{i \in L} B_i^{\downarrow} & \text{for } B_i \subseteq M.
\end{array}$$

Theorem 1 *Let $\mathcal{J} = (G, M, I)$ be an incidence structure. If we put*

$$\mathcal{G}_{\mathcal{J}} = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\}, \qquad \mathcal{M}_{\mathcal{J}} = \{B \subseteq M \mid B = B^{\downarrow\uparrow}\},$$

then the pairs $(G, \mathcal{G}_{\mathcal{J}})$, $(M, \mathcal{M}_{\mathcal{J}})$ are closure spaces. (See [2].)

Remark 2 If $A \subseteq G$, $B \subseteq M$, then $\langle A \rangle = A^{\uparrow\downarrow}$ and $\langle B \rangle = B^{\downarrow\uparrow}$ in $(G, \mathcal{G}_{\mathcal{J}})$ and $(M, \mathcal{M}_{\mathcal{J}})$, respectively.

Definition 5 Let V be a vector space over a field K, $\dim V = n + 1$, $n \geq 2$. The system consisting of all subspaces of V is called a *projective space* and will be denoted by $\mathcal{P}^n$.

Projective dimension of the subspaces of $\mathcal{P}^n$ is defined with a help of dimension of the subspaces in V by the formula

$$\dim_{\mathcal{P}} U = \dim_V U - 1$$

for any subspace U in V.

Then the projective space $\mathcal{P}^n$ has projective dimension n. The subspaces of $\mathcal{P}^n$ with projective dimension 0 $(1, 2, n-1)$ are called *points* (*lines, planes, hyperplanes*).

The empty set is a subspace of $\mathcal{P}^n$ and $\dim_{\mathcal{P}} \emptyset = -1$.

Let us denote $\sum_{i \in L} U_i$ the intersection of all subspaces of $\mathcal{P}^n$ containing the set $\{U_i \mid i \in L\}$ of subspaces. Obviously, $\sum_{i \in L} U_i$ is also a subspace.

In what follows we will consider the notion of dimension of a subspace in the projective sense. However, we put $\dim_{\mathcal{P}} U := \dim U$, i.e. the index $\mathcal{P}$ will be omitted.

Remark 3 If U and V are subspaces from $\mathcal{P}^n$, then

$$\dim U + \dim V = \dim(U + V) + \dim(U \cap V).$$

(See [3].)

Remark 4 Let $\mathcal{B}$ be a set of all points of the projective space $\mathcal{P}^n$. Then $(\mathcal{B}, \mathcal{P}^n)$ is a closure space. The closed set generated by $A \subseteq \mathcal{B}$ is a subspace in $\mathcal{P}^n$, denoted by $[A]$.

Let us consider independent sets and bases in $(\mathcal{B}, \mathcal{P}^n)$ according to Definition 3 and 4. Every basis has cardinality $n+1$ and the closure space has dimension n.

In what follows we will not exactly distinguish between the projective space $\mathcal{P}^n$ and the closure space $(\mathcal{B}, \mathcal{P}^n)$. So, we can speak about independent sets in $\mathcal{P}^n$, bases in $\mathcal{P}^n$ etc.

Remark 5 Let P be a subspace and U be a hyperplane of the projective space $\mathcal{P}^n$ such that $P \not\subseteq U$. Then there exists a point of P which is not contained in U, so $\dim(U + P) = n$. We obtain $n - 1 + \dim P = n + \dim(U \cap P)$, from which $\dim(U \cap P) = \dim P - 1$ follows.

Theorem 2 *Let $U_1, \ldots, U_k$, $1 \leq k \leq n + 1$, be hyperplanes in the projective space $\mathcal{P}^n$. Then*

$$\dim \left(\bigcap_{i=1}^{k} U_i \right) \geq n - k.$$

Proof If $k = 1$, then $\dim U_1 = n - 1 = n - k$. Let $k = 2$. For $U_1 = U_2$ we get $\dim U_1 \cap U_2 = n - 1 > n - 2$. If $U_1 \neq U_2$, then $\dim U_1 \cap U_2 = n - 2 = n - k$ by Remark 5.

Let us assume that the presented inequality is valid for a certain k such that $1 \leq k < n + 1$. Let $U_1, \ldots, U_k, U_{k+1}$ be hyperplanes. We put $W = \bigcap_{1 \leq i \leq k} U_i$ and $V = \bigcap_{1 \leq j \leq k+1} U_j$, so $V = W \cap U_{k+1}$. If $W \subseteq U_{k+1}$, then $V = W$ and $\dim V = \dim W \geq n - k > n - (k + 1)$. For $W \not\subseteq U_{k+1}$ we get $\dim V = \dim(W \cap U_{k+1}) = \dim W - 1 \geq n - (k + 1)$ by Remark 5. □

Theorem 3 *Let $U_1, \ldots, U_k$, $1 \leq k \leq n + 1$, be hyperplanes in $\mathcal{P}^n$ and $n_k = \{1, \ldots, k\}$. Then the following conditions are equivalent:*

$$\forall i \in n_k : \bigcap_{j \in n_k - \{i\}} U_j \not\subseteq U_i \tag{1}$$

$$\dim \bigcap_{j \in n_k} U_j = n - k \tag{2}$$

Proof We denote $A = \{U_1, \dots, U_k\}$ and $V_A = \bigcap_{j \in n_k} U_j$.

(1) $\Longrightarrow$ (2) Obviously, for $k = 1$ we get $\dim U_1 = n - 1 = n - k$. If $k = 2$, then $U_1 \neq U_2$ and $\dim V_A = n - 2 = n - k$.

Let us assume that the condition (1) implies (2) for a certain k, $1 \leq k < n+1$.

Let the set $B = \{U_1, \dots, U_k, U_{k+1}\}$ has the property (1) and consider $U_l \in B$. Then for $W = \bigcap_{j \in n_{k+1} - \{l\}} U_j$ we obtain $\dim W = n - k$ by the assumption. From $W \not\subseteq U_l$ we get $\dim V_B = \dim W \cap U_l = \dim W - 1 = n - (k+1)$ according to Remark 5.

(2) $\Longrightarrow$ (1) Let us take $U_l \in A$ and denote $W = \bigcap_{j \in n_k - \{l\}} U_j$.

We will assume that $W \subseteq U_l$. Hence, because of $V_A = U_l \cap W$, we have $V_A = W$ and $\dim W = n - k$. However, according to Theorem 2, $\dim W \geq n - (k - 1) = n - k + 1$. That is a contradiction. $\square$

Remark 6 Let U be a subspace of dimension k_1 in the projective space $\mathcal{P}^n$ and k_2 be a natural number such that $k_1 + k_2 < n$. Then there exists a subspace V of dimension k_2 such that $U \cap V = \emptyset$.

Now we define an incidence structure $\mathcal{J} = (G_k, M_r, I)$ on the projective space $\mathcal{P}^n$ of dimension $n \geq 3$ in the following way:
G_k contains all points of a subspace of dimension k in $\mathcal{P}^n$, $0 \leq k \leq n$,
M_r contains all subspaces of dimension r in $\mathcal{P}^n$, $0 \leq r \leq n - 1$ and
I is the incidence relation from $\mathcal{P}^n$ restricted to the set $G_k \times M_r$.

Remark 7 Consider a subset $A \subseteq G_k$. Then $A^{\uparrow}$ is a set of subspaces from M_r, which contain A.

If $m \in M_r$, then $m^{\downarrow}$ is a set of points of the subspace m contained in G_k.

We will denote subspaces $m \in M_r$ by usual symbols U, V etc. It means we put $m := U$, where $m^{\downarrow} = U \cap G_k$. So, for $B \subseteq M_r$ we get $B^{\downarrow} = (\bigcap_{U \in B} U) \cap G_k$.

First we will consider a closure space $(G_k, \mathcal{G}_{\mathcal{J}})$, where $\mathcal{G}_{\mathcal{J}} = \{A \subseteq G_k \mid A = A^{\uparrow\downarrow}\}$. For a subset $A \subseteq G_k$ we obtain $[A] \subseteq G_k$ and $\dim[A] \leq k$.

(a) Assume that $\dim[A] \leq r$. Then $A^{\uparrow\downarrow}$ is the intersection of all subspaces containing A from M_r. Thus $A^{\uparrow\downarrow} = [A]$ and it follows $\langle A \rangle = [A]$ for the closure $\langle A \rangle$ from $(G_k, \mathcal{G}_{\mathcal{J}})$.

(b) If $\dim[A] > r$, then $A^{\uparrow} = \emptyset$. This implies $A^{\uparrow\downarrow} = G_k$ and $G_k = \langle A \rangle \neq [A]$ (for $A \subset G_k$).

Theorem 4 *A set $A \subseteq G_k$ is independent in $(G_k, \mathcal{G}_{\mathcal{J}})$ if and only if A is independent in $\mathcal{P}^n$ and $\dim[A] \leq r + 1$.*

Proof Let A be independent in $(G_k, \mathcal{G}_{\mathcal{J}})$. Then $a \notin A_a^{\uparrow\downarrow}$ for an arbitrary $a \in A$. Thus $A_a^{\uparrow\downarrow} \neq G_k$ and $\dim[A_a] \leq r$ according to (b). From (a) we obtain $[A_a] = A_a^{\uparrow\downarrow}$ and $a \notin [A_a]$. Hence A is independent in $\mathcal{P}^n$ and we get $\dim[A_a] + 1 = \dim[A] \leq r + 1$.

Conversely, let A be independent in $\mathcal{P}^n$ and $\dim[A] \leq r+1$. Then $\dim[A_a] \leq r$ for $a \in A$ and $[A_a] = A_a^{\uparrow\downarrow}$ according to (a). Therefore, $a \notin [A_a]$ implies $a \notin A_a^{\uparrow\downarrow}$ and A is independent in $(G_k, \mathcal{G}_{\mathcal{J}})$. $\square$

Theorem 5 *In the closure space $(G_k, \mathcal{G_J})$ there always exist bases and all of them have the same cardinality.*

If $r < k$, then cardinality of bases is $r+2$. If $k \leq r$, then cardinality of bases is $k+1$.

Proof 1. Let $r < k$. First we will prove the existence of a basis in $(G_k, \mathcal{G_J})$.

Because of $r+1 \leq k \leq n$, there exists a set $A \subseteq G_k$, $|A| = r+2$, which is independent in $\mathcal{P}^n$. Then $\dim[A] = r+1$ and A is independent in $(G_k, \mathcal{G_J})$ by Theorem 4. We obtain $A^{\uparrow\downarrow} = G_k$ according to (b) and so A is a basis in $(G_k, \mathcal{G_J})$.

Now we show that every basis consists of $r+2$ elements. Let A be a basis in $(G_k, \mathcal{G_J})$. Then A is independent in $(G_k, \mathcal{G_J})$. Thus A is independent in $\mathcal{P}^n$ by Theorem 4 and $\dim[A] \leq r+1$.

Consider $\dim[A] \leq r$. Then $[A] = A^{\uparrow\downarrow}$ according to (a) and since A is the basis in $(G_k, \mathcal{G_J})$ we obtain $A^{\uparrow\downarrow} = [A] = G_k$. Hence $\dim[A] = k$ and $k \leq r$, a contradiction.

So $\dim[A] = r+1$ and $|A| = r+2$.

2. Let $k \leq r$. If A is a basis of the subspace G_k in $\mathcal{P}^n$, then A is independent in $\mathcal{P}^n$ and $[A] = G_k$, $|A| = k+1$. A is independent in $(G_k, \mathcal{G_J})$ (Theorem 4) and $[A] = A^{\uparrow\downarrow}$ according to (a). From this $A^{\uparrow\downarrow} = G_k$ and A is a basis of $(G_k, \mathcal{G_J})$.

Every basis in $(G_k, \mathcal{G_J})$ has cardinality $k+1$: Let A be a basis in $(G_k, \mathcal{G_J})$. Then $A \subseteq G_k$, A is independent in $(G_k, \mathcal{G_J})$ and A is also independent in $\mathcal{P}^n$ by Theorem 4. We know that $\dim[A] \leq k$. Assume that $\dim[A] < k$. Then $\dim[A] < r$ and $A^{\uparrow\downarrow} = [A]$ according to (a). Hence $A^{\uparrow\downarrow} \neq G_k$, which implies that A is not a basis in $(G_k, \mathcal{G_J})$, a contradiction.

So $\dim[A] = k$ and $|A| = k+1$. □

Now we will deal with the closure space $(M_r, \mathcal{M_J})$.

For $B \subseteq M_r$ we denote $V_B = \bigcap_{U \in B} U$ and $W_B = \bigcap_{U \in B} U \cap G_k = V_B \cap G_k$. Then $B^{\downarrow} = W_B$ and $B^{\downarrow\uparrow} = \{U \in M_r \mid W_B \subseteq U\}$.

If $B = \{U_1, \ldots, U_l\} \subseteq M_r$, then we put

$$W_B^i = \bigcap_{j \in n_l - \{i\}} U_j \cap G_k$$

for $i \in n_l = \{1, \ldots, l\}$.

We get $B_i^{\downarrow\uparrow} = \{U \in M_r \mid W_B^i \subseteq U\}$ for $B_i = B - \{U_i\}$.

Theorem 6 *The set $B = \{U_1, \ldots, U_l\} \subseteq M_r$ is independent in $(M_r, \mathcal{M_J})$ if and only if $W_B^i \neq W_B$ for all $i \in n_l$.*

Proof If B is independent, then $U_i \notin B_i^{\downarrow\uparrow}$ for all $i \in n_l$. Assume that $W_B^i = W_B$ for certain i. Since $W_B \subseteq U_i$, we obtain $W_B^i \subseteq U_i$ and $U_i \in B_i^{\downarrow\uparrow}$, a contradiction.

Conversely, let $W_B^i \neq W_B$ for all $i \in n_l$. We will assume that $U_i \in B_i^{\downarrow\uparrow}$ for certain i. Then $W_B^i \subseteq U_i$. Since $W_B^i \subseteq U_j \cap G_k$ for all $j \in n_l$, $j \neq i$, we get $W_B^i \subseteq U_p \cap G_k$ for all $p \in n_l$ and $W_B^i \subseteq W_B$. Hence $W_B^i = W_B$, which is a contradiction. So $U_i \notin B_i^{\downarrow\uparrow}$ and B is independent in $(M_r, \mathcal{M_J})$. □

Remark 8 An incidence structure $\mathcal{J}_1 = (G, M_r, I)$, where G is a set of all points of the projective space $\mathcal{P}^n$, is a special case of the structure $\mathcal{J} = (G_k, M_r, I)$. Then $k = n$.

If we put $V_B^i = \bigcap_{j \in n_l - \{i\}} U_j$ for $B = \{U_1, \ldots, U_l\}$, then we get a criterion of independence as the special case of Theorem 6:

The set B is independent in $(M_r, \mathcal{M}_{\mathcal{J}})$ if and only if $V_B^i \neq V_B$ for all $i \in n_l$. From this Theorem 7 follows immediately.

Theorem 7 *If a set B is independent in $(M_r, \mathcal{M}_{\mathcal{J}})$, then it is independent in $(M_r, \mathcal{M}_{\mathcal{J}_1})$.*

Theorem 8 *The maximal basis in $(M_r, \mathcal{M}_{\mathcal{J}})$, i.e. the basis of maximal cardinality, has $r+2$ elements for $r < k$ and $k+1$ elements for $k \leq r$.*

Proof 1. Let $r < k$. First we will construct a basis of cardinality $r+2$: There exists a subspace R in G_k, $\dim R = r+1$. Hyperplanes in R are subspaces of dimension r and so they belong to M_r. By Theorem 3, there exists a set $B = \{U_1, \ldots, U_{r+2}\}$ of hyperplanes in R such that $V_B = \bigcap_{i \in n_{r+2}} U_i = \emptyset$ and $V_B^i = \bigcap_{j \in n_{r+2} - \{i\}} U_j \neq \emptyset$ for all $i \in n_{r+2} = \{1, \ldots, r+2\}$.

Since $U_i \subseteq G_k$ for all $i \in n_{r+2}$, we obtain $V_B = W_B$ and $V_B^i = W_B^i$ for all $i \in n_{r+2}$. Hence B is independent in $(M_r, \mathcal{M}_{\mathcal{J}})$ by Theorem 6. Because of $B^{\downarrow\uparrow} = W_B^{\uparrow} = \emptyset^{\uparrow} = M$, B is a basis in $(M_r, \mathcal{M}_{\mathcal{J}})$.

Now we show that every basis in $(M_r, \mathcal{M}_{\mathcal{J}})$ has at most $r+2$ elements: Consider $U_1, U_2 \in M_r$, $U_1 \neq U_2$. If we put $W_2 = U_1 \cap U_2$, then $\dim W_2 < r$. Let us take $U_3 \in M_r$ such that $\{U_1, U_2, U_3\}$ is an independent set in $(M_r, \mathcal{M}_{\mathcal{J}_1})$. Thus $W_3 = U_1 \cap U_2 \cap U_3 = W_2 \cap U_3$ and $W_2 \neq W_3$. This implies $\dim W_3 < \dim W_2$ and so $\dim W_3 < r - 1$.

If we continue in the same way, then $\dim W_i = r - (i-2)$. $\{U_1, \ldots, U_i\}$ is a basis in $(M_r, \mathcal{M}_{\mathcal{J}_1})$ if $\dim W_i = -1$. Thus $i < r+3$. So, a basis in $(M_r, \mathcal{M}_{\mathcal{J}_1})$ has maximal cardinality $r+2$.

Let $B = \{U_1, \ldots, U_{r+3}\}$ be a basis in $(M_r, \mathcal{M}_{\mathcal{J}})$. Then B is independent in $(M_r, \mathcal{M}_{\mathcal{J}})$ and it is independent also in $(M_r, \mathcal{M}_{\mathcal{J}_1})$ by Theorem 7. However, that is a contradiction because a basis is the maximal independent set.

2. Let $k \leq r$. First we will construct a basis of cardinality $k+1$. Because of $r \leq n-1$ there exists a subspace R containing G_k in $\mathcal{P}^n$, $\dim R = r+1$. Hyperplanes in R belong to M_r. Each subspace in R is an intersection of hyperplanes in R. We can take hyperplanes $U_1, \ldots, U_m$ in R such that they satisfy conditions from Theorem 3 and $G_k = \bigcap_{j \in n_m} U_j$ where $r + 1 - m = k$. There exist (by Theorem 3 again) hyperplanes $U_{m+1}, \ldots, U_{m+k+1}$ such that for $C = \{U_1, \ldots, U_{m+k+1}\}$ we obtain $V_C = \emptyset$ and $V_C^i \neq \emptyset$ for all $i \in n_{m+k+1}$. Let us put $B = \{U_{m+1}, \ldots, U_{m+k+1}\}$. Then $W_B = V_B \cap G_k = \bigcap_{j \in n_{m+k+1}} U_j = V_C = \emptyset$ and $W_B^i = V_B^i \cap G_k = V_C^i \neq \emptyset$ for all $i \in n_{m+k+1}$. It means (by Theorem 6) that B is a basis of cardinality $k+1$ in $(M_r, \mathcal{M}_{\mathcal{J}})$.

Finally we show that every basis in $(M_r, \mathcal{M}_{\mathcal{J}})$ has at most $k+1$ elements: Consider $U_1 \in M_r$. Then obviously $\dim U_1 \cap G_k \leq k$. If $\dim U_1 \cap G_k = k$, then $G_k \subseteq U_1$. For $B = \{U_1\}$ we get $B_1 = B - \{U_1\} = \emptyset$, $B_1^{\downarrow} = \emptyset^{\downarrow} = G_k$ and $B_1^{\downarrow\uparrow} = \{U \in M_r \mid G_k \subseteq U\}$. Thus $U_1 \in B_1^{\downarrow\uparrow}$ and B is not independent in $(M_r, \mathcal{M}_{\mathcal{J}})$.

Let us assume that $\dim U_1 \cap G_k < k$, $U_2 \in M_r$ and the set $\{U_1, U_2\}$ is independent in $(M_r, \mathcal{M}_{\mathcal{J}})$. If we put $W_2 = (G_k \cap U_1) \cap U_2$, then $W_2 \neq G_k \cap U_1$ and $\dim W_2 < k - 1$. We can continue in the same way. In the end we will obtain $\dim W_i < k - (i - 1)$ which for $\dim W_i = -1$ implies $i < k + 2$. □

Theorem 9 *The minimal basis in the closure space $(M_r, \mathcal{M}_{\mathcal{J}})$ has cardinality $d = [\frac{k}{n-r}] + 1$ where $[\frac{k}{n-r}]$ denotes the integer part of the number $\frac{k}{n-r}$.*

Proof We will construct the minimal basis.

1. Let $r + k < n$. Then (by Remark 6) there exists a subspace $U_1 \in M_r$ such that $U_1 \cap G_k = \emptyset$. $B = \{U_1\}$ is a basis in $(M_r, \mathcal{M}_{\mathcal{J}})$ and $d = 1$.

2. Let $r + k \geq n$. Then $r + k = \dim(U_1 + G_k) + \dim(U_1 \cap G_k) \geq n$ for any $U_1 \in M_r$. Hence each subspace from M_r has a nonempty intersection with G_k. Consider U_1 such that $\dim U_1 \cap G_k$ is minimal. Thus $\dim(U_1 + G_k) = n$ and $\dim(U_1 \cap G_k) = r + k - n$. We put $W_1 = U_1 \cap G_k$.

3. Let $r + (r + k - n) < n$. Then there exists a subspace $U_2 \in M_r$ such that $U_2 \cap W_1 = \emptyset$ by Remark 6 again. We put $B = \{U_1, U_2\}$. Since $W_B = U_1 \cap U_2 \cap G_k = W_1 \cap U_2 = \emptyset$, $W_B^1 = U_2 \cap G_k \neq \emptyset$ and $W_B^2 = U_1 \cap G_k \neq \emptyset$, B is a basis in $(M_r, \mathcal{M}_{\mathcal{J}})$ according to Theorem 6. In this case $d = 2$.

4. Let $2r + k - n \geq n$. Then $2r - n + k = \dim(U_2 + W_1) + \dim(U_2 \cap W_1) \geq n$ for any subspace $U_2 \in M_r$. Consider $U_2 \in M_r$ such that dimension of $W_2 = U_2 \cap W_1$ is minimal. Thus $\dim(U_2 + W_1) = n$ and $\dim W_2 = 2r - 2n + k \geq 0$. This implies that every subspace from M_r has a nonempty intersection with W_1.

5. Let $3r - 2n + k < n$. Then there exists a subspace $U_3 \in M_r$ such that $U_3 \cap W_2 = \emptyset$. We put $B = \{U_1, U_2, U_3\}$. Hence $W_B = (U_1 \cap U_2 \cap U_3) \cap G_k = W_2 \cap U_3 = \emptyset$. It is easy to see that $\dim W_1 \leq \dim(U_2 \cap G_k)$ and $U_3 \cap (U_2 \cap G_k) \neq \emptyset$. Thus $W_B^1 \neq \emptyset$, $W_B^2 = U_3 \cap W_1 \neq \emptyset$ and $W_B^3 = U_2 \cap W_1 \neq \emptyset$. B is independent in $(M_r, \mathcal{M}_{\mathcal{J}})$ by Theorem 6 and because of $W_B = \emptyset$ it is a basis in $(M_r, \mathcal{M}_{\mathcal{J}})$ of cardinality $d = 3$.

6. Let $3r - 2n + k \geq n$. Consider $U_3 \in M_r$ such that dimension of $W_3 = U_3 \cap W_2$ is minimal.

If we continue in the same way, then we get the following inequalities for cardinality d of the minimal basis:

$$dr - (d-1)n + k < n \quad \text{and} \quad (d-1)r - (d-2)n + k$$

It follows that

$$d > \frac{k}{n-r} \quad \text{and} \quad d \leq 1 + \frac{k}{n-r}.$$

We obtained

$$d = \left[\frac{k}{n-r}\right] + 1. \qquad \square$$

Example 1 Let us consider the incidence structure $\mathcal{J} = (G_k, M_r, I)$ and let us find the maximal and the minimal basis in the closure space $(M_r, \mathcal{M}_\mathcal{J})$.

(a) Let $n = 3$, $r = k = 1$. Then G_k is a line and subspaces from M_r are also lines. The minimal basis consists of one line U_1 which is disjoint with G_k.

Cardinality of the maximal basis is $k + 1 = 2$. This basis consists of two lines U_1, U_2 which intersect in one point and each of them intersect G_k.

(b) Let $n = 4$, $k = 3$, $r = 2$. Then the minimal basis has 2 elements and the maximal basis has 4 elements. Let us take a plane U_1 such that $W_1 = U_1 \cap G_k$ is a line. Then take a plane U_2 disjoint with W_1. Obviously $\{U_1, U_2\}$ is a basis in $(M_r, \mathcal{M}_\mathcal{J})$.

The maximal basis consists of 4 planes, all of them are contained in G_k. They all have the empty intersection but an intersection of every 3 of them is not empty.

It is easy to see that there exists also a basis of cardinality 3.

Theorem 10 *All bases in the closure space $(M_r, \mathcal{M}_\mathcal{J})$ have the same cardinality (i.e. the maximal basis is equal to the minimal one) if and only if some of these conditions is satisfied:*

(1) $r = n - 1$

(2) $k = 0$

(3) $k = n,\ r = 0$

Proof We denote $D(d)$ cardinality of the maximal (minimal) basis. First we will show that the conditions are sufficient:

(1) Let $r = n - 1$. Then $d = k + 1$. For $k \leq n - 1$ we get $D = k + 1 = d$. If $k = n$, then $D = r + 2 = n + 1 = k + 1 = d$.

(2) From $k = 0$ follows $d = 1 = D$ immediately.

(3) Let $k = n$ and $r = 0$. Then $d = 2$ and $D = r + 2 = 2$.

Conversely, let us assume that $d = D$.

(a) Consider $k \leq r$. Then from the equality

$$\left[\frac{k}{n-r}\right] + 1 = k + 1$$

follows

$$\left[\frac{k}{n-r}\right] = k \leq \frac{k}{n-r}.$$

This yields $k = 0$ or $n - r \leq 1$, thus $r \geq n - 1$. Because of $r \leq n - 1$, we get $r = n - 1$.

(b) Let $k > r$. We distinguish two cases:

1. If $k = n$, then the equality $d = D$ means

$$r + 2 = \left[\frac{n}{n-r}\right] + 1.$$

From this

$$r + 1 \leq \frac{n}{n-r} \quad \text{which is equivalent to } nr + n - r^2 - r \leq n.$$

This yields $r(n-1-r) \leq 0$. Since $r \leq n-1$, only two possibilities remain: $r = 0$ or $r = n-1$.

2. For $k < n$ we obtain

$$r+1 = \left[\frac{k}{n-r}\right].$$

This is equivalent to

$$r+1 \leq \frac{k}{n-r} < \frac{n}{n-r}$$

and from this

$$nr + n - r^2 - r < n \quad \text{if and only if} \quad r(n-1-r) < 0$$

which is not possible for any r because $0 \leq r \leq n-1$. □

Remark 9 If some of the three conditions from Theorem 10 is satisfied, then all bases in the closure spaces $(G_k, \mathcal{G}_{\mathcal{J}})$ and $(M_r, \mathcal{M}_{\mathcal{J}})$ induced by the incidence structure $\mathcal{J} = (G_k, M_r, I)$ have the same cardinality.

References

[1] Ganter B., Wille R.: *Formale Begriffsanalyse.* Mathematische Grundlagen, Springer-Verlag, 1996.

[2] Machala F., Slezák V.: *Incidence Structures and Closure Spaces.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **36** (1997), 149–156.

[3] Baer R.: *Linear Algebra and Projective Geometry.* New York, 1952.

[4] Ohn C.: *Notes on Dimensional Closure Spaces.* J. Combin. Theory A 55 (1990), 140–142.